Continuations beyond the singularity, loss of phase, stochastic interactions, and universality

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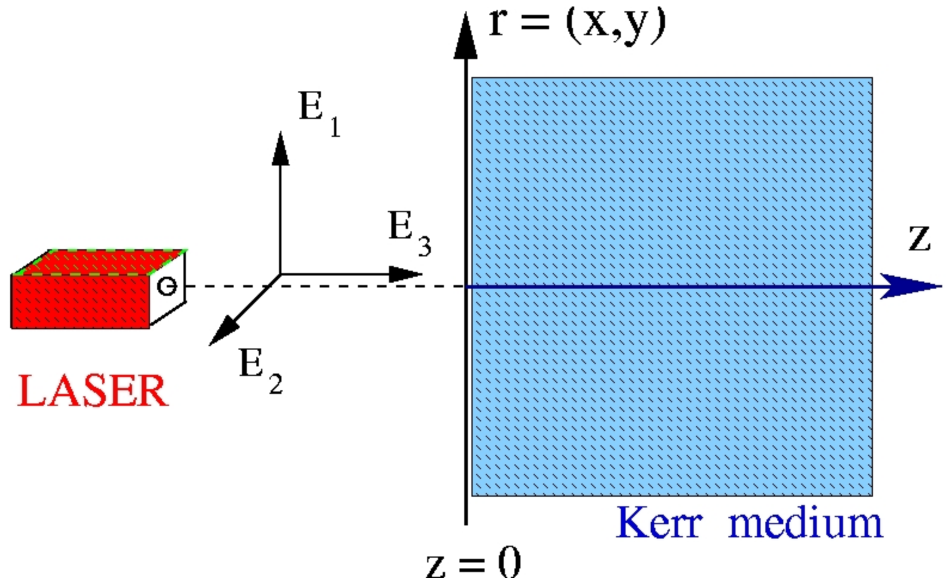
Physical setup

- Propagation of intense laser beams in homogeneous medium (air, water, glass, …)

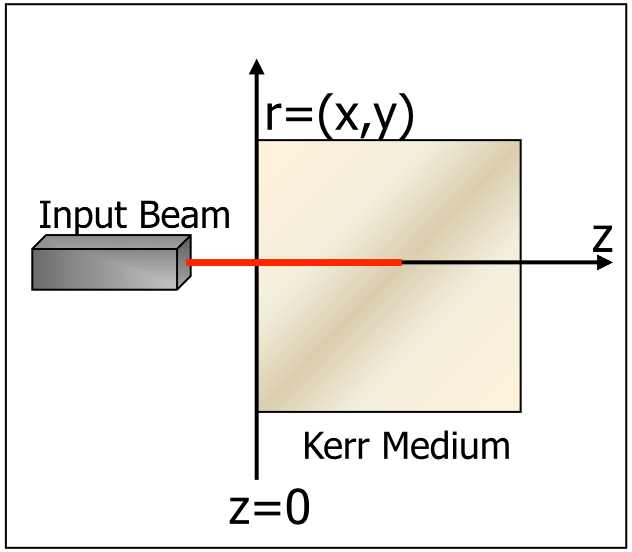


- *z* is direction of propagation

Mathematical model

- Propagation governed by nonlinear Maxwell's eqs.
- Approximate by nonlinear Schrödinger eq. (**NLS**)

$$i\psi_z(z,x,y) + \Delta\psi + |\psi|^2\psi = 0, \qquad \Delta\psi = \psi_{xx} + \psi_{yy}$$



- *z*"="*t* (evolution variable)

$$\psi(z=0,x,y) = \psi_0(x,y)$$

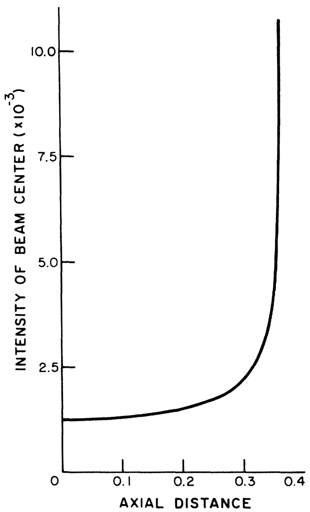
- Initial value problem in *z*

Finite-time singularity

$z \to t$

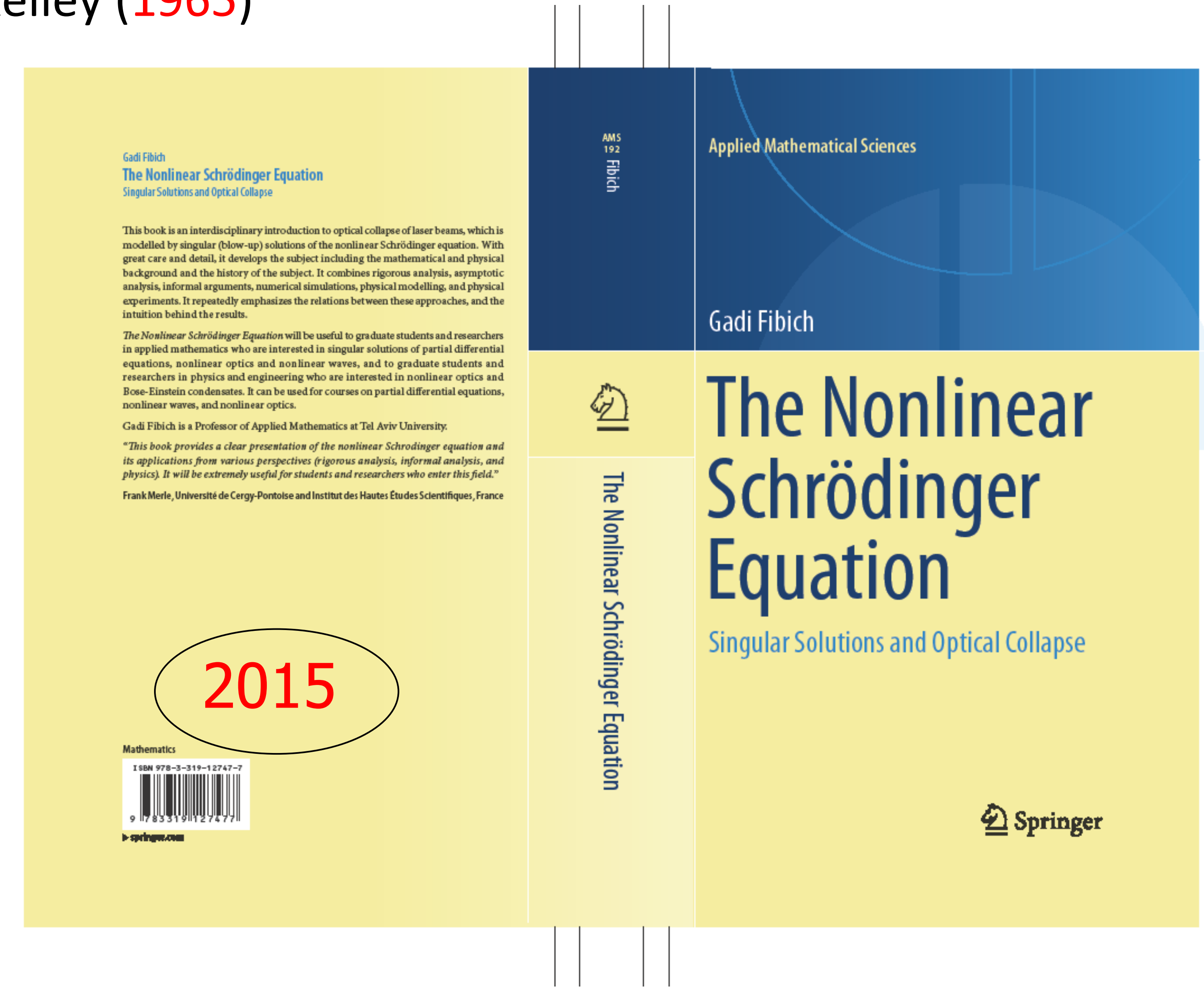
$$i\psi_t(t,x,y) + \Delta\psi + |\psi|^2\psi = 0, \quad \psi(t=0,x,y) = \psi_0(x,y)$$

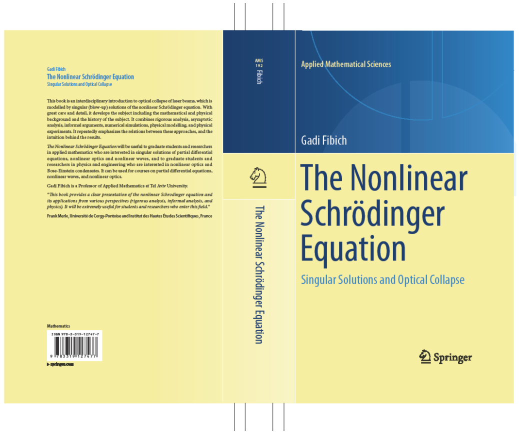
- Kelley (1965) : NLS Solutions can become singular (blowup) in finite distance/time T_c



Optical collapse

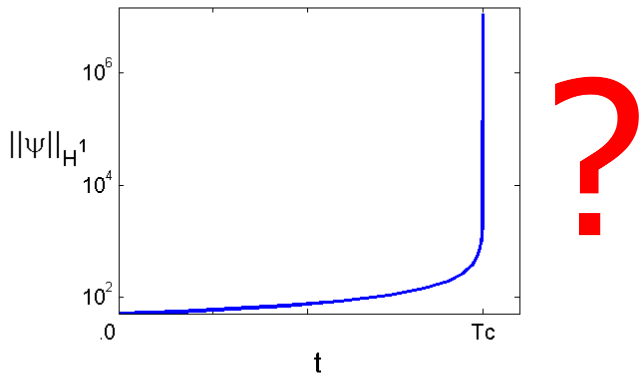
- Kelley (1965)





Applied Mathematical Sciences
Gadi Fibich
The Nonlinear Schrödinger Equation
Singular Solutions and Optical Collapse
Springer

Beyond the singularity



- No singularities in nature
- Laser beam propagates past T_c
- NLS is only an approximate model
- Common approach: Retain effects neglected in NLS model
  - Nonparaxiality, quintic nonlinearity, dispersion, …
  - Many studies, mostly numerical
  - …

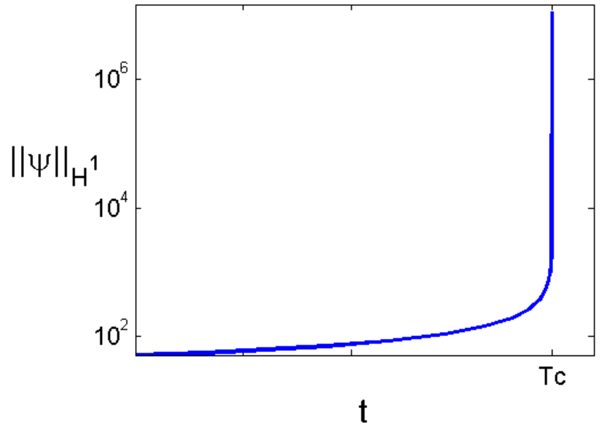
Compare with hyperbolic conservation laws

- Solutions can become singular (shock waves)
- Singularity arrested in the presence of viscosity

- Huge literature on **continuation of the singular inviscid solutions**:
  - Riemann problem
  - Vanishing-viscosity solutions
  - Rankine-Hugoniot jump conditions
  - …

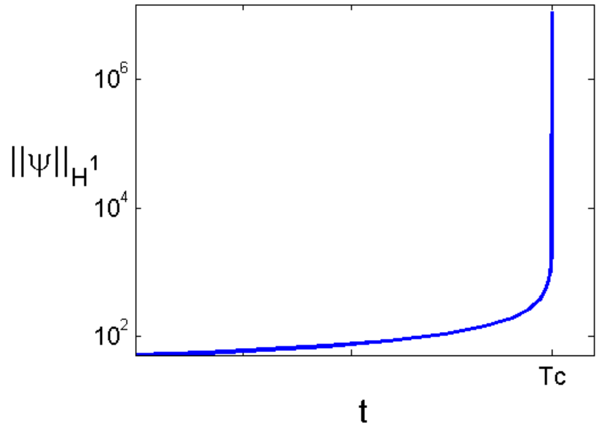
Goal – develop a ``similar’’ theory for the NLS

Continuation of singular NLS solutions



$i\psi_t + \Delta\psi + |\psi|^2\psi = 0$ NLS

Continuation of singular NLS solutions



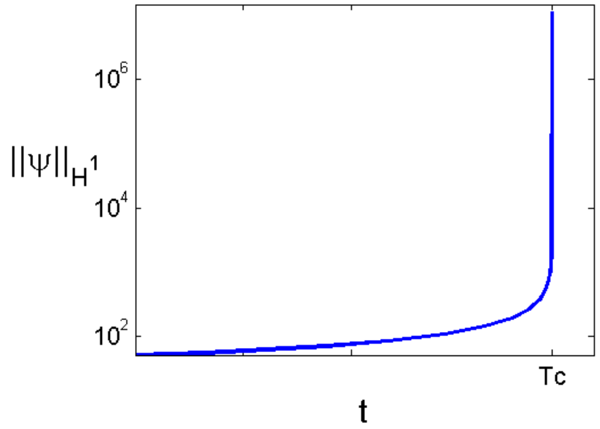
no ``viscous'' terms

$i\psi_t + \Delta\psi + |\psi|^2\psi = 0$ NLS

NLS $i\psi_t + \Delta\psi + |\psi|^2\psi = 0$

T_c

Continuation of singular NLS solutions



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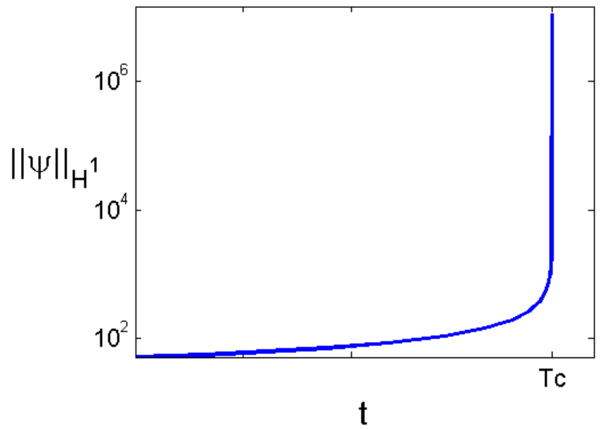
$i\psi_t + \Delta\psi + |\psi|^2 \psi = 0$ NLS

NLS $i\psi_t + \Delta\psi + |\psi|^2 \psi = 0$

T_c

``jump'' condition

Continuation of singular NLS solutions



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T_c

``jump'' condition

- 2 key papers by Merle (1992)
- Less than 10 papers

Talk plan

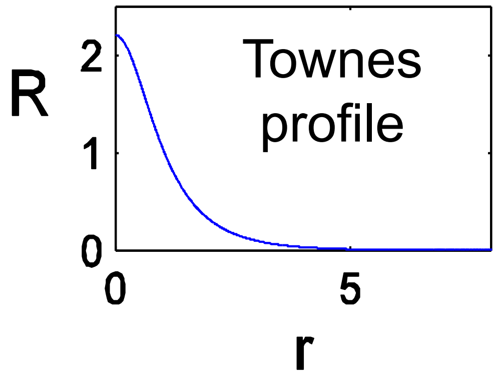
1. Review
2. Nonlinear damping continuation
3. Sub threshold power continuation
4. Loss of phase
5. Universality of stochastic interactions
6. Numerical methods

Ground state

- NLS admits solitary waves $\psi = e^{it} R(r)$

$$\Delta R(r) - R + R^3 = 0, \qquad r = \sqrt{x^2 + y^2}$$

- Enumerable number of solutions
- Of most interest is the **ground state:**
  - Positive sol. with minimal power (L_2 norm)



Thm (Weinstein, 83): A necessary condition for collapse is

$$\|\psi_0\|_2^2 \geq P_{\mathrm{cr}}, \qquad P_{\mathrm{cr}} = \|R\|_2^2$$

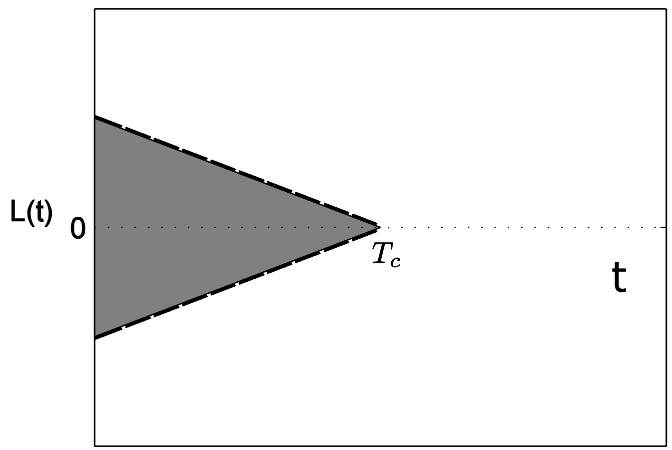
- P_{cr} - critical **power**/L_2-norm for collapse

Explicit blowup solutions

- Apply quasi-conformal (Talanov) transformation to $\psi = e^{it} R(r)$

$$\psi^{\text{explicit}}_{R,\alpha}(t,r) = \frac{1}{L(t)} R\left(\frac{r}{L(t)}\right) e^{i\tau(t) + i\frac{L_t}{L}\frac{r^2}{4}}, \quad L(t) = \alpha(T_c - t), \quad \tau = \int_0^t L^{-2}$$

- Width $L(t) \to 0$ as $t \to T_c$
- $\psi_{R,\alpha}^{\text{explicit}}$ becomes singular at T_c
  - Linear blowup rate



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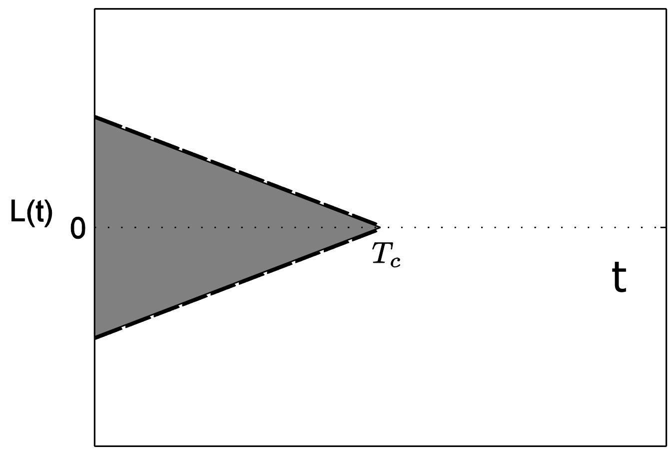
- Width $L(t) \rightarrow 0$ as $t \rightarrow T_c$
- $\psi_{R,\alpha}^{\text{explicit}}$ becomes singular at T_c
  - Linear blowup rate

$$\left\|\psi_{\mathrm{R},\alpha}^{\mathrm{explicit}}\right\|_2^2 = \|R\|_2^2 = P_{\mathrm{cr}}$$



- $\psi_{R,\alpha}^{\text{explicit}}$ is unstable, since any perturbation that reduces its power leads to global existence

Explicit continuation of $\psi_{R,\alpha}^{\text{explicit}}$ (Merle, 92)

- Let ψ^{ε} be the NLS solution with the ic

$$\psi_0^{\varepsilon}(r) = (1-\varepsilon)\psi_{\mathrm{R},\alpha}^{\text{explicit}}(t=0,r), \qquad 0 < \varepsilon = 1$$

- $\left\|\psi_0^{\varepsilon}\right\|_2^2 = (1-\varepsilon)^2 \left\|R\right\|_2^2 < P_{\text{cr}}$

- Ψ^{ε} exists globally
- Merle computed the limit

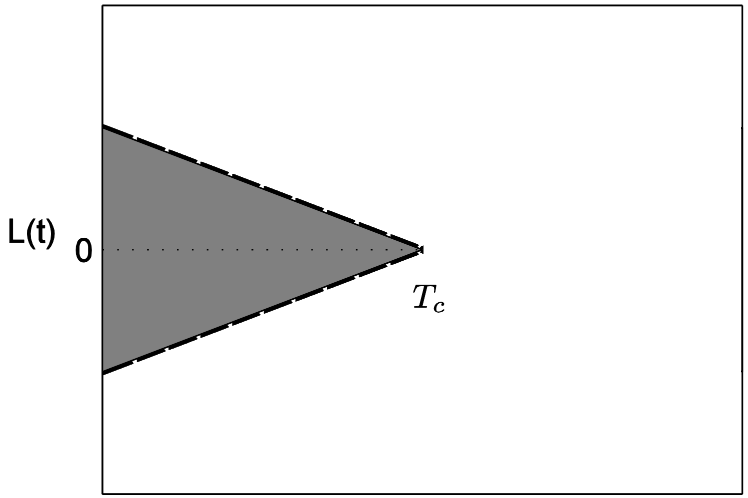
$$\lim_{\varepsilon \to 0+} \psi^{\varepsilon}(t,r), \qquad 0 \le t < \infty$$

Thm (Merle 92)

$$\psi^{\varepsilon}(t=0)=(1-\varepsilon)\psi_{\mathrm{R},\alpha}^{\text{explicit}}(t=0)$$

- Before singularity ($t<T_c$)

$$\lim_{\varepsilon\to 0+}\psi^{\varepsilon}(t,r)=\psi_{\mathrm{R},\alpha}^{\text{explicit}}(t,r)$$



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- Before singularity ($t<T_c$)

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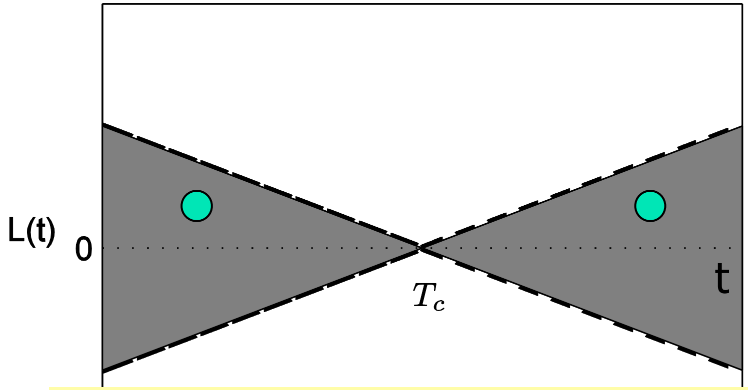
- After singularity ($t>T_c$)

$$\lim_{\varepsilon\to 0+}\left|\psi^{\varepsilon}(T_c+t,r)\right|=\left|\psi^{\text{explicit}}_{\mathrm{R},\alpha}(T_c-t,r)\right|,\qquad 0<t$$



$$\lim_{\varepsilon\to 0}L(t)=\alpha\left|T_c-t\right|,\qquad 0\le t<\infty$$

Symmetry Property:
Continuation symmetric with respect to T_c
- ``Jump condition''

Thm (Merle 92)

$$\psi^{\varepsilon}(t=0)=(1-\varepsilon)\psi_{\mathrm{R},\alpha}^{\text{explicit}}(t=0)$$

- After singularity

For any θ, there exists a sequence $\varepsilon_n \to 0+$ such that

$$\lim_{\varepsilon_n \to 0+} \psi^{\varepsilon_n}(T_c+t,r)=e^{i\theta}\psi_{\mathrm{R},\alpha}^{\text{explicit}\,*}(T_c-t,r), \qquad \forall t>0$$

- Phase-loss Property: Initial phase lost at/after the singularity

Properties of Merle's continuation

1. Symmetry: Continuation symmetric with respect to T_c
2. Phase loss: Initial phase is lost at/after the singularity

Are these properties universal?

Talk plan

1. Review
2. Nonlinear-damping continuation
3. Sub threshold power continuation
4. Loss of phase
5. Universality of stochastic interactions
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Vanishing -``viscosity'' continuations

- Add a small perturbation to the NLS
- Let ψ^{ε} be the solution of

$$i\psi_t^{\varepsilon}(t,\mathbf{x})+\Delta\psi^{\varepsilon}+\left|\psi^{\varepsilon}\right|^2\psi^{\varepsilon}+\underbrace{\varepsilon F\left[\psi^{\varepsilon}\right]}_{\text{``viscosity"}}=0$$

- If ψ^{ε} exists globally for any $0<\varepsilon\ll 1$, can define the vanishing–``viscosity'' continuation

$$\psi^{\text{continuation}}(t,\mathbf{x}):=\lim_{\varepsilon\to 0+}\psi^{\varepsilon}(t,\mathbf{x}),\qquad 0\le t<\infty$$

Vanishing -``viscosity'' continuations

- But, what is `viscosity' for the NLS?
- Should arrest collapse even when it is infinitesimally small
  - Cannot use linear damping (Fibich, 01)
- Plenty of other candidates:
  - Nonlinear saturation (Merle 92)
  - Nonparaxiality (Fibich, 96)
  - Dispersion
  - …

Nonlinear damping

- ``Viscosity'' = nonlinear damping (Fibich & Klein, 2011/2)
- Physical – multi-photon absorption
- Dissipative perturbation
  - Good, same as viscosity!

Different continuation of $\psi_{R,\alpha}^{explicit}$

- 2D cubic NLS with cubic nonlinear damping

- Compute the limit $\delta \to 0+$
- Vanishing nonlinear-damping continuation of $\psi_{R,\alpha}^{explicit}$

Proposition (Fibich & Klein, 2011)

$$\boxed{\psi_0 = \psi^{\text{explicit}}_{\text{R},\alpha}(t=0)}$$

- Before singularity ($t<T_c$)

$$\lim_{\delta \to 0+} \psi^{(\delta)} = \psi^{\text{explicit}}_{\text{R},\alpha}$$

- After singularity ($t>T_c$)

For any θ, there exists a sequence $\delta_n \to 0+$ such that

$$\lim_{\delta_n \to 0+} \psi^{(\delta)}(T_c + t, r) = e^{i\theta} \psi^{*\,\text{explicit}}_{\text{R},\kappa\alpha}(T_c - t, r), \qquad \kappa \approx 1.614$$

- $\kappa = \dfrac{\text{Ai}(0)}{\text{Ai}(s_0)}$, $\quad s_0 \approx -2.7$ is the first negative root of $\sqrt{3}Ai(s) = Bi(s)$

- Asymptotic calculations (rigorous proof still needed)

Properties of nonlinear damping continuation

For any θ, there exists a sequence $\delta_n \to 0+$ such that

$$\lim_{\delta_n \to 0+} \psi^{(\delta)}(T_c + t, r) = e^{i\theta} \psi^{*\,\text{explicit}}_{\mathrm{R},\kappa\alpha}(T_c - t, r), \qquad \kappa \approx 1.614$$

- Continuation has phase-loss property

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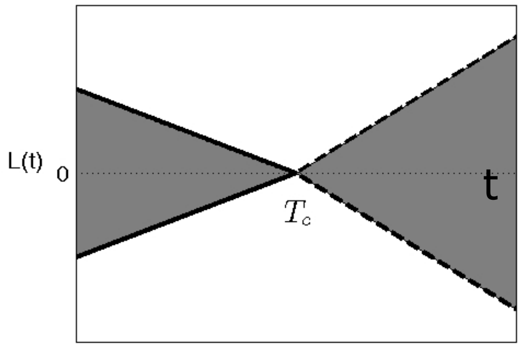
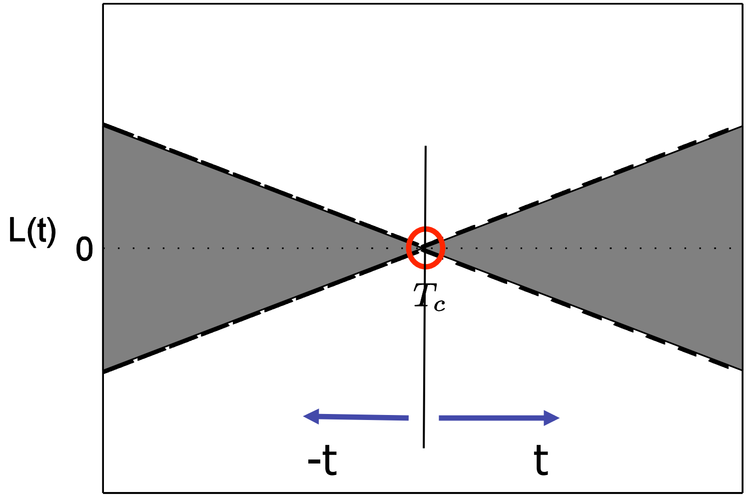
- Continuation has phase-loss property
- Impossible d'afficher l'image. Votre ordinateur manque peut-être de mémoire pour ouvrir l'image ou l'image est endommagée. Redémarrez l'ordinateur, puis ouvrez à nouveau le fichier. Si le x rouge est toujours affiché, vous devrez peut-être supprimer l'image avant de la réinsérer.



- Continuation asymmetric with respect to T_c

Symmetry or asymmetry?

- NLS is invariant under time reversal $t \to -t$ and $\psi \to \psi^*$



- Hence, Merle's continuation is symmetric with respect to T_c
- Nonlinear damping breaks reversibility in time
  - Hence, our continuation is asymmetric

- Symmetry property not universal

Phase-loss Property - Motivation

- Before singularity ($t<T_c$)

$$\lim_{\varepsilon \to 0+} \psi^{(\varepsilon)} = \psi^{\text{explicit}}_{\text{R},\alpha} \qquad \lim_{\delta \to 0+} \psi^{(\delta)} = \psi^{\text{explicit}}_{\text{R},\alpha}$$

$$\lim_{t \to Tc} \arg\left(\psi^{\text{explicit}}_{\text{R},\alpha}\right) = \infty$$

- For $t>T_c$, phase is ``beyond infinity''
- Holds for any continuation

- Loss of phase property is universal

Talk plan

1. Review
2. Nonlinear damping continuation
3. Sub threshold-power continuation
4. Loss of phase
5. Universality of stochastic interactions
6. Numerical methods

Sub threshold-power continuation (Fibich & Klein, 2011)

- Let $f(\boldsymbol{x}) \in H^1, \quad f \neq R$
- Let ψ^c be the NLS solution with $\psi_0^c = c\, f(\boldsymbol{x})$
- Let $c_{th} = \min\{c \mid \psi^c \text{ blows up}\}$
- $\psi(t, \mathbf{x}; c_{th})$ is ``minimal-power'' blowup solution

- Let ψ^ε be the NLS solution with $\psi_0^\epsilon = (1-\epsilon) c_{th} f(\boldsymbol{x})$
- By construction, no collapse for $0<\varepsilon\ll 1$
- Compute the limit of ψ^ε as $\varepsilon \to 0+$
  - Continuation of the ``minimal-power'' blowup solution $\psi(t, \mathbf{x}; c_{th})$
  - Generalization of Merle's continuation of $\psi_{R,\alpha}^{explicit}$
  - Asymptotic calculation (non-rigorous)

Proposition (Fibich and Klein, 2011)

$\psi_0^{\varepsilon} = (1-\varepsilon)K_{th}\, f(\mathbf{x})$

- Before singularity ($t<T_c$)

$$\lim_{\varepsilon \to 0+} \psi^{\varepsilon}(t,r) \qquad 0 < t < T_c$$

- Solution core collapses with $\psi_{R,\alpha}^{\text{explicit}}$ profile
  - Blowup rate is linear
- Solution also has a nontrivial tail

Proposition (Fibich and Klein, 2011)

$\psi_0^{\varepsilon} = (1-\varepsilon)K_{th}\, f(\mathbf{x})$

- Before singularity ($t<T_c$)

$$\lim_{\varepsilon \to 0+} \psi^{\varepsilon}(t,r) = \psi_{\text{B-W}}(t,r), \qquad 0 < t < T_c$$

- Solution core collapses with $\psi_{R,\alpha}^{\text{explicit}}$ profile
  - Blowup rate is linear
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Conclusion:
Bourgain-Wang solutions are ``generic'', in the sense that they are the minimal-power blowup solutions of $\psi_0 = c\, f(\mathbf{x})$

Proposition (Fibich and Klein, 2011)

$\psi_0^{\varepsilon} = (1-\varepsilon)K_{th}\, f(\mathbf{x})$

- After singularity ($t>T_c$)

For any θ, there exists a sequence $\varepsilon_n \to 0+$ such that

$$\lim_{\varepsilon_n \to 0+} \psi^{\varepsilon_n}(T_c + t, r) = e^{i\theta} \psi^{*}_{\text{B-W}}(T_c - t, r), \qquad 0 < t = 1$$

- Continuation of B-W solution is the ``same'' B-W solution
- Phase information is lost at the singularity

$$\lim_{t \to Tc-} \arg\left(\psi_{\text{B-W}}\right) = \lim_{t \to Tc-} \arg\left(\psi^{\text{explicit}}_{\text{R},\alpha}\right) = \infty$$

- Related result by Merle, Raphael, Szeftel (2014)

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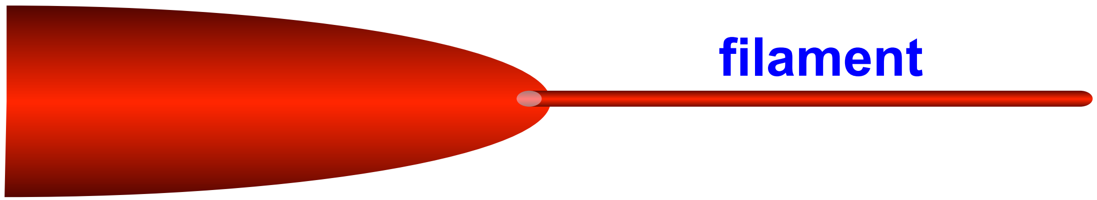
Universality of loss of phase

- Phase of all known singular NLS solutions blows up at the singularity
- Hence, any continuation of singular NLS solutions will have the phase-loss property

- Physically
  - Collapse-arresting mechanism is small, but not zero



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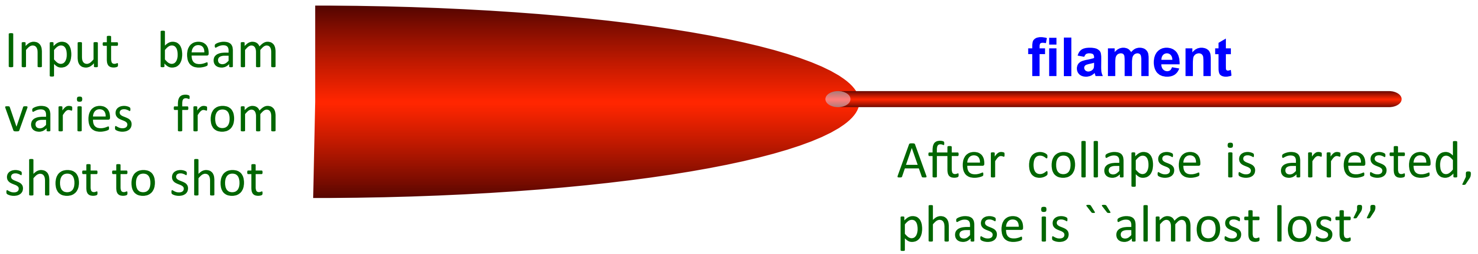
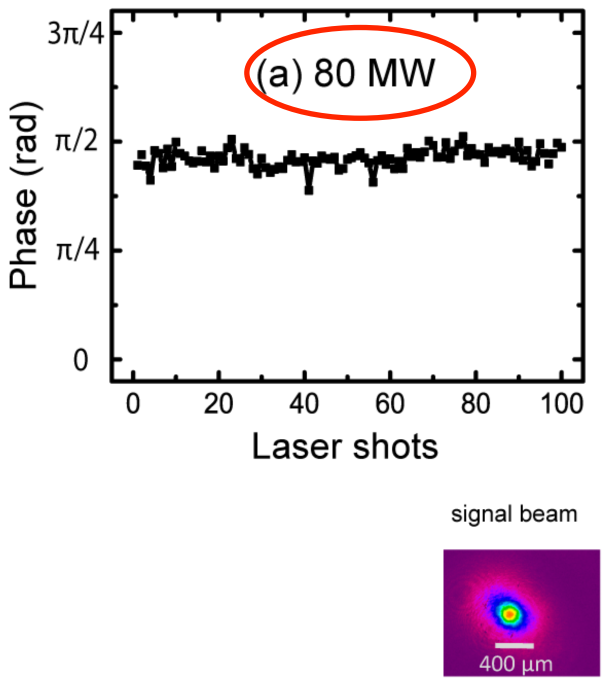
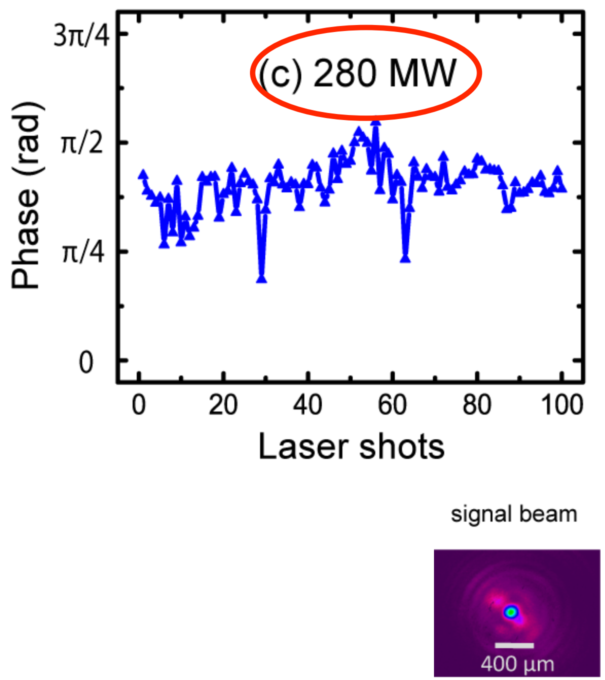


Experiments (Shim et al., 2012)

- Phase of laser beam after propagation of 24cm in water
  - ``Correct'' physical continuation is not known

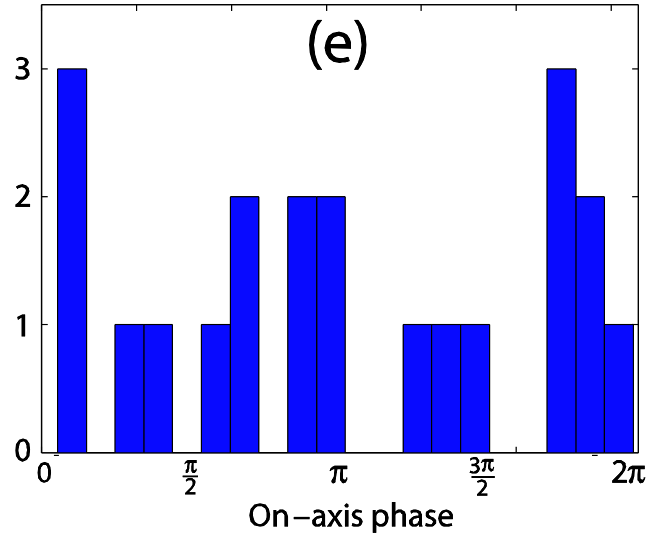




- Post-collapse loss-of-phase observed experimentally

Simulations of propagation in water

- NLS with dispersion, space-time focusing, multiphoton absorption, plasma, …
- Input power randomly chosen between 240 -260 MW
- Compute on-axis phase after propagation of 24cm



- Post-collapse loss-of-phase observed numerically

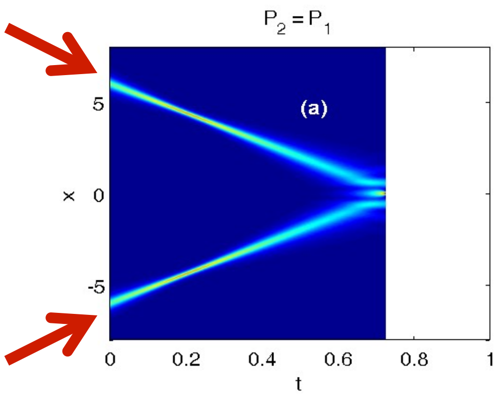
Importance of loss of phase

- NLS solution is invariant under multiplication by $e^{i\theta}$
- Multiplication by $e^{i\theta}$ does not affect the dynamics

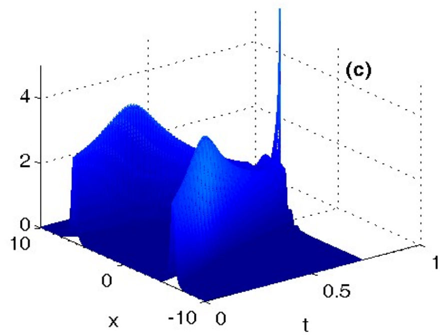
Importance of loss of phase

- NLS solution is invariant under multiplication by $e^{i\theta}$
- Multiplication by $e^{i\theta}$ does not affect the dynamics
- But, relative phase of two beams does affect the dynamics

$\psi\downarrow 0\uparrow (1)$



$\psi\downarrow 0\uparrow (2)$



collapse

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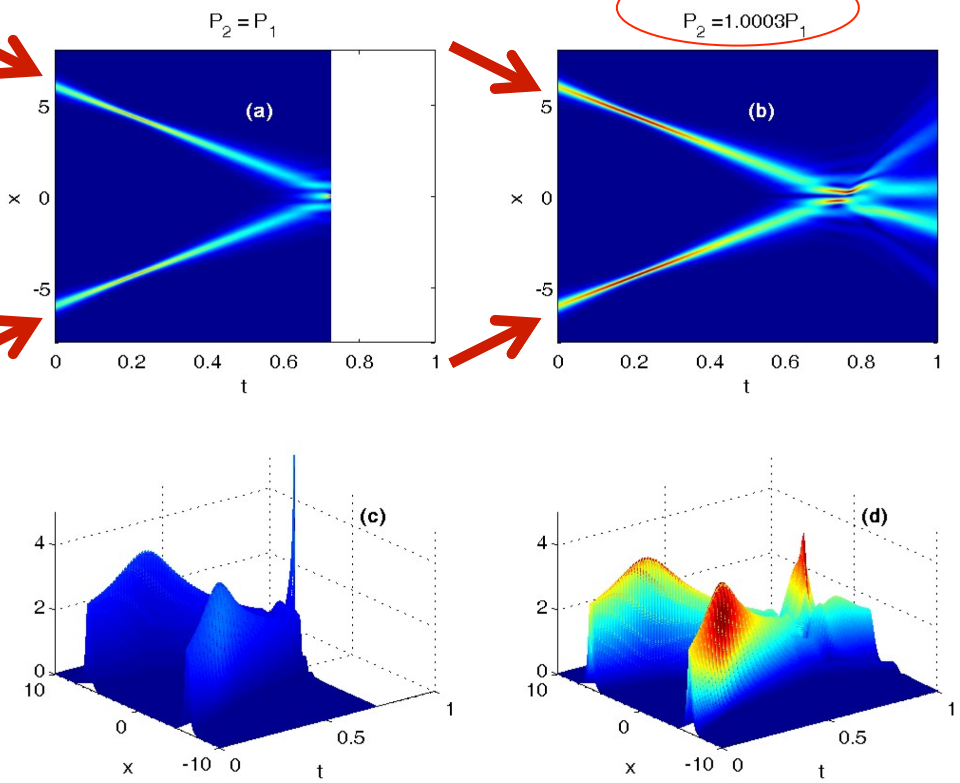
$\psi_0^{(1)}$

$\psi_0^{(2)}$



collapse

no collapse

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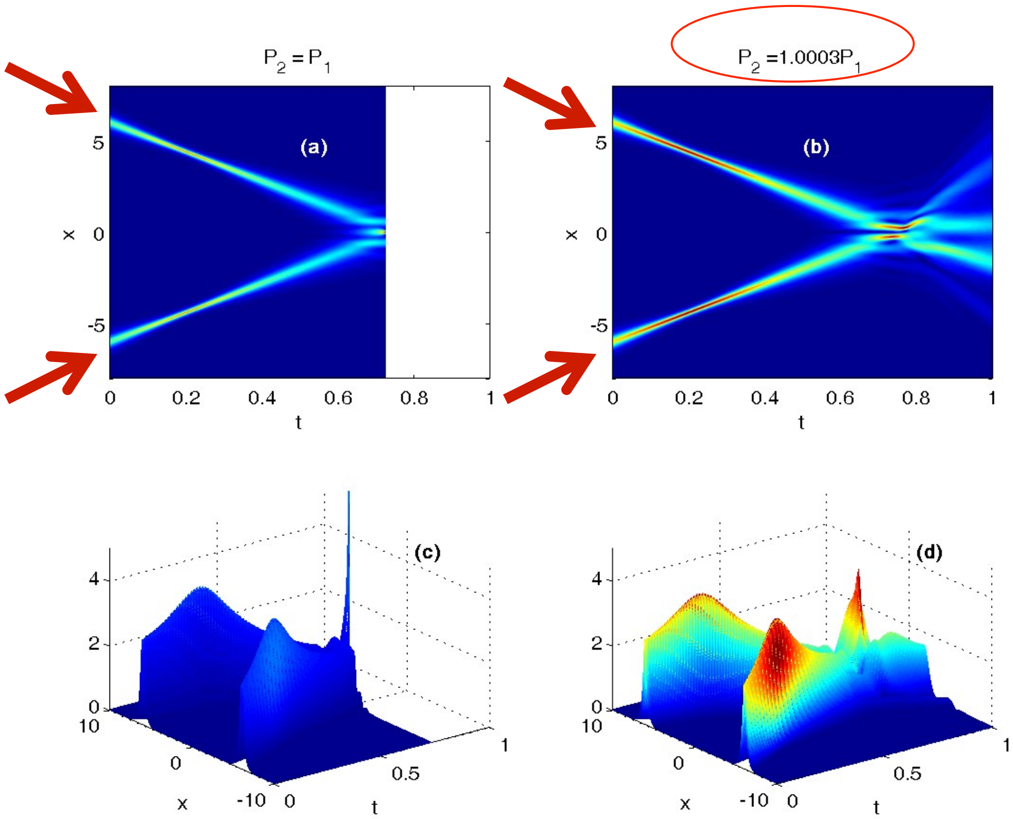
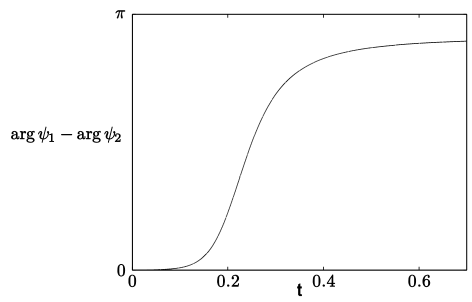
$\psi\downarrow 0\uparrow (1) \psi\downarrow 0\uparrow (2)$





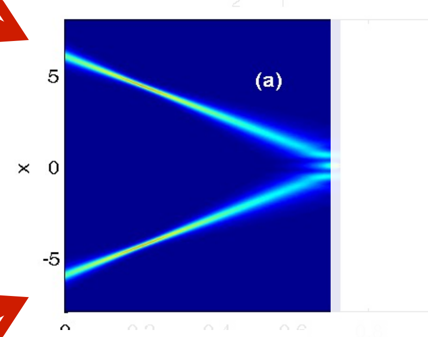
Post-collapse ``chaotic'' interactions

collapse

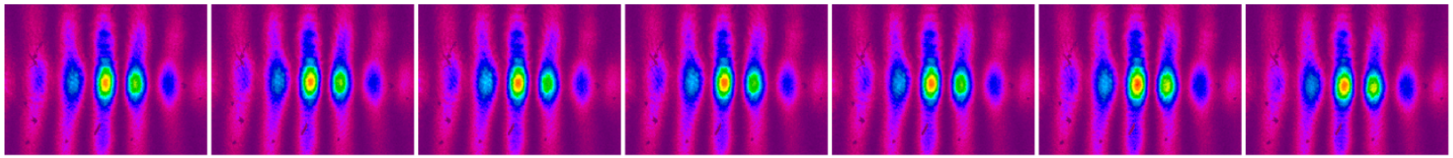
no collapse

Experiments (Shim et al., 2012)

- Interaction between two ``identical'' crossing beams after propagation of 24cm in water - seven consecutive shots

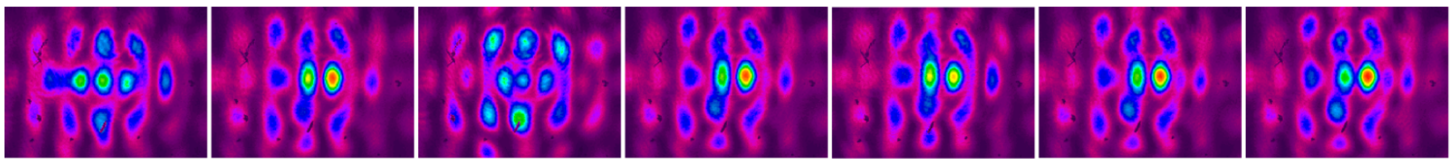


(a) 160 MW



(b) 240 MW

(c) 280 MW



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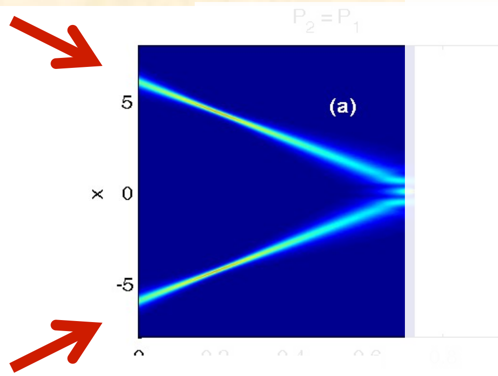
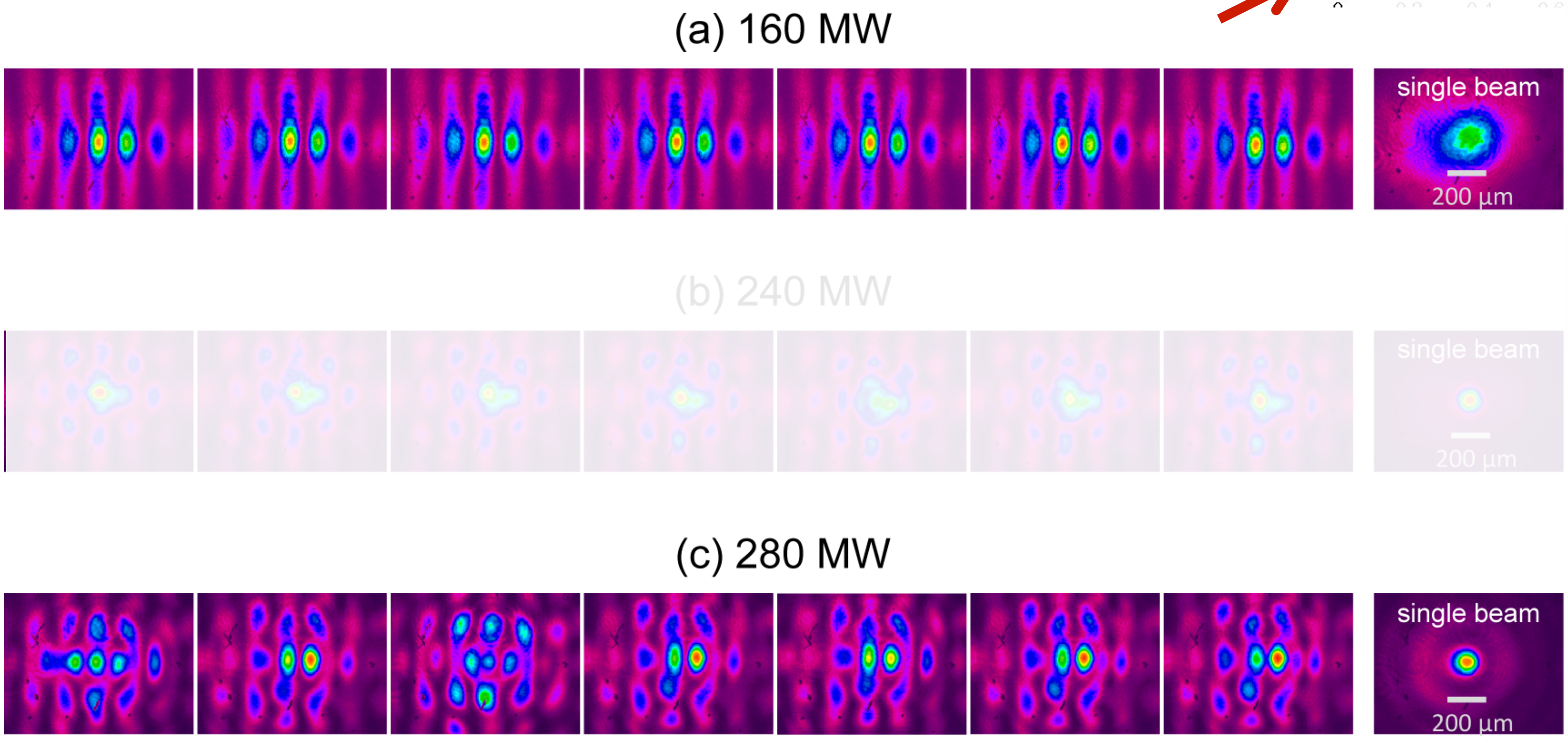
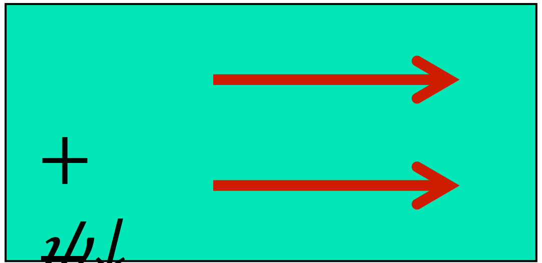
- Interaction between two ``identical'' crossing beams after propagation of 24cm in water - seven consecutive shots



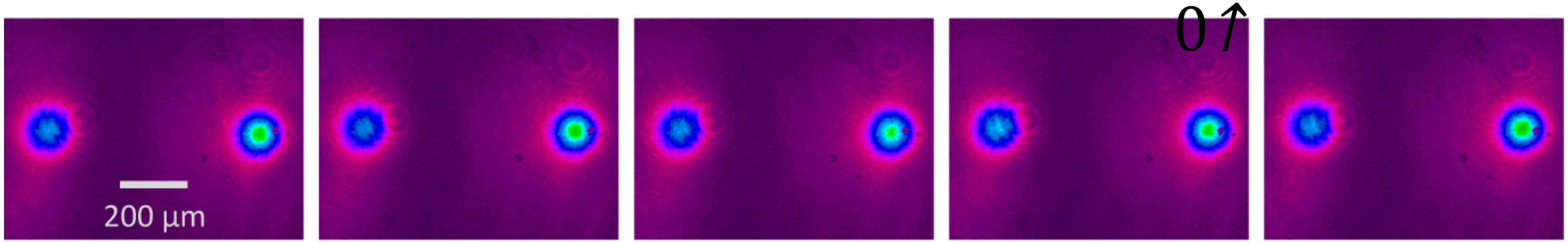


Experiments (Shim et al., 2012)

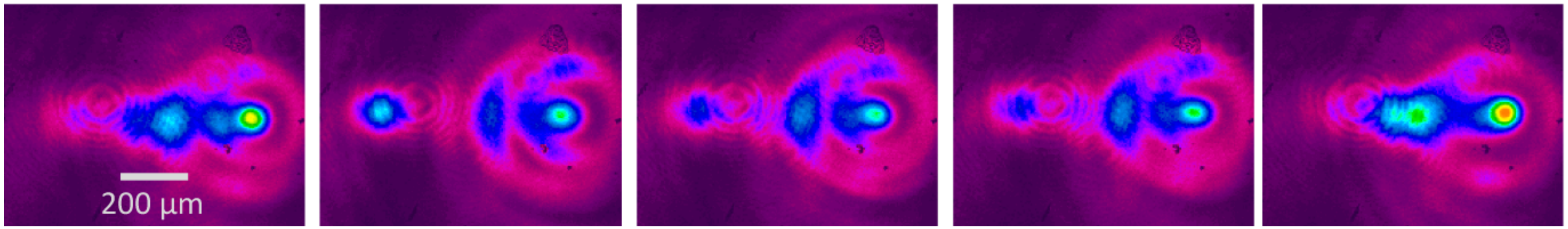
- Interaction between two parallel beams with initial π phase difference, after propagation of 24cm in water - five consecutive shots



(a) 160 MW



(b) 280 MW



Loss of phase – interim summary

- Collapse $\Longrightarrow$ blowup of phase $\Longrightarrow$ loss of phase
- Question 1:

  Can we have loss of phase without collapse?

- Loss of phase $\Longrightarrow$ cannot make deterministic predictions of interactions
- Question 2

  Can we make stochastic predictions?

Loss of phase

- Collapse $\Longrightarrow$ blowup of phase $\Longrightarrow$ loss of phase
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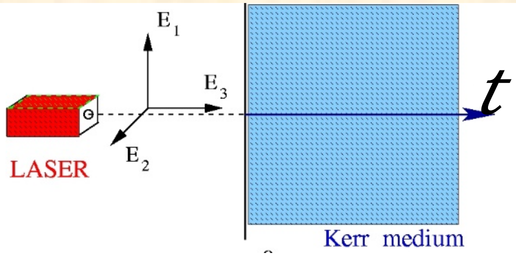
NLS with random ic

Each laser shot is different



NLS with random ic

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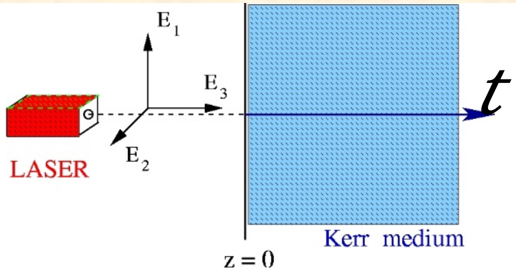


NLS model

random ic

$$\psi_0(\boldsymbol{x};\alpha)$$

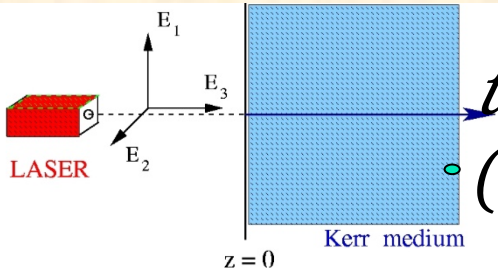
random output

$$\psi(t,\boldsymbol{x};\alpha)$$

α - noise parameter

NLS with random ic

Each laser shot is different



t

$(tf, \boldsymbol{x}f)$

NLS model

random ic

$\psi 0(\boldsymbol{x};\alpha)$

random output

$\psi(t,\boldsymbol{x};\alpha)$

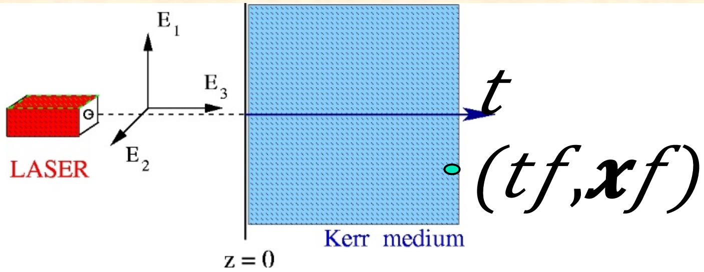
α - noise parameter

Distribution (over many shots) of output phase φ

$= \arg(\psi(tf,\boldsymbol{x}f;\alpha))\, mod(2\pi)$

NLS with random ic

Each laser shot is different



NLS model

random ic

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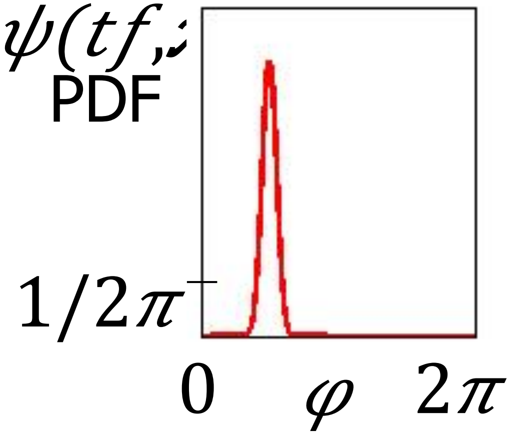
random output

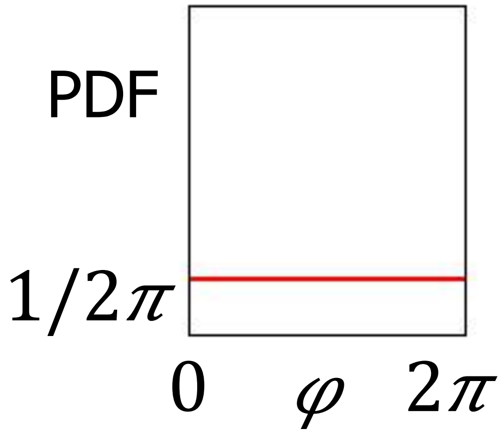
$\psi(t,\boldsymbol{x};\alpha)$

α - noise parameter

Distribution (over many shots) of output phase φ

$=\arg(\psi(tf,$





localized
no loss of phase

$U(0,2\pi)$
complete loss of phase

Example: 1D cubic−quintic NLS

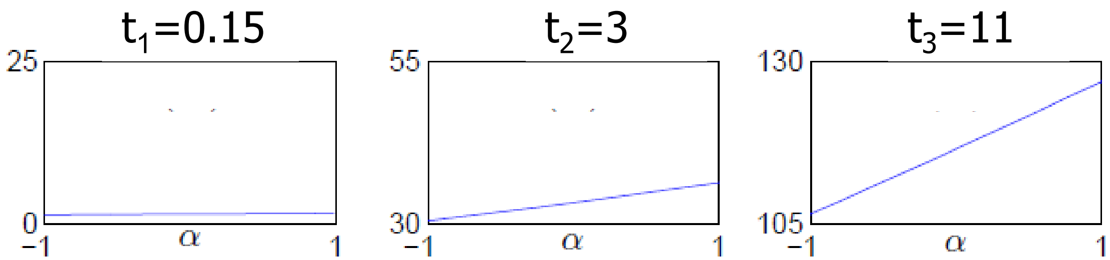
$$i\psi_t(t,x)+\psi_{xx}+|\psi|^2\psi-\epsilon|\psi|^4\psi=0,\ \epsilon=0.001$$

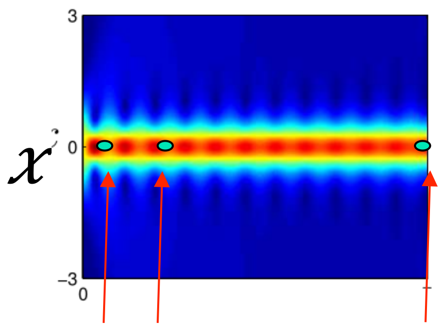
$$\psi_0(x;\alpha)=(1+0.1\alpha)3.4e^{-x^2},$$

$$\alpha\sim U(-1,1)$$

$$\varphi=\arg(\psi(t_i,0;\alpha))$$

cumulative phase





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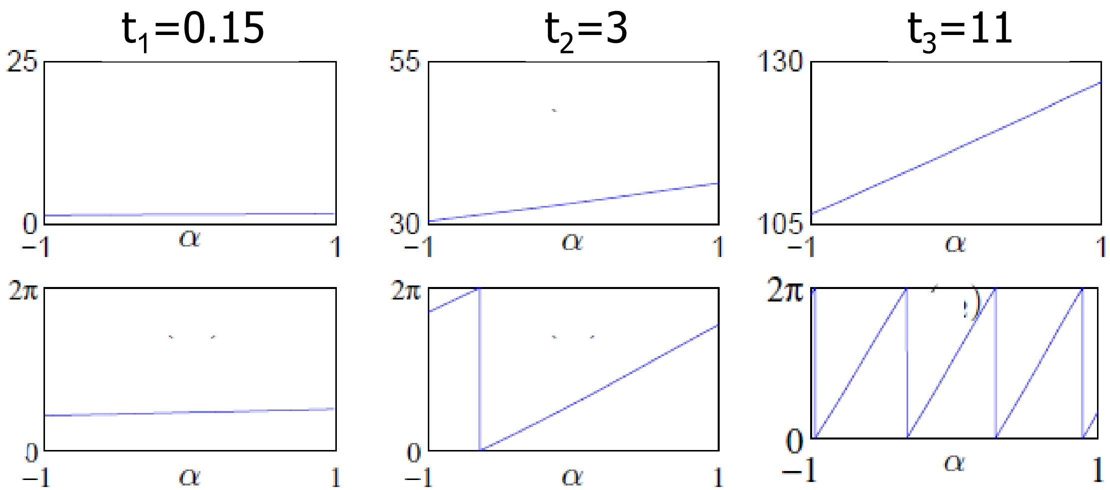
$$\alpha\sim U(-1,1)$$

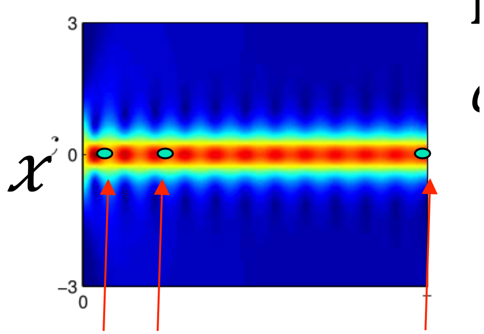
$\varphi=\arg(\psi(t_i,0;\alpha))$

cumulative phase

$\varphi=\varphi\ mod(2\pi)$

non-cumulative phase





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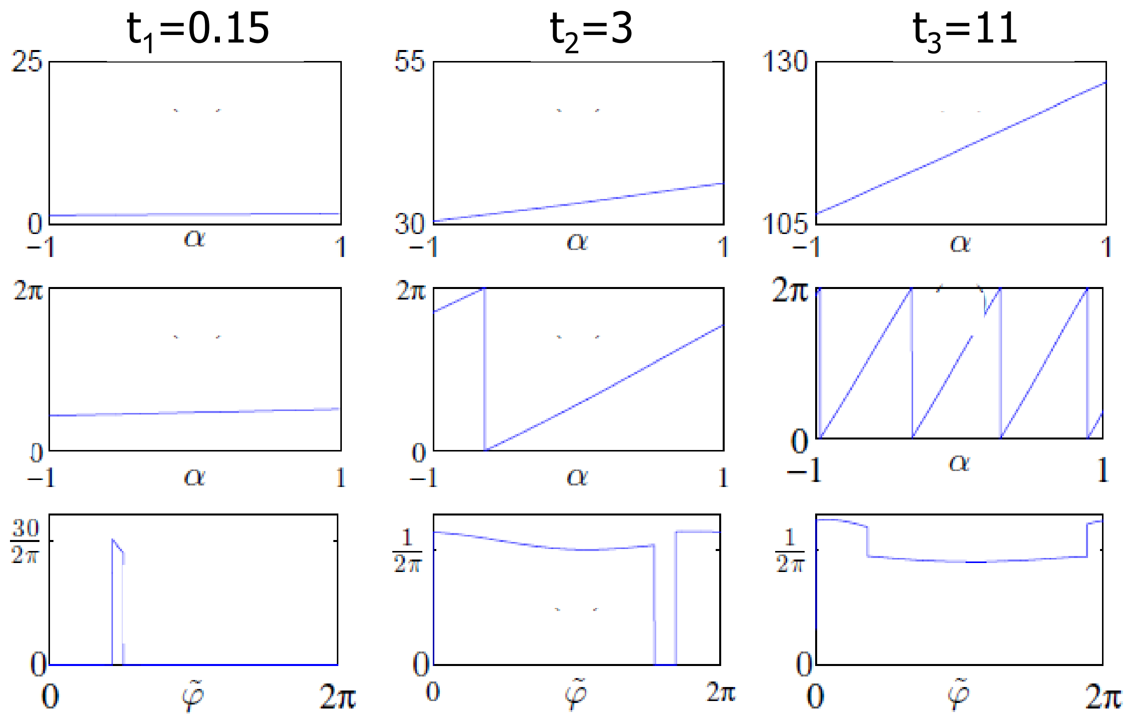
$$\varphi=\arg(\psi(t_i,0;\alpha))$$

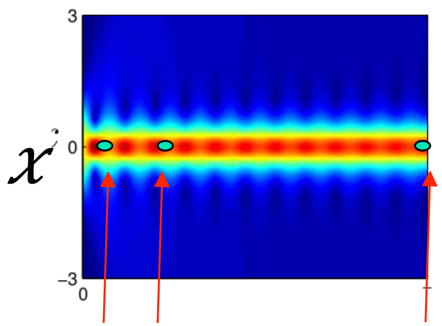
cumulative phase

$$\tilde{\varphi}=\varphi \bmod (2\pi)$$

non-cumulative phase

PDF of $\tilde{\varphi}$





$\tilde{\varphi}$ localized
no loss of phase

$\tilde{\varphi}\sim U(0,2\pi)$
loss of phase

Numerical observations

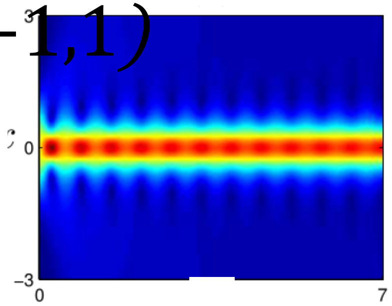
- Phase loss builds up gradually with time/distance
- No collapse necessary
- Same for 2D NLS, other nonlinearities, other noises, …
- Theoretical explanation?

Loss of phase - explanation

$$i\psi_t(t,x)+\psi_{xx}+|\psi|^2\psi-\epsilon|\psi|^4\psi=0,\quad \epsilon=0.001$$

$$\psi_0(x;\alpha)=(1+0.1\alpha)3.4e^{-x^2},\quad \alpha\sim U(-1,1)$$

initial condition $\psi_0(x)\alpha$



t

solitary wave (+radiation)

$\psi\approx e^{i\kappa t}R_\kappa(x)$,

$$(R_\kappa)_{xx}-\kappa R_\kappa+R_\kappa^3-\epsilon R_\kappa^5=0$$

Loss of phase - explanation

$$i\psi_t(t,x)+\psi_{xx}+|\psi|^2\psi-\epsilon|\psi|^4\psi=0,\quad \epsilon=0.001$$

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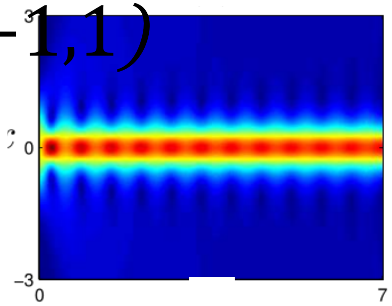

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$$\psi\approx e^{i\kappa t}R_\kappa(x),\quad \boxed{\kappa=\kappa(\alpha)}$$

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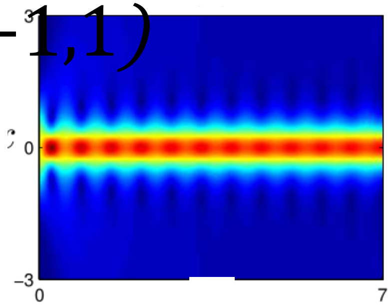

t

solitary wave (+radiation)

$\psi\approx e^{i\kappa t}R_\kappa(x)$, $\boxed{\kappa=\kappa(\alpha)}$

$$\varphi(t;\alpha):=\arg\psi(t,x=0;\alpha)\approx t\kappa(\alpha)$$

Loss of phase - explanation

$$i\psi_t(t,x)+\psi_{xx}+|\psi|^2\psi-\epsilon|\psi|^4\psi=0,\ \epsilon=0.001$$

$$\psi_0(x;\alpha)=(1+0.1\alpha)3.4e^{-x^2},$$

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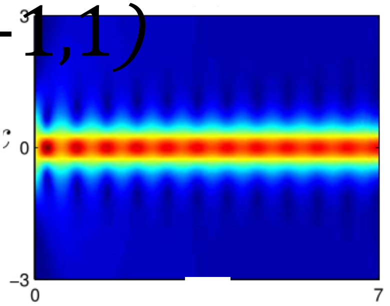


solitary wave (+radiation)

$$\psi\approx e^{i\kappa t}R_\kappa(x),\ \boxed{\kappa=\kappa(\alpha)}$$

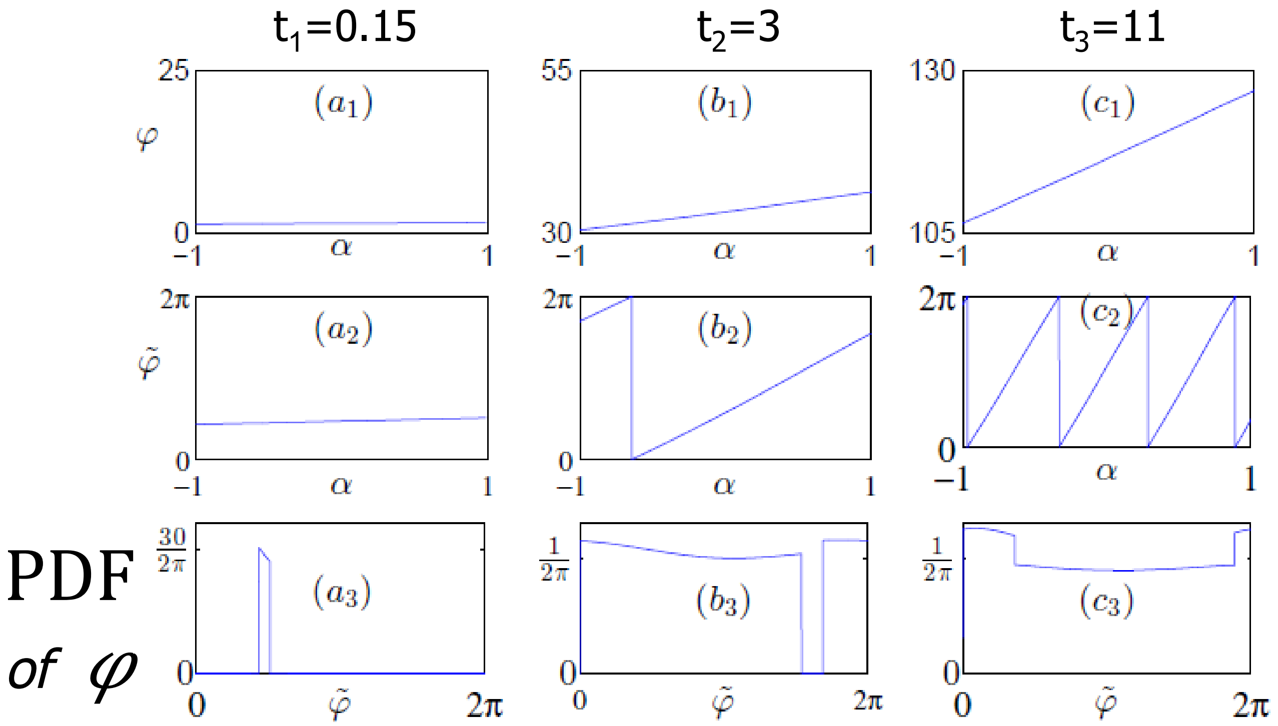
$$\varphi(t;\alpha):=\arg\psi(t,x=0;\alpha)\approx t\kappa(\alpha)$$

Loss of Phase Lemma (Sagiv, Ditkowski, Fibich, 2017)

Let $\varphi(t;\alpha)\approx t\kappa(\alpha)$, where $\kappa(\alpha)\in C^1$ is piece-wise monotonic and α is a random variable with an absolutely continuous distribution. Then

$$\lim_{t\to\infty}\varphi(t;\alpha)\ mod(2\pi)\sim U(0,2\pi)$$

Proof idea



φ localized
no loss of phase

φ
$\sim U(0,2\pi)$
loss of phase

Loss of phase

Loss of Phase Lemma (Sagiv, Ditkowski, Fibich, 2017)

$$\lim_{t\to\infty} \varphi(t;\alpha)\ mod(2\pi) \sim U(0,2\pi)$$

- Phase loss occurs whenever
  - ψ_0 is noisy
  - Evolves into a solitary wave
  - Propagates sufficiently long time/distance
- Generic phenomena
  - Any NLS that supports solitary waves
  - Any noise
  - Collapse not needed
- Phase loss builds up gradually in time/distance
  - Unlike in collapse

Loss of phase

- Collapse $\Longrightarrow$ blowup of phase $\Longrightarrow$ loss of phase
- Question 1:

  Can we have loss of phase without collapse? YES

- Loss of phase $\Longrightarrow$ cannot make deterministic predictions of interactions
- Question 2

  Can we make stochastic predictions?

Loss of phase

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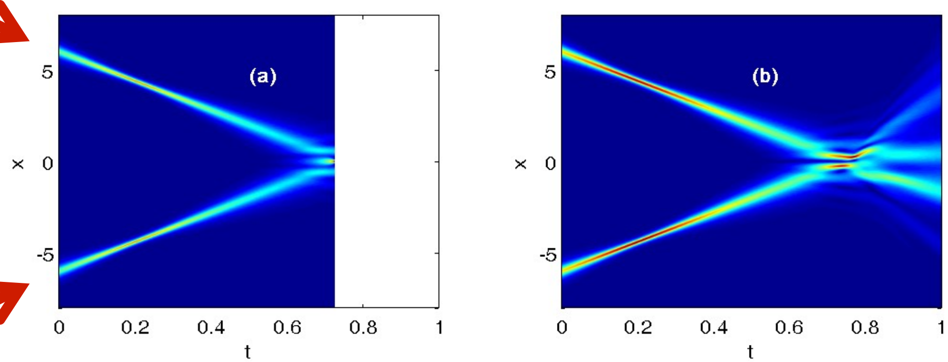
Stochastic interactions

$\psi_0^{(1)}$

$\psi_0^{(2)}$



- Interactions predominantly determined by relative phase at intersection
- **Loss of Phase Lemma** ⇨ relative phase $\sim U(0,2\pi)$
- Interactions statistics becomes universal (independent of noise source)
- Can compute using the universal model

$$\psi_0 = \psi_0^{(1)} + e^{i\alpha}\psi_0^{(2)}, \quad \alpha \sim U(-\pi,\pi)$$

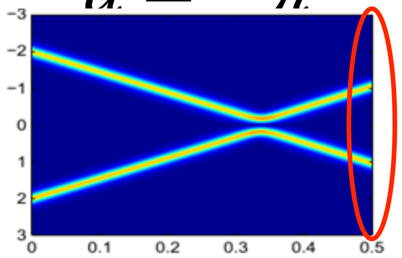
- Even when the noise source/distribution are unknown

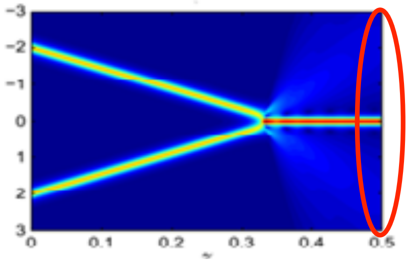
Universal model

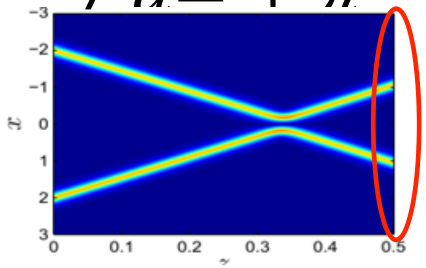
$$i\psi_t(t,x)+\psi_{xx}+|\psi|^2\psi-\epsilon|\psi|^4\psi=0$$

$$\psi_0=R_{\kappa_1}(x+d)e^{i\theta x}+e^{i\alpha}R_{\kappa_1}(x-d)e^{-i\theta x},\quad \alpha\sim U(-\pi,\pi)$$

$\alpha=-\pi$ … $\alpha=0$ … $\alpha=+\pi$







x

t t t

Universal model

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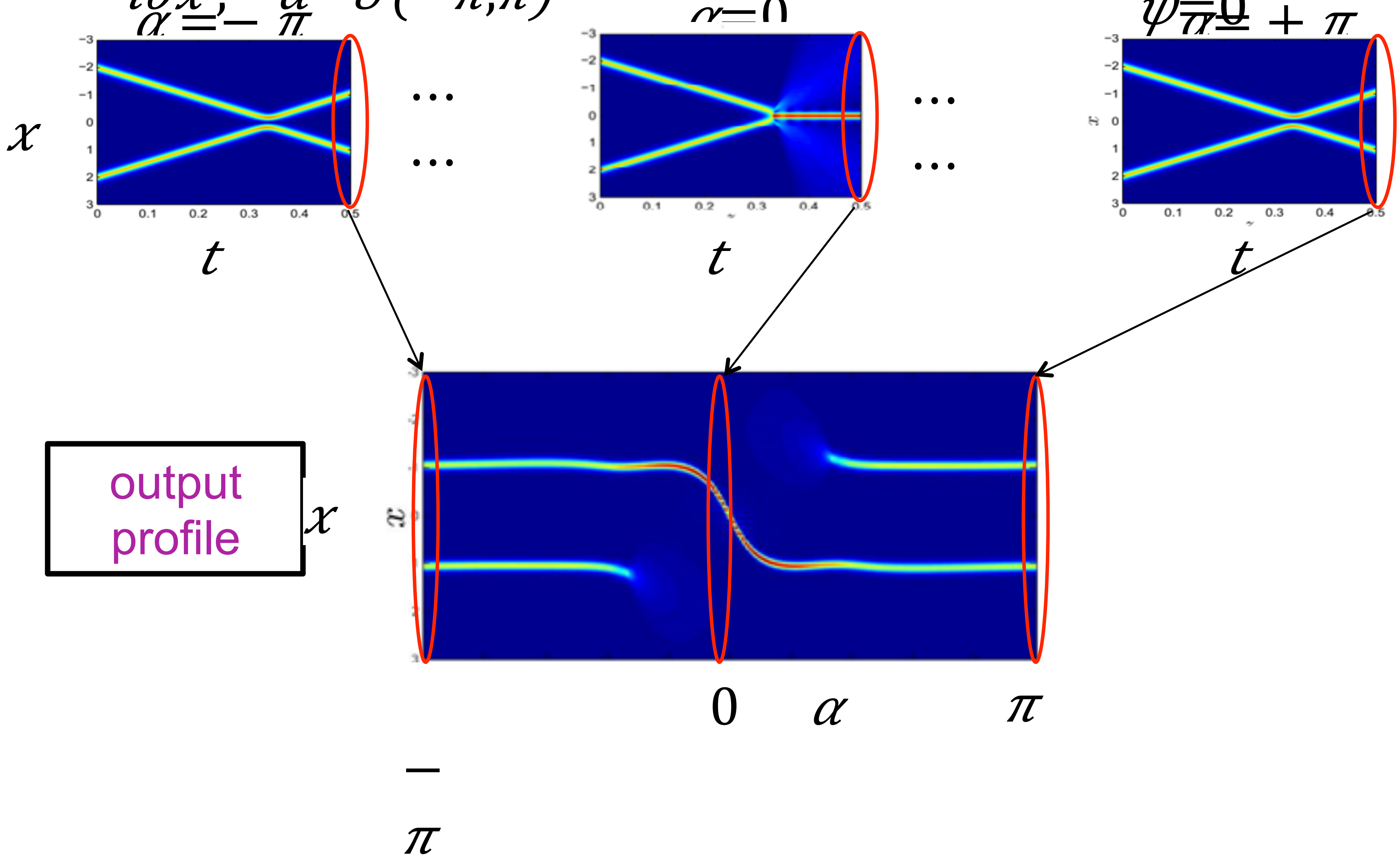

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$$i\psi_t(t,x)+\psi_{xx}+|\psi|^2\psi-\epsilon|\psi|^4\psi=0$$

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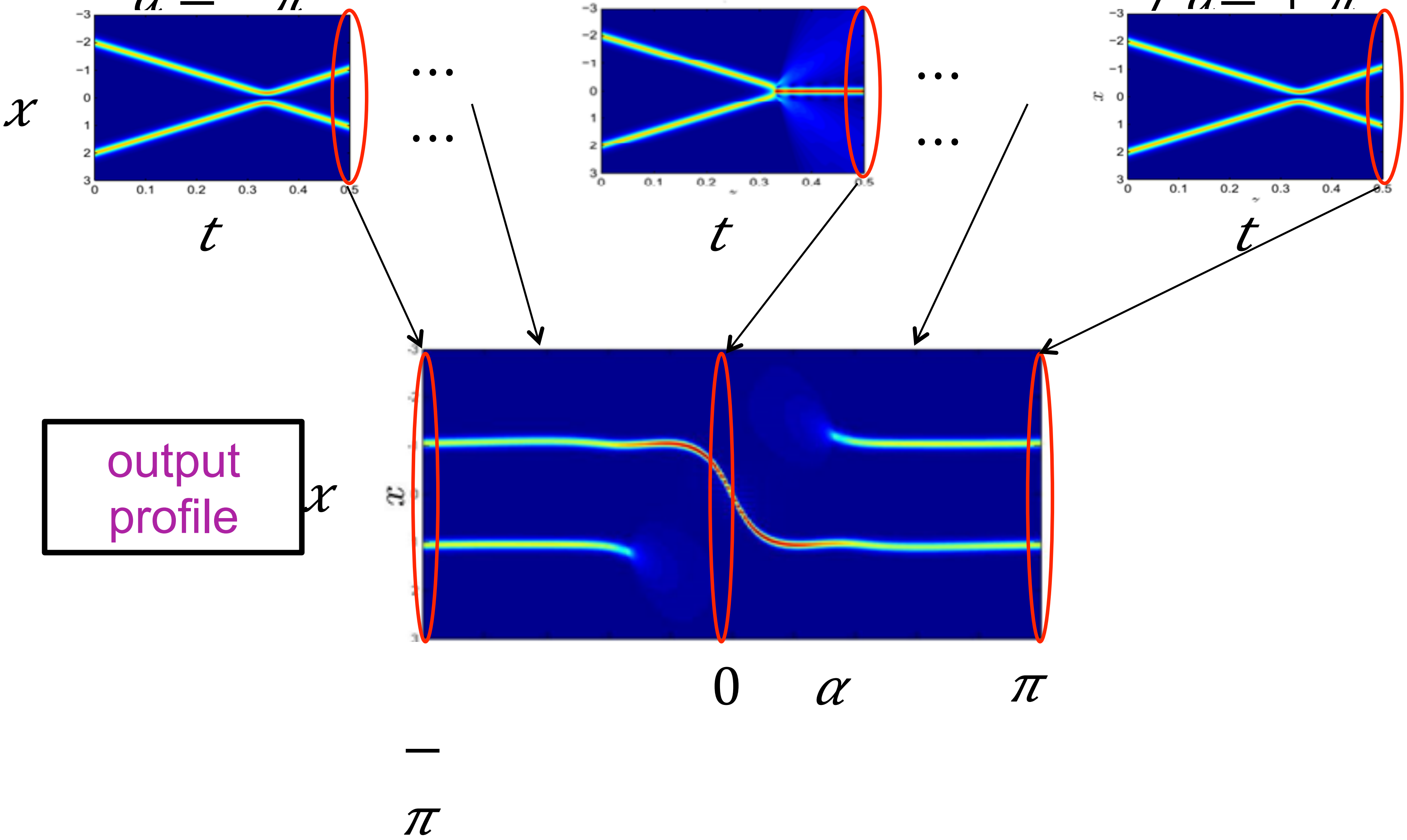
$\alpha=-\pi$ $\alpha=0$ $\alpha=+\pi$



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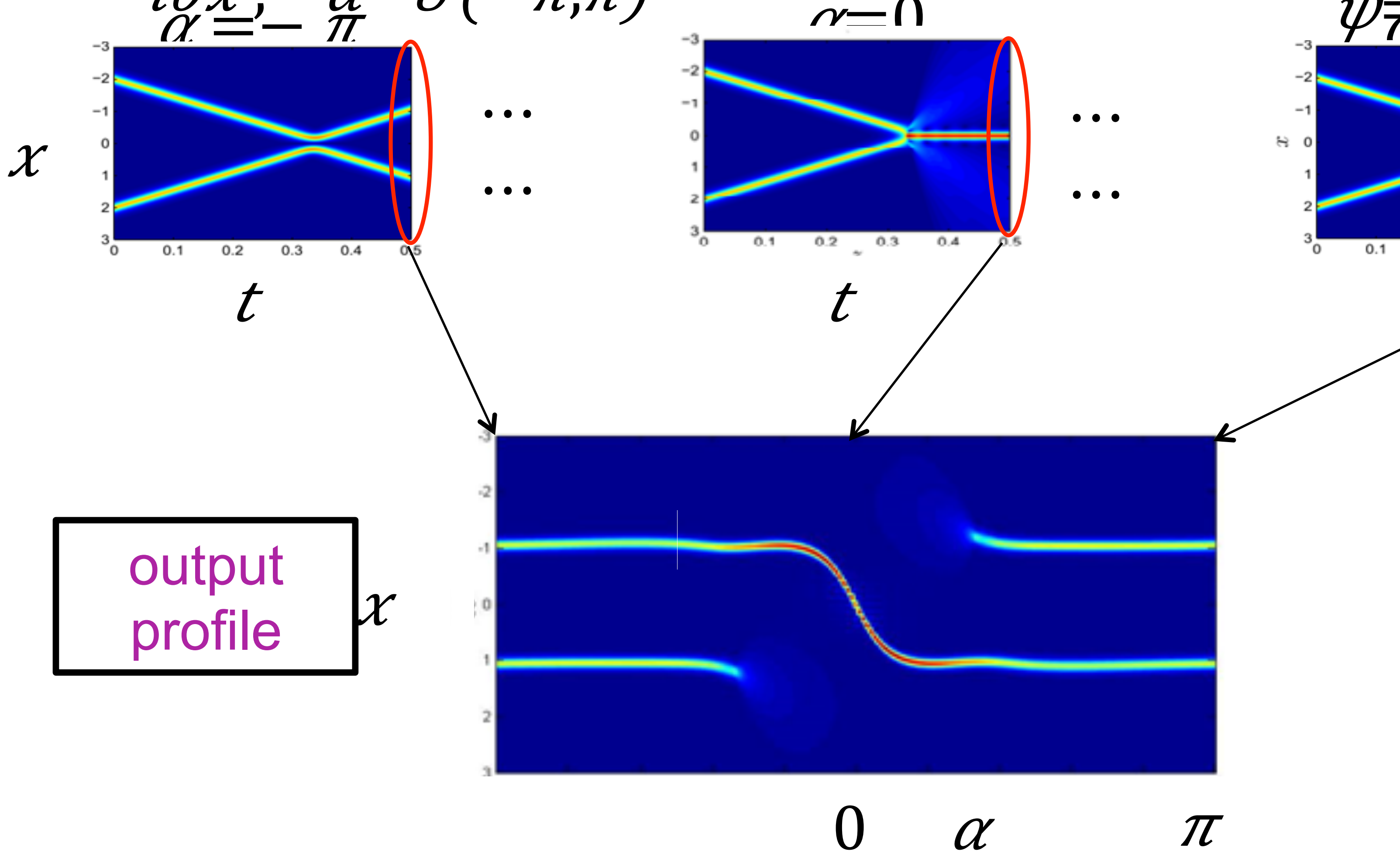


can compute statistics of interactions

NLS with 4 noise models

$$i\psi_t(t,x)+\psi_{xx}+|\psi|^2\psi-\epsilon|\psi|^4\psi=0$$

1. Perturbed profile

$$\psi_0 = R_{\kappa_1}(x+d)e^{i\theta x} + R_{\boldsymbol{\kappa}(\boldsymbol{\alpha})}(x-d)e^{-i\theta x}$$

2. Perturbed profile, out of phase

$$\psi_0 = R_{\kappa_1}(x+d)e^{i\theta x} - R_{\boldsymbol{\kappa}(\boldsymbol{\alpha})}(x-d)e^{-i\theta x}$$

3. Perturbed amplitude

$$\psi_0 = R_{\kappa_1}(x+d)e^{i\theta x} + (1+0.1\alpha)R_{\kappa_1}(x-d)e^{-i\theta x}$$

4. Uniformly distributed random phase (universal model)

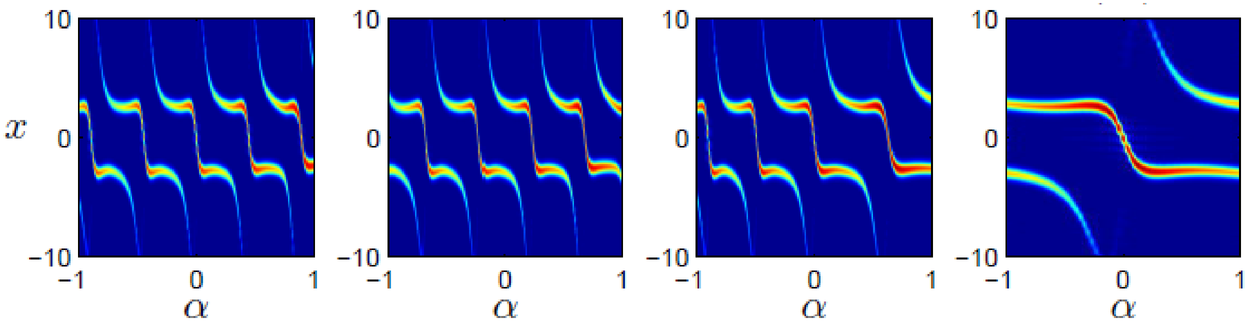
$$\psi_0 = R_{\kappa_1}(x+d)e^{i\theta x} + e^{i\alpha}R_{\kappa_1}(x-d)e^{-i\theta x}, \quad \alpha\sim U(-\pi,\pi)$$

Universality of stochastic interactions

4 noise models

universal model

output profile



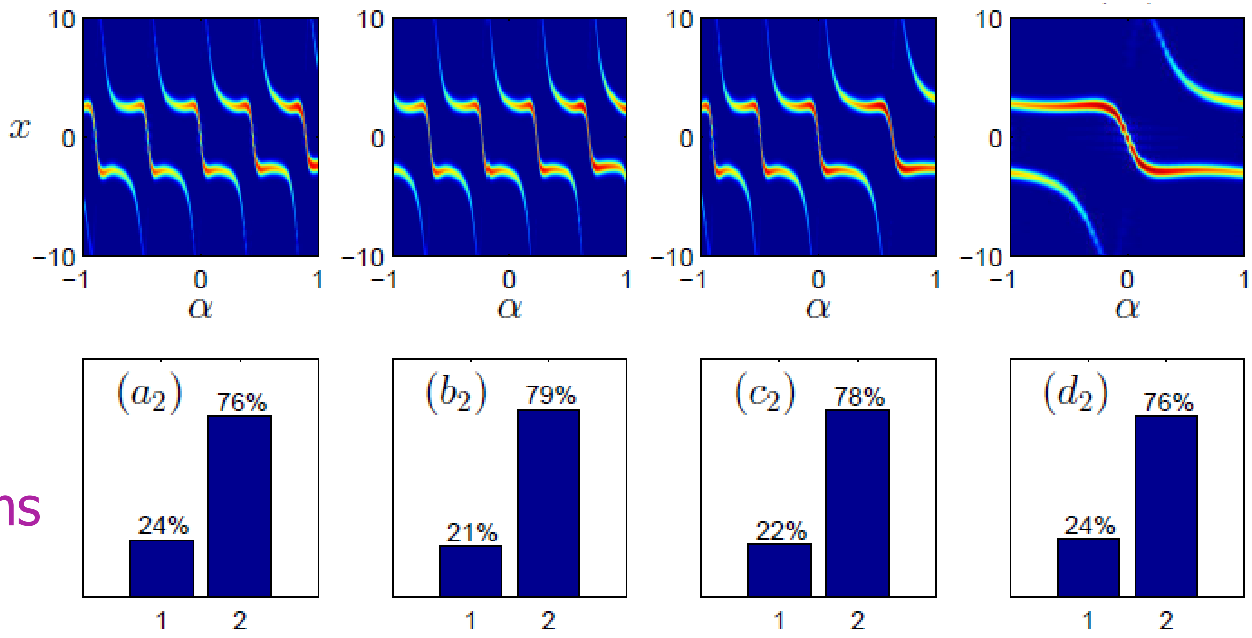
Universality of stochastic interactions

4 noise models

universal model

output profile

num. of output beams



universal statistics

Universality of stochastic interactions

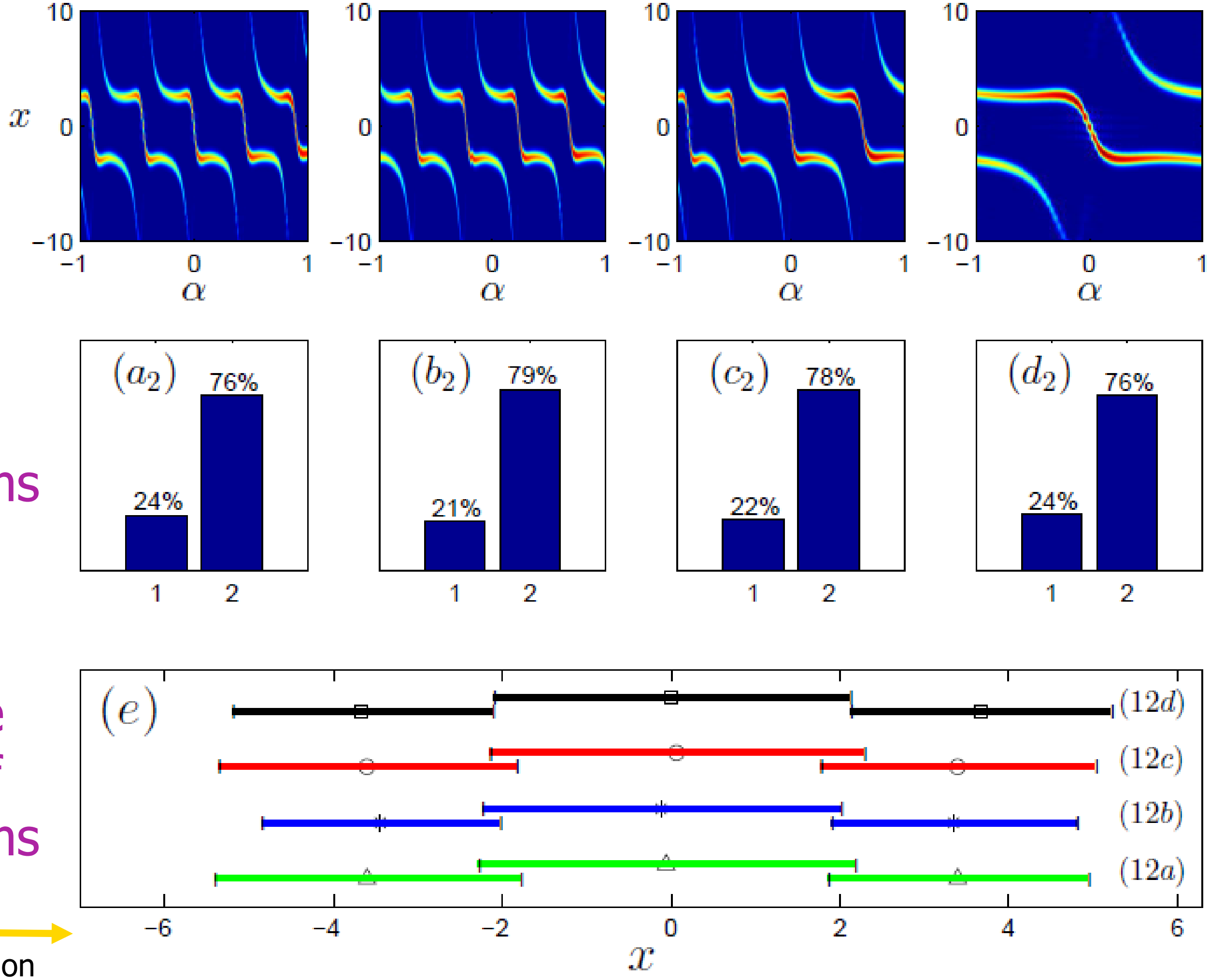
4 noise models

universal model

output profile

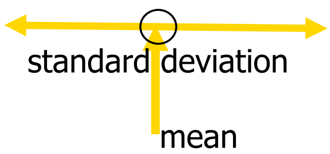
num. of output beams

Transverse location of output beams



universal statistics

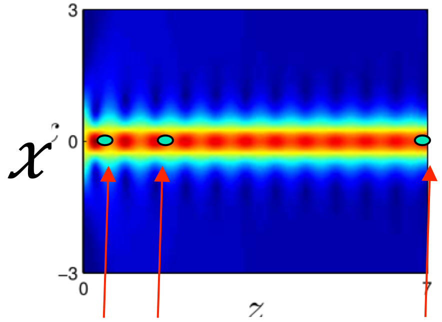
universal statistics



Talk plan

1. Review
2. Nonlinear damping continuation
3. Sub threshold power continuation
4. Loss of phase
5. Universality of stochastic interactions
6. Numerical methods

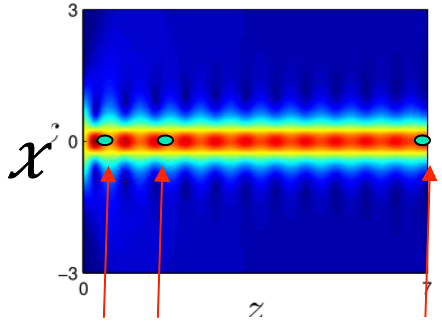
Numerical methods



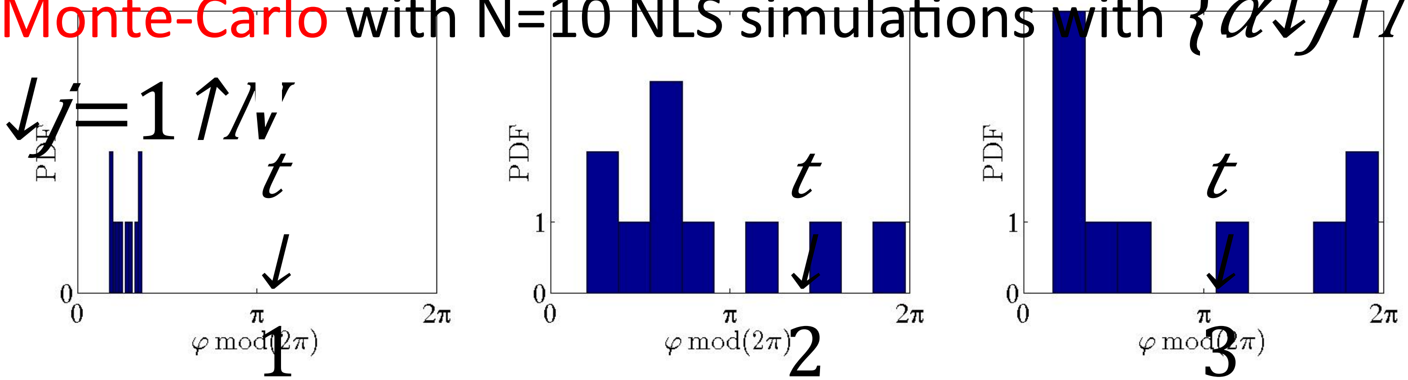
$t_1 \quad t_2 \quad\quad\quad t_3$

- Goal: compute distribution of $\varphi(t_i,\alpha) := \arg(\psi(t_i,0)) \bmod (2\pi)$

Numerical methods

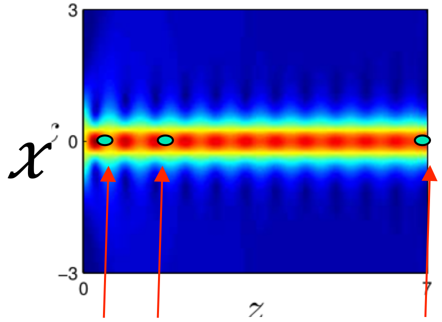


- Goal: compute distribution of $\varphi(t_i,\alpha):=\arg(\psi(t_i,0))mod(2\pi)$ at t_1, t_2, t_3
- Monte-Carlo with N=10 NLS simulations with $\{\alpha_j\}_{j=1}^{N}$

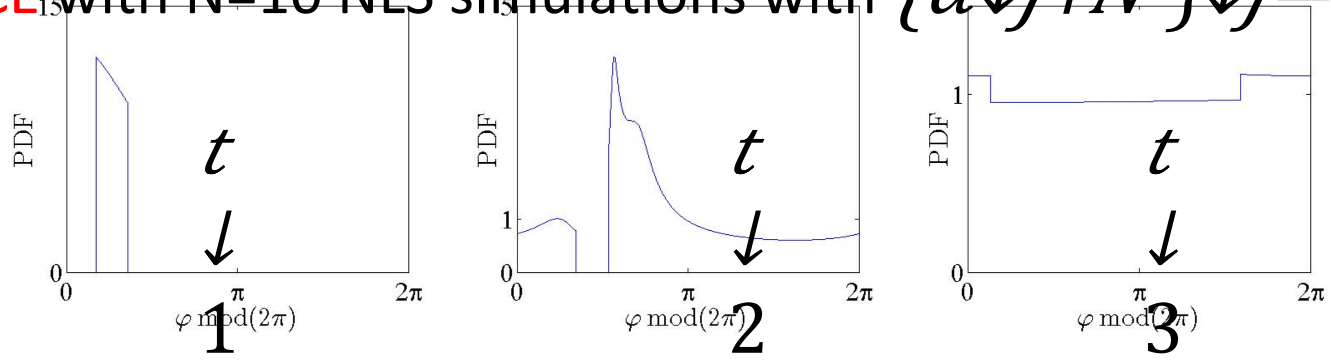


- N=10 simulations insufficient to determine distribution
- Monter-Carlo error $\sim 1/\sqrt{N}$

Numerical methods



- Goal: compute distribution of $\varphi(t_i,\alpha) := \arg(\psi(t_i, 0))\, mod(2\pi)$ — t_1, t_2, t_3
- PCE with N=10 NLS simulations with $\{\alpha_j^N\}_{j=1}^N$



- N=10 simulations sufficient to determine distribution
- PCE has spectral accuracy

Polynomial Chaos Expansion (PCE)

Step 1 Obtain Gauss-Legendre quadrature formula $\{\alpha_j^N, w_j^N\}_{j=1}^N$

Step 2 Solve the NLS at the N quadrature points, i.e., compute

$$\{\psi(t,x;\alpha_j^N)\}_{j=1}^N$$

Step 3 Approximate $\psi(t,x;\alpha)$ with

$$\psi^N(t,x;\alpha)=\sum_{n=0}^{N-1} \psi^N(t,x;n)\, P_n(\alpha)$$

where P_n are Legendre polynomials, and

$$\psi^N(t,x;n)=\sum_{j=1}^{N} w_j^N\, \psi(t,x;\alpha_j^N)\, p_n(\alpha_j^N), \quad n=0,1,,\ldots,N-1$$

Summary

- Vanishing nonlinear-damping continuation
  - Vanishing-viscosity approach
  - Viscosity = nonlinear damping
  - Explicit continuation of $\psi_{R,\alpha}^{\text{explicit}}$
  - Asymmetric w.r.t. T_c
  - Rigorous pf needed

- Sub threshold-power continuation
  - ``Minimal-power'' blowup solution $\psi(t, \mathbf{x}; K_{th})$ is B-W sol
  - B-W solutions are generic
  - Continuation of B-W solution is ``same'' B-W solution

Summary

- Loss of phase
    - At blowup
    - Also without blowup
        - ψ_0 is noisy
        - Evolves into a solitary wave
        - Propagates sufficiently long time/distance
    - $\lim_{t\to\infty} \varphi(t;\alpha)\ mod(2\pi) \sim U(0,2\pi)$
- Stochastic interactions
    - Cannot make deterministic predictions
    - Can make stochastic predictions using universal model

Summary

- Numerical method
  - Monte-Carlo is inefficient
  - Can use Polynomial Chaos Expansion (PCE)

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Loss of phase of collapsing beams

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A. Sagiv, A. Ditkowski, G. Fibich

Loss of phase and universality of stochastic interactions between laser beams

ArXiv 1705.01137