Poisson-saddlepoint approximation for spatial point processes

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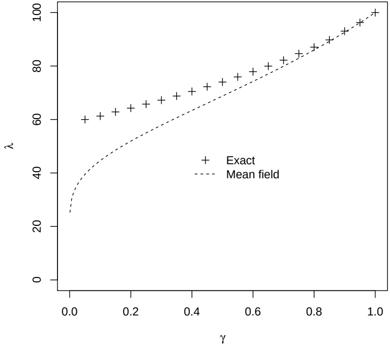
- Many important properties of a Gibbs process are **not known** as an explicit function of the model parameters
  - intensity
  - pair correlation function
  - ...
- Approximations exist, but have limitations.

Example: mean field approximation

Classical "mean field" approximation to λ for a Strauss process with parameters $\beta = 100$, $r = 0.05$ for various values of γ



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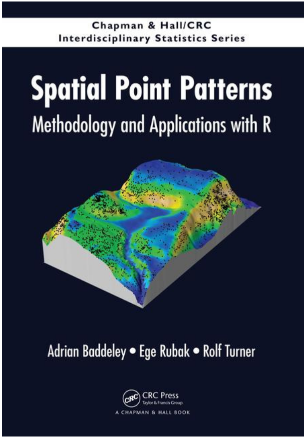
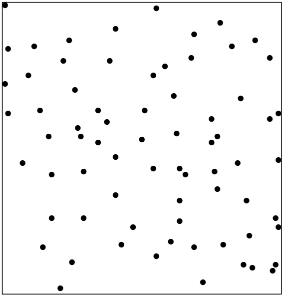


Software for spatial point process data
spatstat.org

Dataset swedishpines



```
library(spatstat)
fit <- ppm(swedishpines ~ 1, Strauss(9))
parameters(fit)
X <- simulate(fit)
```

Setting

- $\mathbf{X}$ is a stationary Gibbs point process on $\mathbb{R}^d$, with intensity $\lambda > 0$ and conditional intensity $\Lambda_{\mathbf{X}}(u; \mathbf{X})$.

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- Typically the expectation on the RHS of (1) is intractable.

Pairwise interaction Gibbs processes

A stationary pairwise interaction process has conditional intensity

$$\Lambda_{\mathbf{X}}(u; \mathbf{x}) = \beta \prod_i g(x_i - u),$$

where $\beta > 0$ is a parameter and $g : \mathbb{R}^d \to [0, \infty)$ is the interaction function.

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The RHS of (2) is not known explicitly (it involves the probability generating functional of $\mathbf{X}$). "Self-consistency equation"

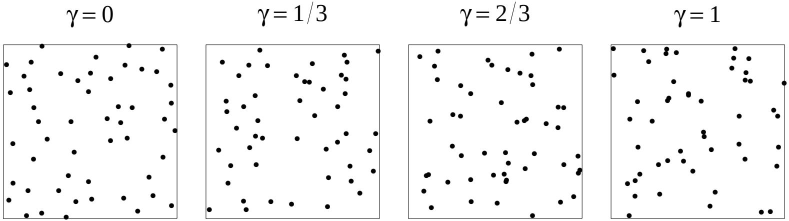
Special case: Strauss process

The Strauss process is the pairwise interaction process with

$$g(u) = \begin{cases} \gamma & \text{if } \|u\| \leq r \\ 1 & \text{otherwise} \end{cases}$$

where $r > 0$ is a fixed threshold, and $0 \leq \gamma \leq 1$ is the interaction parameter.

Special case: Strauss process



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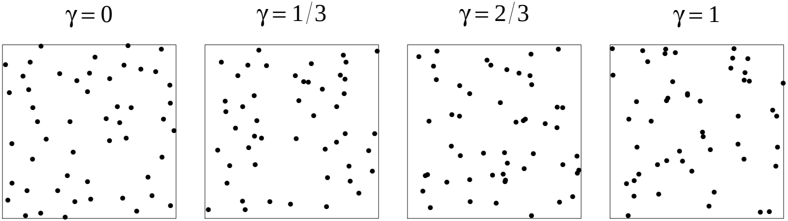

$$\lambda \approx 50$$

The conditional intensity is

$$\Lambda_{\mathbf{X}}(u; \mathbf{x}) = \beta \gamma^{t(u,\mathbf{x})}$$

where

$$t(u, \mathbf{x}) = \sum_i \mathbf{1}\{\|x_i - u\| \leq r\}.$$

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2. Poisson-saddlepoint approximation

- Replace $\mathbb{E}[\Lambda_{\mathbf{X}}(0, \mathbf{X})]$ by $\mathbb{E}[\Lambda_{\mathbf{X}}(0, \Pi_\lambda)]$ where Π_λ is the homogeneous Poisson process with intensity λ.

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B & N, *Electronic J. Statist.* 2012

Case of pairwise interaction

For a stationary pairwise interaction process $\mathbf{X}$ with conditional intensity $\Lambda_{\mathbf{X}}(u, \mathbf{X}) = \beta \prod g(u - x_i)$,

$$
\begin{aligned}
M(\lambda) &= \beta\, \mathbb{E}_{\mathsf{Pois}(\lambda)}[\prod_i g(x_i)] \\
&= \beta \exp\left(\lambda \int_{\mathbb{R}^d} [g(u) - 1]\, \mathrm{d}u \right) \\
&= \beta \exp(-\lambda G)
\end{aligned}
$$

where

$$
G = \int_{\mathbb{R}^d} [1 - g(u)]\, \mathrm{d}u
$$

is the "second Mayer cluster integral".

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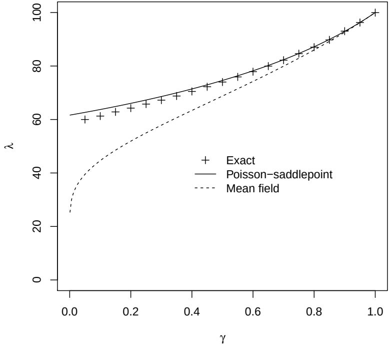
Coincides with the *Poisson-Boltzmann-Emden approximation*

Example: Strauss process

Approximation to λ for a Strauss process with parameters $\beta = 100$, $r = 0.05$ for various values of γ



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Rather, a functional of $\mathbf{X}$ is approximated by a tilted version of the functional of the Poisson process.

(The *mean field* approximation comes from Poisson process approximation: it is the value of λ which minimises Kullback-Leibler divergence $K(\Pi_\lambda \| \mathbf{X})$.)

Bounds

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and

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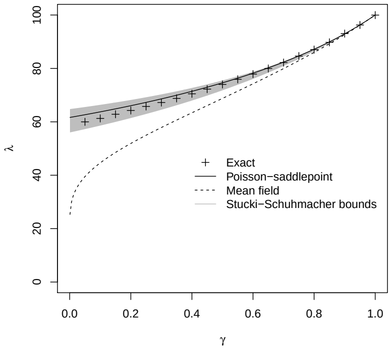
Approximation to λ for a Strauss process with parameters $\beta = 100$, $r = 0.05$ for various values of γ



Implementation in R

```
  >  library(spatstat)
  >  fit <- ppm(swedishpines ~ 1, Strauss(9))
  >  parameters(fit)
  beta    gamma    r
0.0546    0.264    9
  >  intensity(fit)
0.0094
```

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so that the Poisson-saddlepoint approximation is the solution of

$$\lambda = M(\lambda, \theta).$$

Existence and uniqueness

Suppose the conditional intensity is **monotone increasing** in the sense that, for two point patterns $\mathbf{x}$ and $\mathbf{y}$,

$$\mathbf{x} \subset \mathbf{y} \;\Rightarrow\; \lambda_\theta(u, \mathbf{x}) \le \lambda_\theta(u, \mathbf{y}).$$

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Then $M(\lambda, \theta) = \mathbb{E}[\Lambda_\theta(0, \Pi_\lambda)]$ is a nondecreasing, continuous function of λ, by a simple coupling argument. This can be used to prove that λ^{PS} exists, and under additional conditions, that it is unique.

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$$M(\lambda, \theta) = e^{-\mu} \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \mathbb{E}[\Lambda_\theta(0, \mathbf{Z}_n)],$$

where $\mu = \lambda \pi R^2$, and $\mathbf{Z}_n = \mathbf{Z}_{n,R}$ is the binomial point process consisting of exactly n points independently and uniformly distributed in $b(0, R)$.

Series expansion

$$M(\lambda, \theta) = e^{-\mu} \sum_{n=0}^{\infty} \frac{\mu^n}{n!} m_n(\theta, R)$$

where

$$m_n(\theta, R) = \mathbb{E}[\Lambda_\theta(0, \mathbf{Z}_{n,R})]$$

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5. Solve $\lambda = M(\lambda, \theta)$ to obtain λ^{PS}.

Loglinear form

Assume conditional intensity has loglinear form

$$\Lambda_\theta(u, \mathbf{x}) = \exp(\theta^\top T(u, \mathbf{x})),$$

where $T(u, \mathbf{x})$ is a vector-valued statistic with finite range R:

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Then

$$m_n(\theta, 1) = \mathbb{E}[\exp(\theta^\top T(0, \mathbf{Z}_{n,1}))]$$

is the moment generating function of $T_n = T(0, \mathbf{Z}_{n,1})$.

Strategy for loglinear case

- Tabulate the cumulative distribution function of $T_n = T(0, \mathbf{Z}_{n,1})$ for each $n = 0, 1, \ldots, N$ up to a moderately large N.

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- Find the asymptotic behaviour of T_n as $n \to \infty$.
- Use these results to approximate the m.g.f. of T_n for all n.

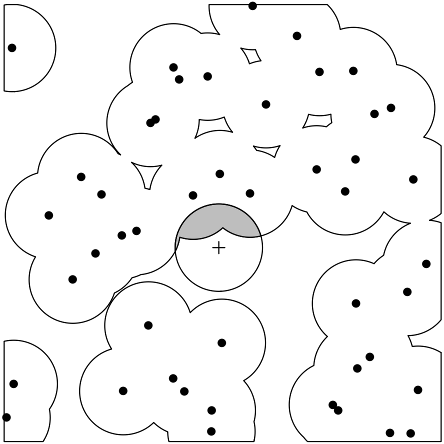
4. Area-interaction process

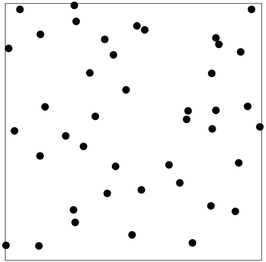
The **area-interaction** process with parameters $\beta, \eta, r \geq 0$ has conditional intensity

$$\Lambda(u, \mathbf{x}) = \beta \eta^{C(u,\mathbf{x},r)},$$

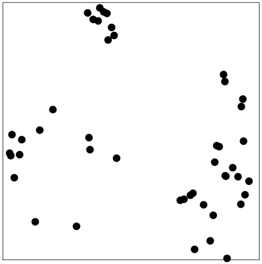
where

$$C(u, \mathbf{x}, r) = \frac{1}{\pi r^2} \left| b(u, r) \cap \bigcup_{i=1}^{n(\mathbf{x})} b(x_i, r) \right|$$





$\eta = 0.5$



$\eta = 2$

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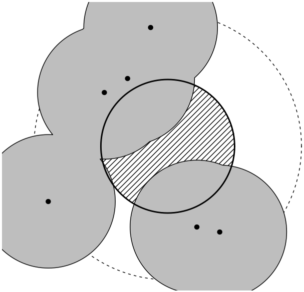
$$m_n(\theta, 1) = \mathbb{E}[\eta^{-A_n}]$$

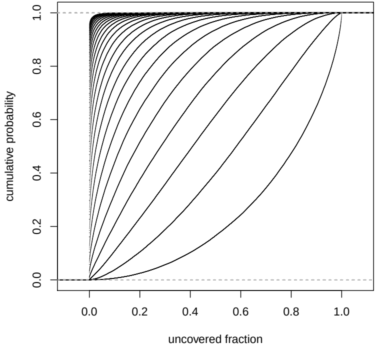
where

$$A_n = 1 - C(0, \mathbf{Z}_{n,2}, 1)$$

is the uncovered area fraction (area fraction of the unit disc which is not covered by n unit discs whose centres are independently uniformly distributed in the disc of radius 2).



Stage 1: Tabulate distribution of A_n for small n



Stage 2: Asymptotics of A_n as $n \to \infty$

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Peter Hall

Roger Miles

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Using two theorems of Peter Hall (1988), as $n \to \infty$

$$\mathbb{P}\left\{A_n = 0\right\} \sim 1 - \min\left\{1, 3\left(1 + \frac{n^2}{16\pi}\right) e^{-n/4}\right\}$$

$$\mathbb{E}[A_n \mid A_n > 0] \sim 16n^{-2} \exp\{n(\frac{1}{4} - \log\frac{4}{3})\}$$

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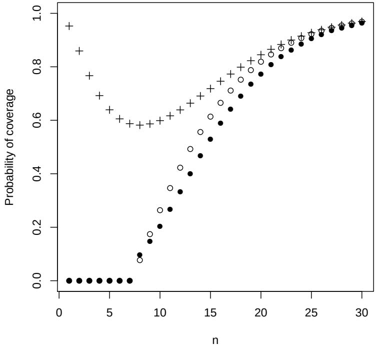
For better performance for very small n, we replace the latter expression by

$$\mathbb{E}[A_n \mid A_n > 0] \sim 16(n+3)^{-2} \exp\{n(\frac{1}{4} - \log\frac{4}{3})\}$$

Probability of complete coverage $\mathbb{P}\{A_n = 0\}$



Hall's limit

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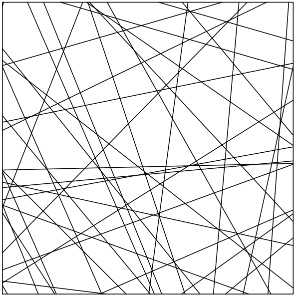
A famous theorem of Peter Hall (1988, Theorem 3.7) states that, in a high-intensity Boolean model, a typical chink (connected component of the uncovered region) is asymptotically equivalent to the typical polygon of the Poisson line tessellation.

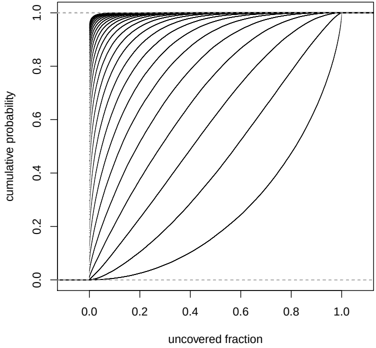
Hall's limit



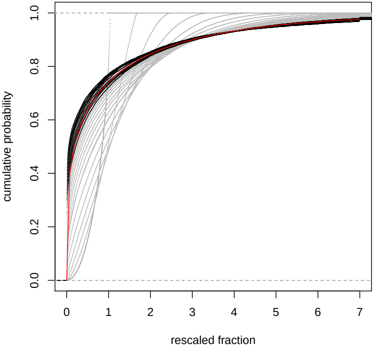
Poisson line tessellation

Roger Miles (1973, Section 8) showed that the area A of the typical polygon in the Poisson line tessellation has $\mathrm{var}\,[A]\,/\mathbb{E}[A]^2 = \pi^2/2 - 1$.

Conditional distribution of A_n given $A_n > 0$



Conditional distribution of $A_n/\mathbb{E}(A_n)$ given $A_n > 0$



Grey: $n = 1, 2, \ldots, 25$
Black: $n = 25, \ldots, 50$
Red: gamma distribution with mean 1 and variance $\zeta = \pi^2/2 - 1$

Putting it together

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- If $N > 25$, then for $25 < n \leq N$, approximate

$$\mathbb{E}[\eta^{-A_n}] \approx p_n + (1 - p_n)(1 - \zeta m_n \log \eta)^{-1/\zeta}$$

where p_n, m_n are the asymptotic approximations to $\mathbb{P}\{A_n > 0\}$ and $\mathbb{E}[A_n \mid A_n > 0]$ while $\zeta = \pi^2/2 - 1$;

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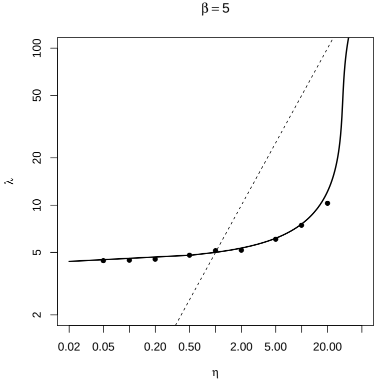
- For $n > N$, approximate $\mathbb{E}[\eta^{-A_n}] \approx 1$.
- Compute the series expansion of $M(\lambda)$.

Implementation in R

```
>  library(spatstat)
>  fit <- ppm(swedishpines ~ 1, AreaInter(3))
>  parameters(fit)
 beta     eta   r
0.0123  0.0123  3
>  intensity(fit)
0.0078
```

Performance of approximation



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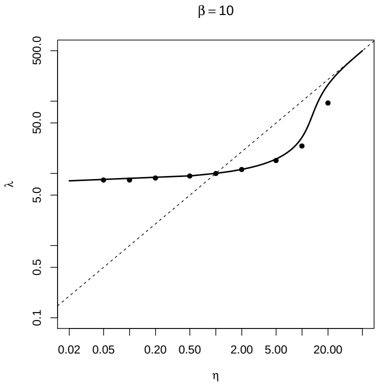

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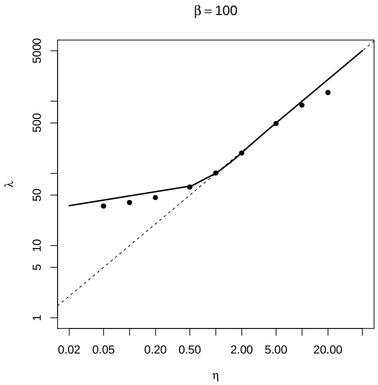

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- Also works for Geyer saturation process
- Can be extended to non-stationary Gibbs models

References

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