On the intersections of $\times 2$ and $\times 3$ invariant sets – a conjecture of Furstenberg

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Problem of Furstenberg (1969): intersections of Cantor sets

- Dynamical system (X, f): X compact metric space, $f : X \to X$.
- Principal example : $X = [0, 1]$, $f : x \mapsto 2x \mod 1$.
- f-invariant sets:

$$\{A \subset X : A \textit{ compact } , f(A) \subset A\} .$$

$\rightarrow$ display dynamical properties of f.

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$\to$ display dynamical properties of f.

- $(X, f), (X, g)$: compare the two systems $\to$ compare their invariant sets.
- Particularly: when f et g are "independent" $\to$ few common dynamical structures $\to$ the intersections of their invariant sets should be "as small as possible".
- independence: arithmetical or geometrical
- Ex: $f : x \to 2x \mod 1$, $f : x \to 3x \mod 1$ ($\log 2/\log 3 \notin \mathbb{Q} \to$ multiplicatively independent)

Transversality of dynamical systems (Furstenberg 1969)

- (X, f), (X, g); dim $\to$ a dimension function (ex. $\dim_{\mathrm{H}}$)
- Furstenberg: f et g are called transverse if for all $A = f$-invariant et $B = g$-invariant, we have

$$\dim A \cap B \leq \max\{0, \dim A + \dim B - \dim X\}$$

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Let $p, q \in \mathbb{N}_{\geq 2}$ with $\log p / \log q \notin \mathbb{Q}$. If $A \subset [0, 1]$ is $\times p$-invariant and $B \subset [0, 1]$ is $\times q$-invariant, then for all $u, v \in \mathbb{R}$, we have

$$\dim_{\mathrm{H}} (uA + v) \cap B \leq \max\{0, \dim_{\mathrm{H}} A + \dim_{\mathrm{H}} B - 1\}.$$

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Object of this talk.

Landmark paper of Furstenberg (1967):

"Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation"

- Disjointness of dynamical systems.
- The celebrated $\times 2, \times 3$-rigidity result: if $A \subset [0,1]$ is simultaneously $\times 2$ and $\times 3$-invariant and $\sharp A = \infty$, then $A = [0,1]$.
- $\to$ if $x \in [0,1] \setminus \mathbb{Q}$, then $\{2^n 3^m x \mod 1\}_{n,m\in\mathbb{N}}$ is dense in $[0,1]$.
- The famous $\times 2, \times 3$ conjecture: if μ is a non-atomic measure, ergodic and simultaneously $\times 2$ and $\times 3$-invariant, then $\mu = \mathcal{L}$.
  (Partial solution: Rudolph; influenced: homogeneous dynamics and diophantine questions (e.g. Einsiedler-Katok-Lindenstrauss work))

Problem 1967 $\to$ "intersections of invariant measures";
Problem 1969 $\to$ "intersections des invariant sets".

Known results about Conjecture 1 (intersection Conjecture)

- Observation: for $A, B \subset \mathbb{R}$, $u, v \in \mathbb{R}$, we have

$$(uA + v) \cap B \xrightarrow{\textit{affine copy}} (A \times B) \cap \ell_{u,v},$$

where $\ell_{u,v} = \{(x, y) : y = ux + v\}$.

- Classical result of Marstrand on the sections of fractals:

Theorem (Marstrand, 1954)

Let $E \subset \mathbb{R}^2$ be a Borel set. Then for each $u \in \mathbb{R}$,

$$\dim_{\mathrm{H}}(E \cap \ell_{u,v}) \leq \max\{0, \dim_{\mathrm{H}} E - 1\} \text{ for } \mathcal{L}\text{-almost every } v.$$

- Consequence: if $A = \times p\text{-inv}$ and $B = \times q\text{-inv}$, then for each u

$$\dim_{\mathrm{H}}(uA + v) \cap B \leq \max\{0, \dim_{\mathrm{H}} A + \dim_{\mathrm{H}} B - 1\} \text{ for } \mathcal{L}\text{-a.e. } v$$

Proof: $E = A \times B$; $\dim_{\mathrm{H}} A \times B = \dim_{\mathrm{H}} A + \dim_{\mathrm{H}} B$.

- Marstrand: Furstenberg is true for "almost all" sections.

Dual problem of slices: Projection of fractals

- Heuristically: small slices/fibers $\longleftrightarrow$ large projections.

Conjectured by Furstenberg (late 60'), **proved** by Hochman-Shmerkin (2011) (special cases by Peres-Shmerkin, 2008)

If A is $\times 2$-invariant and B is $\times 3$-invariant, then

$$\dim_{\mathrm{H}} \pi(A \times B) = \min\{1, \dim_{\mathrm{H}} A + \dim_{\mathrm{H}} B\}$$

for any $\pi \in \Pi_{2,1} \setminus \{\pi_x, \pi_y\}$.

Partial results concerning all sections: Furstenberg

Theorem (Furstenberg, 1969)

Under the hypothesis of Conjecture 2, if there exist u_0, v_0 such that $\overline{\dim}_{\mathrm{B}}(u_0 A + v_0) \cap B = \gamma > 0$, then for $\mathcal{L}$-a.e. u, $\exists$ v such that

$$\dim_{\mathrm{H}}(uA + v) \cap B \geq \gamma.$$

As a (simple) corollary, for all u, v, we have

$$\overline{\dim}_{\mathrm{B}}(uA + v) \cap B \leq \max\{0, \dim_{\mathrm{H}} A + \dim_{\mathrm{H}} B - \frac{1}{2}\}.$$

Method (Furstenberg): CP-process/CP-chain.

Consequence: Conjecture 2 is true under the condition

$$\dim_{\mathrm{H}} A + \dim_{\mathrm{H}} B < \frac{1}{2}.$$

Some recent results (Feng-Huang-Rao, 2013)

- Feng-Huang-Rao: Affine embeddings of self-similar sets.
- $A =$ p-Cantor set if $\exists D \subset \{0, \cdots, p-1\}$ such that
$$A = \{\sum_{k\geq 1} p^{-k} x_k : x_k \in D\}.$$
Remark: A is a self-similar and $\times p$-invariant set.

Theorem (Feng-Huang-Rao, 2013)

Let $p, q \in \mathbb{N}_{\geq 2}$ with $\log p / \log q \notin \mathbb{Q}$. If $A = p$-Cantor and $B = q$-Cantor, then there is no affine embedding between A and B.

As a consequence, $\exists\ \delta = \delta(A, B) > 0$ (non-effective) such that
$$\dim_{\mathrm{H}}(uA + v) \cap B \leq \min\{\dim_{\mathrm{H}} A, \dim_{\mathrm{H}} B\} - \delta.$$

Remark: for the Conjecture of Furstenberg, one expects
$$\delta = 1 - \max\{\dim_{\mathrm{H}} A, \dim_{\mathrm{H}} B\}.$$

- Recent work of Feng (2015).

The intersection conjecture of Furstenberg is true.

Theorem (M.W, 2016)

Let $p, q \in \mathbb{N}_{\geq 2}$ with $\log p / \log q \notin \mathbb{Q}$. If $A =$ p-invariant and $B =$ q-invariant, then for all $u, v \in \mathbb{R}$,

$$\overline{\dim}_{\mathrm{B}}(uA + v) \cap B \leq \max\{0, \dim_{\mathrm{H}} A + \dim_{\mathrm{H}} B - 1\}.$$

We have a more general version of the theorem on the intersections of incommensurable homogeneous self-similar sets.

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Remark: The intersection conjecture of Furstenberg has been simultaneously and independently proved by P. Shmerkin using completely different (additive combinatorial) methods. (inspired by work of Hochman)

Another conjecture of Furstenberg on expansions of numbers in different bases.

- For $x \in [0,1]$, we denote the orbit of x under $\times m$ by
$$\mathcal{O}_m(x) = \{m^k x \mod 1 : k \in \mathbb{N}\}.$$
$C_m(x) := \dim_{\mathrm{H}} \overline{\mathcal{O}_m(x)} \to$ complexity of x in base m.
- The complexity of a given number (e.g. $\pi, \sqrt{2}$): fundamental and widely open question!
- Easy fact: if $x \in \mathbb{Q} \Rightarrow \dim_{\mathrm{H}} \overline{\mathcal{O}_m(x)} = 0$;
- If $\log p / \log q \in \mathbb{Q} \Rightarrow C_p(x) = C_q(x)$. e.g.: $C_{10}(x) = C_{100}(x)$.
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- Furstenberg: if $\log p / \log q \notin \mathbb{Q}$, then $\mathcal{O}_p(x)$ and $\mathcal{O}_q(x)$ can not both be "simple"!

Conjecture 2 (Furstenberg, 1969)

If $\log p / \log q \notin \mathbb{Q}$, then for all $x \in [0,1] \setminus \mathbb{Q}$, we have

$$\dim_{\mathrm{H}} \overline{\mathcal{O}_p(x)} + \dim_{\mathrm{H}} \overline{\mathcal{O}_q(x)} \geq 1.$$

$\Rightarrow$ Low complexity in one base implies high complexity in another base.

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- Conjecture 1 *vs* Conjecture 2: If $A = \times 2\text{-inv}$ and $B = \times 3\text{-inv}$, and if $\dim_{\mathrm{H}} A + \dim_{\mathrm{H}} B < 1$, then Conjecture 1 $\Rightarrow \dim_{\mathrm{H}} A \cap B = 0$, while Conjecture 2 $\Rightarrow A \cap B \subset \mathbb{Q}$.
- Applying Conjecture 2 to $\sqrt{2}, e, \pi, \ln 2, \ldots$

$$\textit{Exemple}: \quad \dim_{\mathrm{H}} \overline{\mathcal{O}_2(\pi)} + \dim_{\mathrm{H}} \overline{\mathcal{O}_{10}(\pi)} \geq 1 \quad !$$

- The conjecture goes beyond this:
  instead of p-adic expansion $\rightarrow$ continued fraction.
- arithmetical independence $\rightarrow$ geometrical independence.

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Conjecture: For $p \geq 2$ and all $x \in [0,1] \setminus \mathbb{Q}$, we have

$$\dim_{\mathrm{H}} \overline{\mathcal{O}_G(x)} + \dim_{\mathrm{H}} \overline{\mathcal{O}_p(x)} \geq 1.$$

Here $\mathcal{O}_G(x)$ is the orbit of x under the Gauss map.

- Example: for $x = \sqrt{2}$, $\mathcal{O}_G(x)$ is eventually periodic, so $\dim_{\mathrm{H}} \overline{\mathcal{O}_G(x)} = 0$,

$$\textit{Conjecture} \Rightarrow \dim_{\mathrm{H}} \overline{\mathcal{O}_p(\sqrt{2})} = 1!$$

- $x = \sum_n a_n p^{-n} \leftrightarrow E(x,p) = (a_n)_n \in \{0, \cdots, p-1\}^{\mathbb{N}}$.
- Bugeaud-Kim: if $\log p / log q \notin \mathbb{Q}$ et $x \in [0,1] \setminus \mathbb{Q}$, then the sequences $E(x,p)$ and $E(x,q)$ can not both be sturmian.

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Theorem (M.W, 2016) (A recent question of Furstenberg)

Suppose $\log p / \log q \notin \mathbb{Q}$. There exists a set $E \subset [0,1]$ with $\dim_{\mathrm{H}} E = 0$ ($\dim_{\mathrm{P}} E = 0$) such that for all $x \in [0,1] \setminus E$, we have

$$\dim_{\mathrm{H}} \overline{\mathcal{O}_p(x)} + \dim_{\mathrm{H}} \overline{\mathcal{O}_q(x)} \geq 1.$$

Recall that $\mathbb{Q} \subset E$.

Results on sections of self-similar sets with rotation

- Self-similar set: IFS $\{f_i(x) = \lambda_i O_{\theta_i} x + b_i\}_{i=1}^m$ on $\mathbb{R}^2$, $0 < \lambda_i < 1$ and O_{θ_i} = rotation of angle θ_i,

$$X = \bigcup_i f_i(X).$$

- Strong separation condition (SSC): $f_i(X) \cap f_j(X) = \emptyset$, $\forall i \neq j$.
- Homogeneous: for each $1 \leq i \leq m$, $f_i(x) = \lambda O_\theta x + b_i$.

Theorem 2 (M.W, 2016)

Let X be a self-similar as above. Suppose that θ is irrational and X satisfies the SSC. Then for each line $\ell \subset \mathbb{R}^2$,

$$\overline{\dim}_{\mathrm{B}} X \cap \ell \leq \max\{0, \dim_{\mathrm{H}} X - 1\}.$$

- The irrationality of θ is necessary. Ex: $C_{1/3} \times C_{1/3}$.

Strategy of proof for Theorem 2.

- Overall Strategy: if there exists ℓ such that $\overline{\dim}_{\mathrm{B}} X \cap \ell = \gamma > 0$, then

$$\Longrightarrow \dim_{\mathrm{H}} X \geq 1 + \gamma; \quad \leftrightarrow \dim_{\mathrm{B}} X \geq 1 + \gamma.$$

- For $\dim_{\mathrm{B}} X \geq 1 + \gamma$, it is sufficient to show: $\forall \epsilon > 0$, $\exists r_0 > 0, n_0 \in \mathbb{N}$ such that: for all $n \geq n_0$, $\exists B_n = B(x, e^{-n})$ and a set of directions $F_n \in S^1$ with $N_{e^{-n}}(F_n) \geq e^{n(1-\epsilon)}$ satisfying: $\forall \xi \in F_n$, $\exists \ell = \ell(\xi)$ such that
  - (1) ℓ has direction ξ;
  - (2) $\ell \cap B_n \neq \emptyset$;
  - (3) $\inf_y N_{e^{-n}}((X \cap \ell) \setminus B(y, r_0)) \geq e^{n(\gamma-\epsilon)}$.

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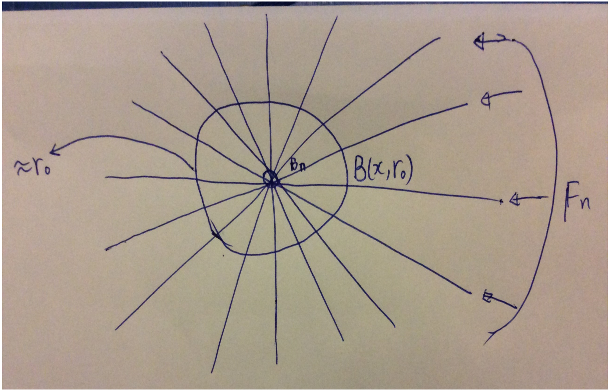

We will show:

$\forall \epsilon > 0$, $\exists r_0 > 0, n_0 \in \mathbb{N}$ such that: for all $n \geq n_0$, $\exists B_n \subset X$ with $N_{e^{-n}}(B_n) \leq e^{n\epsilon}$ and a set of directions $F_n \in S^1$ with $N_{e^{-n}}(F_n) \geq e^{n(1-\epsilon)}$ satisfying: $\forall \xi \in F_n$, $\exists \ell = \ell(\xi)$ such that

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Strategy of proof – dynamical system

- System: (X, T) with $T :=$ inverse of $\{f_i\}_{i=1}^m$,

$$T|_{f_i(X)} = f_i^{-1} = \frac{1}{\lambda} O_{-\theta}(x) + c_i.$$

  T is expanding and rotating.

- $T : \ell \cap X \mapsto \{\ell'_j \cap X\}_j$,
  if the direction of $\ell \cap X = \xi$ and the direction of $\ell'_j \cap X = \xi'$,
  then $\xi' = \xi - \theta$ for each j.

Strategy of proof (continued)

- For ℓ with $\ell \cap X \neq \emptyset$ and $z \in \ell \cap X$, we denote

$$\ell(n,z) = \textit{ unique section de } T^n(\ell \cap X) \text{ contenant } T^n(z).$$

Remark: direction changes $\to O_\theta^n$.

- Find a $\ell \cap X$ and a $z \in \ell \cap X$ such that $\exists B_n$ satisfying:
  - (1) $N_{e^{-n}}(B_n) \leq e^{n\epsilon}$;
  - (2) $N_{e^{-n}}\{n\theta : T^n(z) \in B_n\} \geq e^{n(1-\epsilon)}$;
  - (3) For most of $n \in \{n\theta : T^n(z) \in B_n\}$, we have

$$\inf_y N_{e^{-n}}(\ell(n,z) \setminus B(y, r_0)) \geq e^{n(\gamma-\epsilon)}.$$

Ergodic methods: Construction of a "good" T-invariant measure

- Construct a measure ν_∞: T-invariant (ergodic) such that for ν_∞-a.e. z, $\exists\ \ell \cap X$ s.t. for most k, we have

$$\inf_y N_{e^{-n}}(\ell(n,z) \setminus B(y, r_0)) \geq e^{n(\gamma-\epsilon)}.$$

- Construction: two steps:
  - CP-process of Furstenberg: starting from one ℓ_0 with $\overline{\dim}_{\mathrm{B}} X \cap \ell = \gamma > 0 \to$ a family of "good" measures supported on sections.
  - a nice argument of Hochman-Shmerkin (which relates the small-scale structure of a measure to the distribution of its T-orbits) $\to$ construct a T-invariant measure ν_∞ based on a good measure from the CP-process of Furstenberg.
  - Show that the measure ν_∞ satisfies the desired condition.

Ergodic methods II

- ν_∞: T-invariant (ergodice), for ν_∞-a.e. z, $\exists\ \ell$ s.t. $\ell(n,z)$ is "good" for most n.
- Third ingredient - an ergodic result ($\rightarrow$ Sinai's factor theorem): there exists a family of disjoint sets $\{B_n^j\}_j$ s.t.
    - $\nu_\infty(X \setminus \bigsqcup_j B_n^j) < \epsilon$;
    - $N_{e^{-n}}(B_n^j) \leq e^{n\epsilon}$;
    - $\mathcal{L}(\overline{\{n\theta : T^n(z) \in B_n^j\}}) \geq 1 - \epsilon$.

Merci de votre attention!