Solitons vs Collapses

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French-American conference on dispersive nonlinear PDEs,

June 15, 2017, CIRM, Marseilles, France

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OUTLINE

- Introduction
- Solitons in the dispersive Hamiltonian systems
- Linear stability of solitons (for NLS and KDV systems)
- Nonlinear stability of solitons (the Sobolev embedding inequalities)
- Wave collapse and its criteria: unboundedness of the Hamiltonian, virial theorem and its generalization, role of wave radiation
- Conclusion
- Open problems

Introduction

- This talk is devoted to solitons and wave collapses which can be considered as two alternative scenarios pertaining to the evolution of nonlinear wave systems describing by a certain class of dispersive PDEs of the Hamiltonian type (see, for instance, review: V.E. Zakharov and E.A. Kuznetsov, *Solitons and collapses - two scenarios of the evolution of nonlinear wave systems* , Physics Uspekhi **55**, 535 - 556 (2012)).
- Solitons are solitary waves propagating in a nonlinear medium with a constant velocity without changing their form. Usually solitons are stationary points of the Hamiltonian for fixed another integral of motion (number of waves or quasi-particles, momentum, etc.).

Introduction

- For this case, the soliton realizing minimum (or maximum) of the Hamiltonian is Lyapunov stable. The latter means it is sufficient the Hamiltonian to be bounded from below (or above). The extremum is approached via the radiation of small-amplitude waves, a process absent in systems with finitely many degrees of freedom.
- The framework of the nonlinear Schrodinger equation, the ZK equation and the three-wave system will be used to show how the boundedness of the Hamiltonian H can be understood by using simple argumentation based on the scaling transformations.
- The linear stability for solitons in the NLS with arbitrary nonlinearity, as it was shown by Vakhitov and Kolokolov (1973), can be effectively analyzed that leads to the so-called VK criterion.

Introduction

- In this lecture it will be shown how such criterion can be derived for the generalized ZK equation.
- The stability of the soliton minimizing H can be proved rigorously using the integral estimate method based on the Sobolev embedding theorems - Lyapunov stability.
- Wave collapse is the process of singularity formation in a finite time for smooth initial conditions. Historically the notation 'collapse' in physics was first intended in general relativity for catastrophic compression of astrophysical objects. In seventies-eighties of the last century the word 'collapse' became to be applied to wave systems. First time it was implied in 1972 by Zakharov (1972) in his famous paper about collapse of Langmure waves in isotropic plasma.

Introduction

- Later this notation became to be used widely also in nonlinear optics not only for stationary selffocusing but also for the nonstationary self-compression of light pulses. See reviews: Zakharov (1984), Rasmussen & Rypdal (1986), Kuznetsov (1996), Berge (1998) and the book: C.Sulem and P.L. Sulem, *The Nonlinear Schrodinger Equation* (Springer-Verlag, New York, 1999), G. Fibich, *The Nonlinear Schrodinger Equation* (Springer, 2015).
- What kind of singularities appear as the result of the collapse development depends on a physical model. For instance, in gas-dynamics collapse is connected with wave breaking resulting in the formation of shocks. For water waves collapse leads to the formation of wedges of fluid surface. For self-focusing of light the intensity of

Introduction

- For many wave systems collapse is associated with the Hamiltonian unboundedness from below. In this case the wave system evolution to collapse can be considered as the fall of a particle in an unbounded potential. The radiation of small-amplitude waves promotes collapse in this case.
- As known, exact sufficient criterion for collapse is based on the virial theorem (which was applied first time to the critical NLS equation by Vlasov, Petrishchev & Talanov, 1971) and its generalizations. Such generalizations, for example, concern criteria for the sub-critical NLS (Zakharov, 1972; Turitsyn, 1993; K., Turitsyn, Rypdal, Rasmussen, 1995) and KP equations (Turitsyn, Falkovich, 1985) which will be considered in this lecture.

Solitons in NLS, 3-wave system and KZ equation

Let us start from the NLS

$$i\psi_t + \frac{1}{2}\Delta\psi + |\psi|^2\psi = 0.$$

In the nonlinear optics ψ has the meaning of the wave envelope of the electric field with definite polarization. The second term describes both diffraction and the group velocity dispersion (anomalous dispersion). The nonlinear term corresponds to account of the Kerr effect.

The NLS belongs to the Hamiltonian type:

$$i\psi_t = \frac{\delta H}{\delta \psi^*}.$$

Solitons in NLS, 3-wave system and KZ equation

Here the Hamiltonian

$$H = \frac{1}{2}\left(\int |\nabla\psi|^2 d\mathbf{r} - \int |\psi|^4 d\mathbf{r}\right) \equiv \frac{1}{2}(I_1 - I_2).$$

Besides H, this equation has two another simple integrals: number of waves $N = \int |\psi|^2 d\mathbf{r}$ and momentum.
The standing soliton describes by the solution of the NLS $\psi_s = \psi_0(\mathbf{r})ei\lambda^2/2t.$, where $-\lambda^2/2$ has a meaning of the energy of soliton as a bound state. The moving soliton hence can be easily constructed by applying the Gallilean transform.
Solitons are stationary point of H for the fixed N:

$$\delta(H + \frac{\lambda^2}{2}N) = 0.$$

Solitons in NLS

This is equivalent to the stationary NLSE:

$$-\lambda^2\psi + \Delta\psi + 2|\psi|^2\psi = 0.$$

Hence one can get the dependence of N on the soliton solutions

$$N_s = \lambda^{2-D} N_0,\;\; N_0 = \int |f(\xi)|^{\mathbf{2}} \mathbf{d}\xi,$$

where f obeys the equation: $-f + \Delta f + 2|f|^2 f = 0$. This is the key dependence for the linear stability criterion - the Vakhitov-Kolokolov (VK) criterion (1973). If

$$\partial N_s/\partial\lambda^2 < 0,$$

then such solitons are unstable. In the opposite case solitons will be stable.

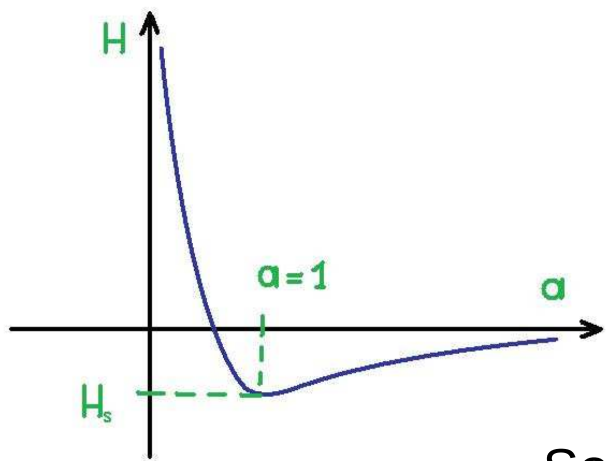
Solitons in NLS

To understand stability problem for solitons perform scaling transformation remaining the number of waves, $N =$ const:

$$\psi_s(\mathbf{r}) \to a^{-D/2}\psi_s\left(\frac{\mathbf{r}}{a}\right).$$

Under this transform H becomes the function of the scaling parameter$a$:

$$H(a) = \frac{I_{1s}}{2a^2} - \frac{I_{2s}}{2a^D}.$$



Solitons in NLS

Hence one can see that for $D = 1$ the function has a minimum at $a = 1$ corresponding to one-dimensional soliton that hints its stability.

A rigorous proof of the stability can be obtained on the base of Sobolev inequality, which follows from the general Sobolev embedding theorem. The theorem states that the space L_p can be embedded in the Sobolev space W_2^1 if

$$D < \frac{2}{p}(p+4).$$

This implies that between the norms

$$\|u\|_p = \left[\int |u|^p d^D x\right]^{1/p}, \ (p > 0), \ \ \|u\|_{W_2^1} = \left[\int (\mu^2 |u|^2 + |\nabla u|^2) d^D x\right]^{1/2}, \ (\mu^2$$

Solitons in NLS

there is the inequality:

$$\|u\|_p \leq M\|u\|_{W_2^1},$$

where constant $M > 0$. For $D = 1$ and $p = 4$, this inequality reads as

$$\int_{-\infty}^{\infty} |\psi|^4 dx \leq M_1 \left[\int_{-\infty}^{\infty} (\mu^2|\psi|^2 + |\psi_x|^2) dx\right]^2 .$$

Hence one can easily derive the so-called multiplicative Gagliardo-Nirenberg inequality if one performs the scaling transformation and then seek for the corresponding minimum:

$$I_2 \leq C N^{3/2} I_1^{1/2}$$

where C is a new constant.

Solitons in NLS

This inequality can be improved by finding the minimal value of the constant C. To find it, we need to consider all extrema of

$$J\{\psi\} = \frac{I_2}{N^{3/2} I_1^{1/2}},$$

which are defined from $\delta J = 0$. This variational problem reduces to finding solution of the stationary NLS

$$-\lambda^2 \psi + \psi_{xx} + 2|\psi|^2 \psi = 0.$$

Hence the best constant C_{best} is a value of $J\{\psi\}$ on 1D soliton:

$$C_{best} = \frac{I_{2s}}{N_s^{3/2} I_{1s}^{1/2}} = \frac{2 I_{1s}^{1/2}}{N_s^{3/2}}.$$

Solitons in NLS

For H it gives the estimate

$$H \geq \frac{1}{2}[I_1 - C_{best} I_1^{1/2} N^{3/2}] = H_s + \frac{1}{2}(I_1^{1/2} - I_{1s}^{1/2})^2.$$

This inequality becomes precise at the soliton solution, which proves that the NLS soliton is Lyapunov stable not only relative to small but also against finite perturbations. Similarly, one can prove the stability of the ‘ground-state’ soliton (a radially symmetric solution without nodes) for a multidimensional NLS equation with a power nonlinearity which Hamiltonian has the form

$$H = \frac{1}{2}\int (|\nabla\psi|^2 - |\psi|^{2\sigma}) d^D x \equiv I_1 - I_\sigma.$$

Solitons in NLS

The ground-state soliton solution $\psi_s = e^{i\lambda^2 t/2}\lambda^{1/(\sigma-2)} g(\lambda r)$ with $g(\xi)$ satisfying Eq.

$$-g + \nabla_\xi^2 g + \sigma |g|^{2\sigma-2} g = 0,$$

represents stationary point of H for fixed N,

$$\delta(H + \lambda^2 N/2) = 0.$$

Then scaling transformation, $\psi_s(r) \to a^{-D/2}\psi_s(r/a)$, shows existence of minimum for

$$H(a) = \frac{1}{2}\left[\frac{I_1}{a^2} - \frac{I_\sigma}{a^{(\sigma-1)D}}\right]$$

if

$$(\sigma - 1)D < 2.$$

Solitons in NLS

To get the stability proof of this soliton one needs to use the corresponding multiplicative Nirenberg-Gagliardo inequality. At the next step it is necessary to find the best constant that finally gives a proof of Lyapunov stability for ground soliton.

NOTE 1: The obtained stability criterion can be considered as the energy principle.

NOTE 2: Stability proof for solitons based on boundedness of Hamiltonian turns out to be more ”simple” than a linear stability analysis.

Solitons for anisotropic KDV (ZK equation)

Consider the next example, i.e. the anisotropic KdV equation (ZK equation) derived by Zakharov & K. in 1974:

$$u_t + \frac{\partial}{\partial x}\Delta u + 6uu_x = 0.$$

This equation describes three-dimensional ion-acoustic solitons $u = u_s(x - Vt, r_\perp)$**?** propagating along the magnetic field (parallel to the x-axis) in a strongly magnetized plasma, where the plasma thermal pressure nT is small compared to the magnetic field pressure $B^2/(8\pi$. These solitons are stationary points of the Hamiltonian

$$H = \frac{1}{2}\int (\nabla u)^2 d\mathbf{r} - \int u^3 d\mathbf{r}$$

with fixed momentum $P = \frac{1}{2}\int u^2 d\mathbf{r}$: $\delta(H - VP) = 0$.

Solitons for anisotropic KDV (ZK equation)

ZK equation belongs to the Hamiltonian type:

$$u_t = \frac{\partial}{\partial x}\frac{\delta H}{\delta u}$$

with symplectic operator $\partial/\partial x$, antisymmetric relative $x \to -x$.

After this introduction one can easily see that 3D solitons will be Lyapunov stable. The corresponding exponent p and dimension $D = 3$ satisfy the inequality (for NLS) leading to stability of the 3D soliton.

Solitons for anisotropic KDV (ZK equation)

For small perturbations $\tilde{u}(x - Vt, r_\perp, t)$ on the background of 3D soliton $u_s(x - Vt, r_\perp)$ linear stability problem reduces to the VK type criterion: if

$$\frac{\partial P}{\partial V} > 0$$

then soliton will be stable and, respectively, unstable in the opposite case. This criterion first time was obtained for the KDV type equation for arbitrary (non-power) nonlinearity (K, 1984). It can be easily generalized this result to multi-dimensional case as well.

Solitons for the 3-wave system

The 3-wave system has the form

$$\begin{aligned} i\frac{\partial\psi_1}{\partial t} - \omega_1\psi_1 + i(\mathbf{v_1}\nabla)\psi_1 + \frac{1}{2}\omega_1^{\alpha\beta}\partial^2_{\alpha\beta}\psi_1 &= V\psi_2\psi_3, \\ i\frac{\partial\psi_2}{\partial t} - \omega_2\psi_2 + i(\mathbf{v_2}\nabla)\psi_2 + \frac{1}{2}\omega_2^{\alpha\beta}\partial^2_{\alpha\beta}\psi_2 &= V\psi_1\psi_3^*, \\ i\frac{\partial\psi_3}{\partial t} - \omega_3\psi_3 + i(\mathbf{v_3}\nabla)\psi_3 + \frac{1}{2}\omega_3^{\alpha\beta}\partial^2_{\alpha\beta}\psi_3 &= V\psi_1\psi_2^*. \end{aligned}$$

where the amplitudes of three wave packets $\psi_l(\mathbf{x},t)$ ($l=1,2,3$)? are slowly varying functions of $\mathbf{x}$ where k_l is the carrier wave vector of the l-th packet, $\mathbf{v}_l = \partial\omega_l(\mathbf{k}_l)/\partial\mathbf{k}_l$ are the group velocities of packets, $\omega_l^{\alpha\beta} = \frac{\partial^2\omega_l(\mathbf{k}_l)}{\partial k_{l\alpha}\partial k_{l\beta}}$ is the dispersion tensor, and V is the three-wave matrix element (real).

Solitons for the 3-wave system

This is the Hamiltonian system:

$$i\frac{\partial \psi_l}{\partial t} = \frac{\delta H}{\delta \psi_l^*}.$$

where

$$H = H_0 + H_1,$$

$$H_0 = \sum_{l=1}^{3} \left[\int \omega_l |\psi_l|^2 d\mathbf{r} - i \int \psi_l^* (v_l \nabla) \psi_l d\mathbf{r} + \frac{1}{2} \int \nabla_\alpha \psi_l^* \omega_l^{\alpha\beta} \nabla_\beta \psi_l d\mathbf{r} \right],$$

$$H_1 = V \int (\psi_1 \psi_2^* \psi_3^* + \psi_1^* \psi_2 \psi_3) d\mathbf{r}.$$

Solitons for the 3-wave system

In the 1D case this system can be reduced as follows

$$
\begin{aligned}
i\frac{\partial\psi_1}{\partial t} - \Omega\psi_1 + \frac{1}{2}\omega_1''\psi_{1xx} &= -\psi_2\psi_3, \\
i\frac{\partial\psi_2}{\partial t} + \frac{1}{2}\omega_2''\psi_{2xx} &= -\psi_1\psi_3^*, \\
i\frac{\partial\psi_3}{\partial t} + \frac{1}{2}\omega_3''\psi_{3xx} &= -\psi_1\psi_2^*.
\end{aligned}
$$

Here $\Omega = \omega_1 - \omega_2 - \omega_3$ is the frequency mismatch which is assumed to be small in comparison with carrying frequencies. These equations have two additional integrals of motion, the so-called Manley-Rowe integrals,

$$
N_1 = \int (|\psi_1|^2 + |\psi_2|^2)dx, \; N_2 = \int (|\psi_1|^2 + |\psi_3|^2)dx.
$$

Solitons for the 3-wave system

Soliton solution in 3-wave system

$$\begin{aligned}\psi_1(x,t) &= \psi_{1s}(x)e^{i(\lambda_1+\lambda_2)t},\\ \psi_2(x,t) &= \psi_{2s}(x)e^{i\lambda_1 t},\\ \psi_3(x,t) &= \psi_{3s}(x)e^{i\lambda_2 t}.\end{aligned}$$

represents stationary point of H for fixed Manley-Rowe integrals, $\delta(H+\lambda_1 N_1+\lambda_2 N_2)=0.$ To prove their stability one needs to consider two spaces, $L_{3,3}$ and W_2^1, with the norms

$$||u||_{L_{3,3}} = \left[\int (|\psi_1|^3+|\psi_2|^3+|\psi_3|^3)d^D x\right]^{1/3},$$

$$||u||_{W_2^1} = \left[\tilde{\lambda}_1 \int (|\psi_1|^2+|\psi_2|^2)d^D x + \tilde{\lambda}_2 \int (|\psi_1|^2+|\psi_3|^2)d^D x\right.$$

Solitons for the 3-wave system

Here constants $\widetilde{\lambda}_{1,2} > 0$ and tensors $\omega_l^{\alpha\beta}$ are assumed to be positive definite. Then the Sobolev inequality reads as

$$||u||_{L_{3,3}} < M||u||_{W_2^1}.$$

Hence one can obtained the Gagliardo-Nirenberg type inequality. In 1D case it is written as

$$J \leq C(N_1 N_2)^{5/8} I^{1/4}.$$

where

$$J = \int (\psi_1^* \psi_2 \psi_3 + c.c.) dx, \quad I = \frac{1}{2} \sum_l \int \omega_l'' |\psi_{xl}|^2 dx.$$

Solitons for the 3-wave system

After finding the best constant C_{best} it is possible to demonstrate that at $\Omega = 0$

$$H \geq I - 2I_s^{3/4} I^{1/4} \geq H_s(\Omega = 0).$$

This inequality becomes precise on the soliton solution that proves its stability. At $\Omega > 0$ we have evident inequality $H \geq H_s(\Omega = 0)$, hence boundedness of H from below follows. At $\Omega < 0$ we have estimate $H \geq H_s(\Omega = 0) - |\Omega| \min(N_1, N_2)$. We have thus proved the stability of ground-state solitons (those without nodes) describing a coupled state of three wave packets. Notably, in the absence of detuning, the soliton realizes a minimum of the Hamiltonian, which is rigorously proved with the help of majorizing Sobolev inequalities.

Collapse in NLS

At $D = 2$ scaling transforms give $H(a) \equiv 0$. This straight line (on $H - a$ – plane) shows that solitons can be treated as separatrices between collapsing and noncollapsing solutions. In the three-dimensional geometry the function $H(a)$ attains its maximum on the three-dimensional soliton that indicates to its instability. Notice also that the Hamiltonian becomes unbounded as $a \to 0$. It is necessary to underline that unboundedness of H represents one of the main criteria for wave collapses. In such a case collapse can be considered as the nonlinear stage of soliton instability.

To clarify the latter we apply the variational approach taking a trial function for the NLSE in the form

$$\psi(\mathbf{r}, t) = a^{-3/2}\psi_s\left(\frac{\mathbf{r}}{a}\right)\exp(i\lambda^2 t + i\mu r^2).$$

Collapse in NLS

Here $a = a(t)$ and $\mu = \mu(t)$. After substitution of this anzats into the action

$$S = \frac{i}{2}\int (\psi_t \psi^* - c.c.)dtd\mathbf{r} - \int H dt$$

and integration over spatial variables we arrive at the Newton equation for a,

$$C\ddot{a} = -\frac{\partial H}{\partial a},$$

where $C = \int \xi^2 |\psi_0(\xi)|^2$ and the function $H(a)$ has a meaning of the potential energy. Behavior of $a(t)$ depends on the total energy,

$$E = C\frac{\dot{a}^2}{2} + H(a)$$

and the dimension D.

Collapse in NLS

At $D = 1$ soliton realizes the minimal value of the potential energy $H(a)$ and it is a reason why 1D soliton is stable. At $D = 3$ if a 'particle' stands at the maximal point of $H(a)$ initially then depending on the its motion direction (toward or upwards the center $a = 0$) the system will collapse ($\psi \to \infty$) or expand ($\psi \to 0$). For collapse (falling at the center) $a(t)$ behaves near singularity like $a(t) \sim (t_0 - t)^{2/5}$. As shown by Zakharov & K.1986, this behavior for $a(t)$ near singular time coincides with that following from the exact semi-classical collapsing solution which asymptotically (as $t \to t_0$) tends to

$$|\psi| \to \lambda\sqrt{1 - \xi^2} \text{ for } \xi = r/a(t) \leq 1$$

with $\lambda \sim (t_0 - t)^{-3/5}$.

Collapse in NLS

However the picture is more complicated than considered above. From the very beginning we have a spatially-distributed system with an infinite number of degrees of freedom and therefore, rigorously speaking, it is hardly feasible to describe such a system by its reduction to a system of ODEs. The NLSE is the wave system and therefore, first of all, here we deal with waves. Waves may propagate, may radiate and so on.

Let us try to understand the influence of wave radiation on the wave collapse. Consider an arbitrary region Ω with $H_\Omega < 0$. Then using the mean value theorem for the integral I_2,

$$\int_\Omega |\psi|^4 d\mathbf{r} \le \max_{x\in\Omega} |\psi|^2 \int_\Omega |\psi|^2 d\mathbf{r},$$

we have

Collapse in NLS

$$\max_{x\in\Omega} |\psi|^2 \geq \frac{|H_\Omega|}{N_\Omega}.$$

This estimate shows that radiation of waves promotes collapse: far from the region Ω radiative waves can be considered almost linear. These waves carry out the positive portion of Hamiltonian making H_Ω more negative with simultaneous vanishing of the number of waves N_Ω that results in growth of the r.h.s. of the inequality (Zakharov 1972). It is why we can say that wave radiation promotes collapse which play the role of friction in the nonlinear wave dynamics. Simultaneously radiation turns out to accelerate compression of the collapsing area with the self-similarity, $r \sim (t_0 - t)^{1/2}$, different from the semiclassical answer.

Collapse in NLS: virial theorem

The exact criterion for singularity formation within the NLSE can be obtained from the virial theorem. In classical mechanics the virial theorem can be easily got if one first calculates the second time derivative from the moment of inertia and then averages the obtained result. It gives the relation between mean kinetic and potential energies of particles if the interaction between particles is of power type. In 1971 Vlasov, Petrishchev and Talanov found that this theorem can be applied also to the 2D NLS:

$$\frac{d^2}{dt^2}\int r^2|\psi|^2 d\mathbf{r} = 8H.$$

This equality is verified by the direct calculation. This Eq. can be integrated twice

Collapse in NLS: virial theorem

that results in

$$\int r^2|\psi|^2 d\mathbf{r} = 4Ht^2 + C_1 t + C_2,$$

Hence the mean square size $\langle r^2 \rangle$ of any field distribution with $H < 0$, independently on $C_{1,2}$ vanishes in a finite time, which, with allowance for the conservation of N, means the formation of a singularity of ψ. This (VPT) criterion, nowadays it is a cornerstone in the theory of wave collapses. This was the first rigorous result for nonlinear wave systems with dispersion, which showed the possibility of the formation of a wave-field singularity in a finite time, despite the presence of the linear dispersion of waves, the effect impeding the formation of point singularities (focii) in the linear optics.

Strong collapse in NLS

Notice that at $D = 2$ $H = 0$ corresponds to the soliton for which number of waves $N = N_s$. Moreover, if $N < N_s$ then $H \geq 1/2 I_{1s}(1 - N/N_s) > 0$ and collapse is impossible (Weinstein 1983). All waves are spread due to dispersion (diffraction) vanishing as $t \to \infty$. Thus, we can say that solitons in this case represent separatrices between collapsing and noncollapsing submanifolds.

From the virial theorem for $H < 0$ one can see that the characteristic size a of the collapsing area behaves like $a \sim (t_0 - t)^{1/2}$ in the correspondence with the self-similar law.

However, the exact analysis (Fraiman 1985) shows that

$$a^2(t) \sim \frac{(t_0 - t)}{\log|\log(t_0 - t)|},$$

Collapse in NLS: virial theorem at D=3

(Recently it was shown by Lushnikov that the collapsing asymptotics has a fine structure.) The power (up to some multiplier, coinciding with N) captured into the singularity occurs *finite*, equal to the power of 2D soliton. It is why such collapse is called as a *strong collapse* (Zakharov, 1982) and, respectively, the 2D NLS as a critical model.
However, the criterion $H < 0$ is not sharp at $D = 3$. As it was shown by Turitsyn 1993, K., Rasmussen, Rypdal, Turitsyn 1995, this criterion can be improved. The sharper criterion of collapse is given by the conditions: $H < H_s$ and $I_1 < I_{1s}$,

Collapse in NLS: virial theorem at D=3

The corresponding virial inequality is of the form,

$$\frac{d^2}{dt^2}\int r^2|\psi|^2 d\mathbf{r} = 8(H - H_s),$$

and, respectively,

$$N\langle r^2\rangle < 4(H - H_s)t^2 + C_1 t + C_2.$$

Here H_s is the value of H on the ground soliton solution (without nodes). This result was obtained by means of the GN inequality.

Conclusion

- The influence of nonlinearity grows with increase spatial dimension D. As a sequence, stable solitons are intrinsic for low dimensional systems while for higher dimensions instead of solitons we have to expect blow-up events.
- One of the main criteria of collapse is unboundedness of the Hamiltonian. In this case, the collapse can be interpreted as the fall off a particle to an attracting center in a self-consistent unbounded potential (Zakharov, Kuznetsov, 1986).
- There are at least two variants of the wave collapse: strong and weak. In both cases radiation of waves promotes collapse. Radiation plays a role of dissipation providing formation of the most rapid collapse, i.e. weak collapse.

Open problems

There are a few models for which the Hamiltonian is unbounded, numerical simulations demonstrated sharp increasing of the wave intensity of the blow-up, but there are unknown mathematical rigorous criteria for collapse.

- 3D KP equation:

$$\frac{\partial}{\partial x}(u_t + u_{xxx} + 6uu_x) = \Delta_\perp u.$$

  This is physically the most important model which describes weakly nonlinear acoustic waves in media with positive dispersion in the real 3D space, for instant in plasma.
  Turitsyn and Falkovich (1985) found the criterion for the KP equation with cubic nonlinearity. The criterion is the

Open problems

- Recently this approach was applied by Litvak, and co. to describe the collapse for the SPE but in 3D with the so-called plasma dispersion. The criterion in this case the same as for the Turitsyn-Falkovich case because for optics the main nonlinearity comes from the Kerr effect.
- 2D Shrira model.

$$u_t + 6uu_x = \frac{\partial}{\partial x}\hat{k}u$$

where the Fourier transform of $\hat{k}$ is $\sqrt{k_x^2 + k_y^2}$. This equation describes the nonlinear behavior of the boundary layer. It represents the critical collapsing model (K. & Dyachenko, 1995).

- Collapse in 3D EULER!

THANKS