The Sine-Gordon regime of the Landau-Lifshitz equation with a strong easy-plane anisotropy

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(joint work with Philippe Gravejat)

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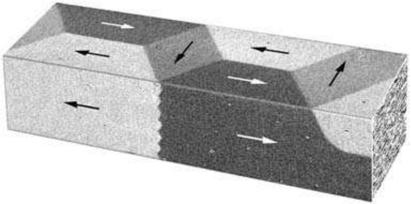
Outline

The Landau–Lifshitz equation

- The dynamics of magnetization in a ferromagnetic material are given by the Landau–Lifshitz equation.
- The magnetization is a direction field:

$$\vec{\mathbf{m}}(x,t) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{S}^2 \subset \mathbb{R}^3, \quad \vec{\mathbf{m}} = (m_1, m_2, m_3)$$

$$|\vec{\mathbf{m}}| = (m_1^2 + m_2^2 + m_3^2)^{1/2} = 1$$

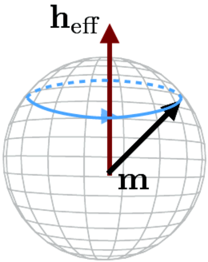


The Landau–Lifshitz equation

$$\partial_t \vec{\mathbf{m}} = \underbrace{\vec{\mathbf{m}} \times \vec{\mathbf{h}}_{\mathit{eff}}(\vec{\mathbf{m}})}_{\textit{precession}}$$

$\vec{\mathbf{h}}_{\mathit{eff}}(\vec{\mathbf{m}})$: effective magnetic field

For instance: $\vec{\mathbf{h}}_{\mathit{eff}}(\vec{\mathbf{m}}) = \Delta \vec{\mathbf{m}}$



The Landau–Lifshitz equation

$$E(\vec{\mathbf{m}}) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla \vec{\mathbf{m}}|^2\, dx + \frac{1}{2}\int_{\mathbb{R}^N} e_{\mathrm{ani}}(\vec{\mathbf{m}})\, dx.$$

Examples of anisotropy

$$\begin{aligned}
&e_{\mathrm{ani}}(\vec{\mathbf{m}}) = 0, && \text{(isotropic - Schrödinger maps)},\\
&e_{\mathrm{ani}}(\vec{\mathbf{m}}) = 1 - m_2^2 = m_1^2 + m_3^2, && \text{(easy-axis anisotropy in the } \vec{\mathbf{e}}_2\text{-direction)},\\
&e_{\mathrm{ani}}(\vec{\mathbf{m}}) = m_3^2, && \text{(easy-plane or planar anisotropy)}\\
&e_{\mathrm{ani}}(\vec{\mathbf{m}}) = \lambda_1 m_1^2 + \lambda_3 m_1^3, \quad \lambda_1 \neq \lambda_3 && \text{(biaxial anisotropy)}.
\end{aligned}$$

In this talk we are interested in the dynamics of biaxial ferromagnets in a regime of strong easy-plane anisotropy $0 < \lambda_1 \ll 1 \ll \lambda_3$. We consider

$$\lambda_1 = \sigma\varepsilon, \quad \text{and} \quad \lambda_3 := \frac{1}{\varepsilon},$$

$\sigma > 0$ (fixed) and $\varepsilon > 0$ (small).

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The biaxial Landau–Lifshitz equation

In this regime, the Landau-Lifshitz equation recasts as

$$\partial_t \vec{\mathbf{m}} + \vec{\mathbf{m}} \times \left(\Delta \vec{\mathbf{m}} - \varepsilon \sigma m_1 \vec{\mathbf{e}}_1 - \frac{m_3 \vec{\mathbf{e}}_3}{\varepsilon} \right) = 0$$

$\vec{\mathbf{m}}(x,t) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{S}^2$, $\vec{\mathbf{m}} = (m_1, m_2, m_3)$.

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In coordinates

$$\begin{aligned}
&\partial_t m_1 + m_2 \Delta m_3 - m_3 \Delta m_2 - \frac{1}{\varepsilon} m_2 m_3 = 0 \\
&\partial_t m_2 + m_3 \Delta m_1 - m_1 \Delta m_3 + (\frac{1}{\varepsilon} - \sigma\varepsilon) m_1 m_3 = 0 \\
&\partial_t m_3 + m_1 \Delta m_2 - m_2 \Delta m_1 + \sigma\varepsilon m_1 m_2 = 0
\end{aligned}$$

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$\vec{\mathbf{m}}(x,t) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{S}^2$, $\vec{\mathbf{m}} = (m_1, m_2, m_3)$.

- The equation is hamiltonian and the energy is (formally) conserved along the flow:

$$E(\vec{\mathbf{m}}(t)) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \vec{\mathbf{m}}(x,t)|^2 + \frac{\sigma\varepsilon}{2} \int_{\mathbb{R}^N} m_1^2(x,t) + \frac{1}{2\varepsilon} \int_{\mathbb{R}^N} m_3^2(x,t).$$

- The energy space is

$$\mathcal{E}(\mathbb{R}^N) = \{ \vec{\mathbf{v}} \in L^1_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^3) : \nabla \vec{\mathbf{v}} \in L^2(\mathbb{R}^N),\ v_1, v_3 \in L^2(\mathbb{R}^N),\ |\vec{\mathbf{v}}| = 1 \text{ a.e. on } \mathbb{R}^N \}.$$

Connection with the Sine–Gordon equation

The connection is well-know and used in the physical literature (e.g. Sklyanin '79). Assume that the map $\check{m} := m_1 + im_2$ does **not vanish**, i.e. $|\check{m}| = \sqrt{1 - m_3^2} \neq 0$, so $\check{m}$ can be lifted

$$\check{m} = m_1 + im_2 = (1 - m_3^2)^{\frac{1}{2}} \big(\sin(\phi) + i \cos(\phi) \big).$$

The variables $u := m_3$ and ϕ give a "hydrodynamical version" of the LL equation:

$$\begin{cases} \partial_t u = \operatorname{div}\left((1-u^2) \nabla \phi \right) - \frac{\varepsilon\sigma}{2}(1-u^2)\sin(2\phi), \\ \partial_t \phi = -\operatorname{div}\left(\frac{\nabla u}{1-u^2} \right) + u \frac{|\nabla u|^2}{(1-u^2)^2} - u|\nabla \phi|^2 + u\left(\frac{1}{\varepsilon} - \varepsilon\sigma \sin^2(\phi) \right). \end{cases} \tag{HLL}$$

Using the rescaled variables: $u(x,t) = \varepsilon U_\varepsilon\left(\sqrt{\varepsilon}x, t\right)$, and $\phi(x,t) = \Phi_\varepsilon\left(\sqrt{\varepsilon}x, t\right)$,

$$\begin{cases} \partial_t U_\varepsilon = \operatorname{div}\left((1-\varepsilon^2 U_\varepsilon^2) \nabla \Phi_\varepsilon \right) - \frac{\sigma}{2}(1-\varepsilon^2 U_\varepsilon^2)\sin(2\Phi_\varepsilon), \\ \partial_t \Phi_\varepsilon = U_\varepsilon\left(1 - \varepsilon^2 \sigma \sin^2(\Phi_\varepsilon)\right) - \varepsilon^2 \operatorname{div}\left(\frac{\nabla U_\varepsilon}{1-\varepsilon^2 U_\varepsilon^2} \right) + \varepsilon^4 U_\varepsilon \frac{|\nabla U_\varepsilon|^2}{(1-\varepsilon^2 U_\varepsilon^2)^2} - \varepsilon^2 U_\varepsilon |\nabla \Phi_\varepsilon|^2 \end{cases}$$

so as $\varepsilon \to 0$, we formally get

$$\partial_t U = \Delta \Phi - \frac{\sigma}{2}\sin(2\Phi), \quad \partial_t \Phi = U, \tag{SGS}$$

i.e. the limit function Φ is a solution to the Sine-Gordon equation

$$\partial_{tt}\Phi - \Delta\Phi + \frac{\sigma}{2}\sin(2\Phi) = 0. \tag{SG}$$

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Connection with the Sine–Gordon equation

- Our goal: to provide a rigorous justification for this Sine-Gordon regime of the Landau-Lifshitz equation and quantify the convergence in Sobolev norms.

- Difficulties:
  - Global/local well-posedness for the Landau-Lifshitz equation (and the HLL system)
  - Global/local well-posedness for the Sine-Gordon equation (for functions with nonvanishing limits at infinity)
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An explicit test: Solitons

The (HLL) system has explicit solitons solutions for $0 < c < 1$ (we consider $\sigma = 1$):

$$U_\varepsilon(x,t) = \frac{1}{\varepsilon}\sqrt{1-a_c^2}\,\mathrm{sech}(\mu_c\xi), \quad \Phi_\varepsilon(x,t) = 2\arctan\left(\frac{\sqrt{a_c^2+\sinh^2(\mu_c\xi)}-\sinh(\mu_c\xi)}{a_c}\right),$$

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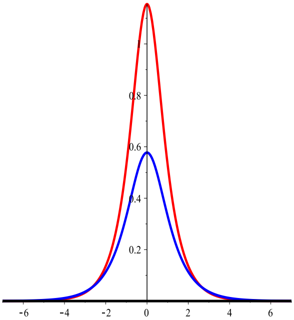
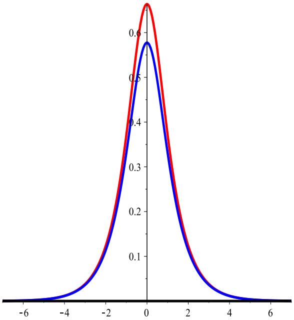
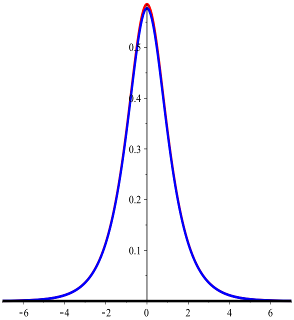
U_ε and U for $\varepsilon = 0.5$



U_ε and U for $\varepsilon = 0.3$



U_ε and U for $\varepsilon = 0.1$



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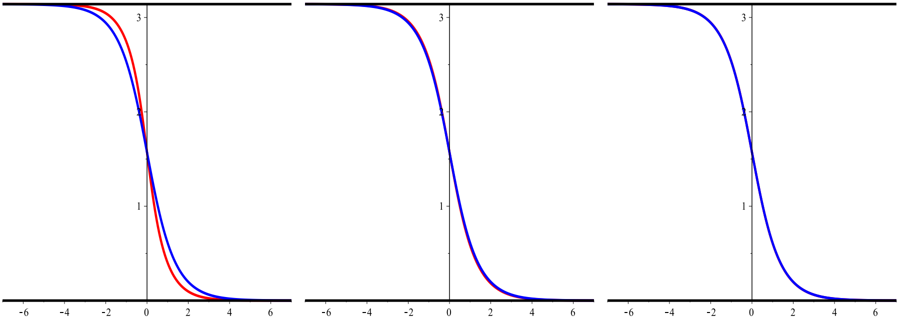

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Moreover, for $k \in \mathbb{N}$,

$$\|U_\varepsilon - U\|_{H^k} + \|\sin(\Phi_\varepsilon - \Phi)\|_{L^2} + \|\Phi_\varepsilon - \Phi\|_{H^{k+1}} \leq C_k\varepsilon^2,$$

and

$$\frac{\|U_\varepsilon - U\|_{L^2}}{\varepsilon^2} \underset{\varepsilon\to 0}{\sim} L, \quad L > 0.$$

Existence and uniqueness

Some results on the literature:

- Zhou and Guo 1984 (parabolic regularization)
- Sulem, Sulem and Bardos 1986 (discrete approximation schemes)
- Ding and Wang, 2001 (parabolic regularization)
- Chang, Shatah and Uhlenbeck, 2000 (generalization of Hasimoto transform)
- Gustafson and Shatah 2002
- Nahmod, Shatah, Vega and Zeng 2006
- McGahagan 2004 (energy estimates, parallel transport, Lipschitz continuity of the flow)

Given $k \geq 1$, we introduce the set

$$\mathcal{E}^k(\mathbb{R}^N) := \{v \in \mathcal{E}(\mathbb{R}^N) : \nabla v \in H^{k-1}(\mathbb{R}^N)\},$$

which we endow with the metric structure provided by the norm

$$\|v\|_{Z^k} := \Big(\|v_1\|_{H^k}^2 + \|v_2\|_{L^\infty}^2 + \|\nabla v_2\|_{H^{k-1}}^2 + \|v_3\|_{H^k}^2\Big)^{\frac{1}{2}},$$

of the vector space

$$Z^k(\mathbb{R}^N) := \{v \in L^1_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^3) : (v_1, v_3) \in L^2(\mathbb{R}^N)^2, v_2 \in L^\infty(\mathbb{R}^N) \text{ and } \nabla v \in H^{k-1}(\mathbb{R}^N)\}.$$

Theorem

Let $\lambda_1, \lambda_3 > 0$, and $k \in \mathbb{N}$, with $k > N/2 + 1$. Given initial condition $m^0 \in \mathcal{E}^k(\mathbb{R}^N)$, there exists $T_{\max} > 0$ and a unique solution $m : \mathbb{R}^N \times [0, T_{\max}) \to \mathbb{S}^2$ to the Landau-Lifshitz equation, satisfying the following statements.
(i) $m \in L^\infty([0,T], \mathcal{E}^k(\mathbb{R}^N))$ and $\partial_t m \in L^\infty([0,T], H^{k-2}(\mathbb{R}^N))$, $\forall T \in]0, T_{\max}[$.
(ii) If $T_{\max} < \infty$, then $\int_0^{T_{\max}} \|\nabla m(\cdot, t)\|_{L^\infty}^2 \, dt = \infty$.
(iii) The flow map $m^0 \mapsto m$ is well-defined and locally Lipschitz continuous
(iv) If $m^0 \in \mathcal{E}^\ell(\mathbb{R}^N)$, $\ell > k$, then $m \in L^\infty([0,T], \mathcal{E}^\ell(\mathbb{R}^N))$.
(v) The Landau-Lifshitz energy is conserved along the flow.

Key element of the proof

Proposition

Let $\lambda_1, \lambda_3 \geq 0$, $k \in \mathbb{N}$, with $k > N/2+1$. Consider two (smooth) solutions m and $\tilde{m}$ to the LL equation. We set

$$u := \tilde{m} - m, \quad v := (\tilde{m}+m)/2, \quad \mathcal{S}^{\ell}_{\mathrm{LL}} = \sum_{j=0}^{\ell} \mathcal{E}^{j}_{\mathrm{LL}}$$

$$\mathcal{E}^0_{\mathrm{LL}}(t) = \int_{\mathbb{R}^N} \left| u(x,t) - u_2^0(x)\, e_2 \right|^2 dx,$$

$$\mathcal{E}^1_{\mathrm{LL}}(t) = \int_{\mathbb{R}^N} \left(|\nabla u|^2 + |u \times \nabla v + v \times \nabla u|^2 \right)(x,t)\, dx,$$

$$\mathcal{E}^{\ell}_{\mathrm{LL}}(t) = \int_{\mathbb{R}^N} \Big(|\partial_t D^{\ell-2} u|^2 + |D^{\ell} u|^2 + (\lambda_1+\lambda_3)\big(|D^{\ell-1}u_1|^2 + |D^{\ell-1}u_3|^2\big) + \lambda_1\lambda_3\big(|D^{\ell-2}u_1|^2 + |D^{\ell-2}u_3|^2\big)\Big)$$

Then $$\mathcal{E}'_{\mathrm{LL}^{\ell}}(t) \leq C(m,\tilde{m})\big(\mathcal{S}^{\ell}_{\mathrm{LL}}(t) + \|u(t)\|^2_{L^\infty}\big),$$

with $C(m,\tilde{m}) \sim 1 + \|\nabla m(\cdot,t)\|^2_{H^{\ell}} + \|\nabla \tilde{m}(\cdot,t)\|^2_{H^{\ell}} + \|\nabla m(\cdot,t)\|^2_{L^\infty} + \|\nabla \tilde{m}(\cdot,t)\|^2_{L^\infty}$

Functional setting

In order to use of the hydrodynamical framework, we need $|m_3| < 1$, i.e.

$$|u| < 1 \quad \text{on } \mathbb{R}^N.$$

Under this condition, the Landau-Lifshitz energy in the hydrodynamical formulation is

$$E_{\mathrm{LL}}(u, \varphi) := \frac{1}{2} \int_{\mathbb{R}^N} \Big(\frac{|\nabla u|^2}{1 - u^2} + (1 - u^2)|\nabla \varphi|^2 + \lambda_1 (1 - u^2) \sin^2(\varphi) + \lambda_3 u^2 \Big).$$

Hence it is natural to use the non-vanishing set

$$\mathcal{NV}(\mathbb{R}^N) := \big\{ (u, \varphi) \in H^1(\mathbb{R}^N) \times H^1_{\sin}(\mathbb{R}^N) : |u| < 1 \text{ on } \mathbb{R}^N \big\},$$

where $H^1_{\sin}(\mathbb{R}^N)$ is the additive group

$$H^1_{\sin}(\mathbb{R}^N) := \big\{ v \in L^1_{\mathrm{loc}}(\mathbb{R}^N) : \nabla v \in L^2(\mathbb{R}^N) \text{ and } \sin(v) \in L^2(\mathbb{R}^N) \big\}.$$

For $k \geq 1$

$$\mathcal{NV}^k(\mathbb{R}^N) := \big\{ (u, \varphi) \in H^k(\mathbb{R}^N) \times H^k_{\sin}(\mathbb{R}^N) : |u| < 1 \text{ on } \mathbb{R}^N \big\}.$$

$$H^k_{\sin}(\mathbb{R}^N) := \big\{ v \in L^1_{\mathrm{loc}}(\mathbb{R}^N) : \nabla v \in H^{k-1}(\mathbb{R}^N) \text{ and } \sin(v) \in L^2(\mathbb{R}^N) \big\}.$$

We can establish a local well-posedness theorem for the (HLL) in this setting.

Main theorem

Our main result provides a quantified convergence of the Landau–Lifshitz equation towards the Sine–Gordon equation in the regime of strong easy-plane anisotropy.

Theorem

Let $N \geq 1$ *and* $k \in \mathbb{N}$, *with* $k > N/2 + 1$, *and* $0 < \varepsilon < 1$. *Consider an initial condition* $(U_\varepsilon^0, \Phi_\varepsilon^0) \in \mathcal{NV}^{k+2}(\mathbb{R}^N)$ *and set*

$$\mathcal{K}_\varepsilon := \big\|U_\varepsilon^0\big\|_{H^k} + \varepsilon\big\|\nabla U_\varepsilon^0\big\|_{H^k} + \big\|\nabla \Phi_\varepsilon^0\big\|_{H^k} + \big\|\sin(\Phi_\varepsilon^0)\big\|_{H^k}.$$

Consider an initial condition $(U^0, \Phi^0) \in L^2(\mathbb{R}^N) \times H^1_{\sin}(\mathbb{R}^N)$, *and denote by* $(U, \Phi) \in \mathcal{C}^0(\mathbb{R}, L^2(\mathbb{R}^N) \times H^1_{\sin}(\mathbb{R}^N))$ *the unique corresponding solution to* (SGS). *Then, there exists* $C > 0$, *depending only on* σ, k *and* N, *s.t., if the initial data satisfies the condition*

$$\varepsilon \mathcal{K}_\varepsilon \leq \frac{1}{C},$$

then the following statements hold.

(i) *There exists*

$$T_\varepsilon \geq \frac{1}{C\mathcal{K}_\varepsilon^2},$$

s.t. there exists a unique solution $(U_\varepsilon, \Phi_\varepsilon) \in \mathcal{C}^0([0, T_\varepsilon], \mathcal{NV}^{k+1}(\mathbb{R}^N))$ *to (HLL) with initial data* $(U_\varepsilon^0, \Phi_\varepsilon^0)$.

Theorem

(continued)

$$T_\varepsilon \geq \frac{1}{C\mathcal{K}_\varepsilon^2}, \quad \mathcal{K}_\varepsilon := \|U_\varepsilon^0\|_{H^k} + \varepsilon\|\nabla U_\varepsilon^0\|_{H^k} + \|\nabla \Phi_\varepsilon^0\|_{H^k} + \|\sin(\Phi_\varepsilon^0)\|_{H^k}.$$

(ii) L^2 *estimate. If* $\Phi_\varepsilon^0 - \Phi^0 \in L^2(\mathbb{R}^N)$, *then*

$$\|\Phi_\varepsilon(t) - \Phi(t)\|_{L^2} \leq C\left(\|\Phi_\varepsilon^0 - \Phi^0\|_{L^2} + \|U_\varepsilon^0 - U^0\|_{L^2} + \varepsilon^2\mathcal{K}_\varepsilon\left(1+\mathcal{K}_\varepsilon\right)^3\right)e^{Ct},$$

for $0 \leq t \leq T_\varepsilon$.

(iii) $L^2 \times H^1_{\sin}$ *estimate. If* $N \geq 2$, *or* $N = 1$ *and* $k > N/2 + 2$, *then*

$$\|U_\varepsilon(t) - U(t)\|_{L^2} + \|\nabla\Phi_\varepsilon(t) - \nabla\Phi(t)\|_{L^2} + \|\sin(\Phi_\varepsilon(t) - \Phi(t))\|_{L^2}$$
$$\leq C\left(\|U_\varepsilon^0 - U^0\|_{L^2} + \|\nabla\Phi_\varepsilon^0 - \nabla\Phi^0\|_{L^2} + \|\sin(\Phi_\varepsilon^0 - \Phi^0)\|_{L^2} + \varepsilon^2\mathcal{K}_\varepsilon\left(1+\mathcal{K}_\varepsilon\right)^3\right)e^{Ct},$$

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$$T_\varepsilon \geq \frac{1}{C\mathcal{K}_\varepsilon^2}, \quad \mathcal{K}_\varepsilon := \big\|U_\varepsilon^0\big\|_{H^k} + \varepsilon\big\|\nabla U_\varepsilon^0\big\|_{H^k} + \big\|\nabla\Phi_\varepsilon^0\big\|_{H^k} + \big\|\sin(\Phi_\varepsilon^0)\big\|_{H^k}.$$

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(iii) $L^2 \times H^1_{\sin}$ *estimate. If* $N \geq 2$, *or* $N = 1$ *and* $k > N/2 + 2$, *then*

$$\begin{aligned}&\big\|U_\varepsilon(t) - U(t)\big\|_{L^2} + \big\|\nabla\Phi_\varepsilon(t) - \nabla\Phi(t)\big\|_{L^2} + \big\|\sin(\Phi_\varepsilon(t) - \Phi(t))\big\|_{L^2}\\ &\leq C\left(\big\|U_\varepsilon^0 - U^0\big\|_{L^2} + \big\|\nabla\Phi_\varepsilon^0 - \nabla\Phi^0\big\|_{L^2} + \big\|\sin(\Phi_\varepsilon^0 - \Phi^0)\big\|_{L^2} + \varepsilon^2\mathcal{K}_\varepsilon\left(1+\mathcal{K}_\varepsilon\right)^3\right)e^{Ct},\end{aligned}$$

for $0 \leq t \leq T_\varepsilon$.

Theorem

(continued)

(iv) *Higher order estimates. Take* $(U^0, \Phi^0) \in H^k(\mathbb{R}^N) \times H^{k+1}_{\sin}(\mathbb{R}^N)$ *and set*

$$\kappa_\varepsilon := \mathcal{K}_\varepsilon + \big\|U^0\big\|_{H^k} + \big\|\nabla\Phi^0\big\|_{H^k} + \big\|\sin(\Phi^0)\big\|_{H^k}.$$

There exists $A > 0$*, depending only on* σ*,* k *and* N*, s.t. the solution* (U, Φ) *lies in* $\mathcal{C}^0([0, T_\varepsilon^*], H^k(\mathbb{R}^N) \times H^{k+1}_{\sin}(\mathbb{R}^N))$*, with*

$$T_\varepsilon \geq T_\varepsilon^* \geq \frac{1}{A\kappa_\varepsilon^2}.$$

Moreover, when $k > N/2 + 3$*,*

$$\begin{aligned}
&\big\|U_\varepsilon(t) - U(t)\big\|_{H^{k-3}} + \big\|\nabla\Phi_\varepsilon(t) - \nabla\Phi(t)\big\|_{H^{k-3}} + \big\|\sin(\Phi_\varepsilon(t) - \Phi(t))\big\|_{H^{k-3}} \\
&\leq A\, e^{A(1+\kappa_\varepsilon)^2 t} \times \\
&\times \Big(\big\|U_\varepsilon - U^0\big\|_{H^{k-3}} + \big\|\nabla\Phi_\varepsilon^0 - \nabla\Phi^0\big\|_{H^{k-3}} + \big\|\sin(\Phi_\varepsilon^0 - \Phi^0)\big\|_{H^{k-3}} + \varepsilon^2\kappa_\varepsilon\big(1+\kappa_\varepsilon\big)^3\Big),
\end{aligned}$$

for $0 \leq t \leq T_\varepsilon^*$*.*

Idea of proof

To obtain good “energy” estimates.

- Shatah and Zeng (wave regime of the LL equation),
- Béthuel, Danchin and Smets (wave regime of the the Gross-Pitaevskii equation)
- Béthuel, Gravejat, Saut, Smets (KdV regime of the the Gross-Pitaevskii equation)
- Chiron (mKdV regime of the planar Landau-Lifshitz equation)

Difficulty: to find the good symmetrization or “energy”.
The LL energy in the scaled hydrodynamical variables is

$$E = \frac{1}{2}\int_{\mathbb{R}^N}\Big(\varepsilon^2\frac{|\nabla U_\varepsilon|^2}{1-\varepsilon^2 U_\varepsilon^2} + U_\varepsilon^2 + (1-\varepsilon^2 U_\varepsilon^2)|\nabla\Phi_\varepsilon|^2 + \sigma(1-\varepsilon^2 U_\varepsilon^2)\sin^2(\Phi_\varepsilon)\Big).$$

Hence, a possible choice of an energy of order $k \in \mathbb{N}^*$ is

$$E_k = \frac{1}{2}\int_{\mathbb{R}^N}\Big(\varepsilon^2\frac{|D^k U_\varepsilon|^2}{1-\varepsilon^2 U_\varepsilon^2} + |D^{k-1}U_\varepsilon|^2 + (1-\varepsilon^2 U_\varepsilon^2)|D^k\Phi_\varepsilon|^2 + \sigma(1-\varepsilon^2 U_\varepsilon^2)|D^{k-1}\sin(\Phi_\varepsilon)|^2\Big).$$

The factors $1-\varepsilon^2 U_\varepsilon^2$ and term corresponding to $\sin(\Phi_\varepsilon)$ provide a better symmetrization of the energy estimates by inducing cancellations in the higher order terms.

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The factors $1 - \varepsilon^2 U_\varepsilon^2$ and term corresponding to $\sin(\Phi_\varepsilon)$ provide a better symmetrization of the energy estimates by inducing cancellations in the higher order terms.

Set

$$E_k = \frac{1}{2}\int_{\mathbb{R}^N}\Big(\varepsilon^2\frac{|D^kU_\varepsilon|^2}{1-\varepsilon^2U_\varepsilon^2} + |D^{k-1}U_\varepsilon|^2 + (1-\varepsilon^2U_\varepsilon^2)|D^k\Phi_\varepsilon|^2 + \sigma(1-\varepsilon^2U_\varepsilon^2)|D^{k-1}\sin(\Phi_\varepsilon)|^2\Big)$$

$$\Sigma_k = \sum_{j=1}^k E_j.$$

Proposition

Let $\varepsilon > 0$ and $k \in \mathbb{N}$, with $k > N/2 + 2$. Consider a solution $(U_\varepsilon, \Phi_\varepsilon)$ to (HLL), with $(U_\varepsilon, \Phi_\varepsilon) \in \mathcal{C}^0([0, T], \mathcal{N}\mathcal{V}^{k+3}(\mathbb{R}^N))$ for a fixed $T > 0$, and assume that

$$\inf_{\mathbb{R}^N\times[0,T]} 1-\varepsilon^2U_\varepsilon^2 \geq \frac{1}{2}.$$

There exists $C > 0$, depending only on σ, k and N, s.t.

$$\begin{aligned}\frac{d}{dt}E_k(t) \leq C\big(1+\varepsilon^4\big)\Big(&\|\sin(\Phi_\varepsilon(t))\|_{L^\infty}^2 + \|U_\varepsilon(t)\|_{L^\infty}^2 + \|\nabla\Phi_\varepsilon(t)\|_{L^\infty}^2\\ &+ \|\nabla U_\varepsilon(t)\|_{L^\infty}^2 + \|d^2\Phi_\varepsilon(t)\|_{L^\infty}^2 + \varepsilon^2\|D^2U_\varepsilon(t)\|_{L^\infty}^2\\ &+ \varepsilon\,\|\nabla\Phi_\varepsilon(t)\|_{L^\infty}\big(\|\nabla\Phi_\varepsilon(t)\|_{L^\infty}^2 + \|\nabla U_\varepsilon(t)\|_{L^\infty}^2\big)\Big)\,\Sigma_k(t),\end{aligned}$$

for any $t \in [0, T]$.

Proof.

Setting $\rho_\varepsilon := 1 - \varepsilon^2 U_\varepsilon^2$, (HLL) recasts as

$$\begin{cases} \partial_t U_\varepsilon = \rho_\varepsilon \, \Delta\Phi_\varepsilon + \nabla\rho_\varepsilon \cdot \nabla\Phi_\varepsilon - \frac{\sigma}{2}\rho_\varepsilon \, \sin(2\Phi_\varepsilon), \\ \partial_t \Phi_\varepsilon = U_\varepsilon - \frac{\varepsilon^2}{\rho_\varepsilon}\,\Delta U_\varepsilon - \frac{\varepsilon^2}{2}\nabla U_\varepsilon \cdot \nabla\big(\frac{1}{\rho_\varepsilon}\big) - \varepsilon^2 U_\varepsilon \, |\nabla\Phi_\varepsilon|^2 - \sigma\varepsilon^2 U_\varepsilon \sin^2(\Phi_\varepsilon), \end{cases}$$

Let $\ell \leq k$, then

$$\frac{d}{dt}E_\ell(t) = \sum_{j=1}^{5} \mathcal{I}_j(t),$$

$$\mathcal{I}_1 = \int D^{\ell-1} U_\varepsilon D^{\ell-1}\Big(\rho_\varepsilon \, \Delta\Phi_\varepsilon + \nabla\rho_\varepsilon \cdot \nabla\Phi_\varepsilon - \frac{\sigma}{2}\rho_\varepsilon \, \sin(2\Phi_\varepsilon)\Big),$$

$$\mathcal{I}_2 = \int \rho_\varepsilon D^\ell \Phi_\varepsilon D^\ell\Big(U_\varepsilon - \frac{\varepsilon^2}{\rho_\varepsilon}\,\Delta U_\varepsilon - \frac{\varepsilon^2}{2}\nabla U_\varepsilon \cdot \nabla\Big(\frac{1}{\rho_\varepsilon}\Big) - \varepsilon^2 U_\varepsilon \, |\nabla\Phi_\varepsilon|^2 - \sigma\varepsilon^2 U_\varepsilon \, \sin^2(\Phi_\varepsilon)\Big),$$

$$\mathcal{I}_3 = \int \frac{\varepsilon^2}{\rho_\varepsilon} D^\ell U_\varepsilon D^\ell\Big(\rho_\varepsilon \, \Delta\Phi_\varepsilon + \nabla\rho_\varepsilon \cdot \nabla\Phi_\varepsilon - \frac{\sigma}{2}\rho_\varepsilon \, \sin(2\Phi_\varepsilon)\Big),$$

$$\mathcal{I}_4 = \int \rho_\varepsilon D^{\ell-1}\big(\sin(\Phi_\varepsilon)\big) D^{\ell-1}\Big(\cos(\Phi_\varepsilon)\,\Big(U_\varepsilon - \frac{\varepsilon^2}{\rho_\varepsilon}\,\Delta U_\varepsilon - \frac{\varepsilon^2}{2}\nabla U_\varepsilon \cdot \nabla\Big(\frac{1}{\rho_\varepsilon}\Big) - \dots$$

$$\mathcal{I}_5 = -\,\varepsilon^2 \int U_\varepsilon \, (\partial_t U_\varepsilon)\,\Big(|\nabla\partial_x^\alpha \Phi_\varepsilon|^2 - \frac{\varepsilon^2}{\rho_\varepsilon^2}|D^\ell U_\varepsilon|^2 + \sigma|D^{\ell-1}\sin(\Phi_\varepsilon)|^2\Big).$$

Moser's tame estimates

Lemma

(i) Let $f, g \in L^\infty(\mathbb{R}^N) \cap \dot{H}^m(\mathbb{R}^N)$*. The product* fg *is in* $\dot{H}^m(\mathbb{R}^N)$*, and there exists* C_m*, depending only on* m*, such that*

$$\|f\,g\|_{\dot{H}^m} \leq C_m\big(\|f\|_{L^\infty}\,\|g\|_{\dot{H}^m} + \|f\|_{\dot{H}^m}\,\|g\|_{L^\infty}\big).$$

(ii) Let $m \in \mathbb{N}^*$*. When* $f \in L^\infty(\mathbb{R}^N) \cap \dot{H}^m(\mathbb{R}^N)$ *and* $F \in \mathcal{C}^m(\mathbb{R}^N)$*, the composition function* $F(f)$ *is in* $\dot{H}^m(\mathbb{R}^N)$*, and there is* C_m*, depending only on* m*, such that*

$$\|F(f)\|_{\dot{H}^m} \leq C_m \max_{1\leq \ell \leq m}\big(\|F^{(\ell)}\|_{L^\infty}\,\|f\|_{L^\infty}^{\ell-1}\big)\,\|f\|_{\dot{H}^m}.$$

Application: since $\rho_\varepsilon := 1 - \varepsilon^2 U_\varepsilon^2$, with $1/2 \leq \rho_\varepsilon \leq 1$ by using

$$F(x) = 1 - x^2 \text{ and } G(x) = \frac{1}{1-x^2}, \qquad \text{for } x \in [0, 1/\sqrt{2}],$$

we deduce that

$$\|\rho_\varepsilon\|_{\dot{H}^m} + \|1/\rho_\varepsilon\|_{\dot{H}^m} \leq C\varepsilon^2\|U_\varepsilon\|_{L^\infty}\,\|U_\varepsilon\|_{\dot{H}^m}.$$

The term $\sin(\Phi_\varepsilon)$ is more delicate to handle. By Moser's estimate:

$$\|\sin(f)\|_{\dot{H}^m} \leq C(1+\|f\|_{L^\infty}^{m-1})\|f\|_{\dot{H}^m},$$

but we cannot use $\|\Phi_\varepsilon\|_{L^\infty}$.

Indeed, if $\phi \in H^1_{\sin}(\mathbb{R})$, the function ϕ is uniformly continuous and bounded on $\mathbb{R}$, and there exist $\ell^+, \ell^- \in \mathbb{Z}$ s.t.

$$\phi(x) \to \ell^\pm \pi, \quad \text{as } x \to \pm\infty.$$

Moreover, the differences $\phi - \ell^\pm\pi \in L^2(\mathbb{R}_\pm)$.

However, there are no positive numbers $A_\pm$ such that

$$\forall \phi \in H^1_{\sin}(\mathbb{R}), \quad \|\phi - \ell^\pm\pi\|_{L^\infty(\mathbb{R}_\pm)} \leq A_\pm \|\phi\|_{H^1_{\sin}(\mathbb{R}_\pm)}.$$

Another possible Moser type estimate:

$$\|\sin(f)\|_{\dot{H}^m} + \|\cos(f)\|_{\dot{H}^m} \leq C_m(1+\|\nabla f\|_{L^\infty}^{m-1})\|\nabla f\|_{H^{m-1}}.$$

We will keep $\|\sin(f)\|_{\dot{H}^m}$ and use the estimate

$$\|\cos(f)\|_{\dot{H}^m} \leq C_m\big(\|\sin(f)\|_{L^\infty} + \|\nabla f\|_{L^\infty}\big)\big(\|\sin(f)\|_{\dot{H}^{m-1}} + \|f\|_{\dot{H}^m}\big).$$

After 8 pages of computations, applying plenty of integration by parts, Moser estimates and Leibniz rule, there is a cancellation between higher order derivatives and we can bound the remainder terms.

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After 8 pages of computations, applying plenty of integration by parts, Moser estimates and Leibniz rule, there is a cancellation between higher order derivatives and we can bound the remainder terms.

Corollary

Let $\varepsilon > 0$, and $k \in \mathbb{N}$, with $k > N/2 + 2$. There exists $C > 0$, depending only on σ, k and N, such that if an initial data $(U_\varepsilon^0, \Phi_\varepsilon^0) \in \mathcal{NV}^{k+2}(\mathbb{R}^N)$ satisfies

$$C\varepsilon\Big(\big\|U_\varepsilon^0\big\|_{H^k} + \varepsilon\big\|\nabla U_\varepsilon^0\big\|_{H^k} + \big\|\nabla\Phi_\varepsilon^0\big\|_{H^k} + \big\|\sin(\Phi_\varepsilon^0)\big\|_{H^k}\Big) \leq 1,$$

then there exists a positive time

$$T_\varepsilon \geq \frac{1}{C_*\big(\|U_\varepsilon^0\|_{H^k} + \varepsilon\|\nabla U_\varepsilon^0\|_{H^k} + \|\nabla\Phi_\varepsilon^0\|_{H^k} + \|\sin(\Phi_\varepsilon^0)\|_{H^k}\big)^2},$$

s.t the unique solution $(U_\varepsilon, \Phi_\varepsilon)$ to (HLL) with initial condition $(U_\varepsilon^0, \Phi_\varepsilon^0)$ satisfies the uniform bound

$$\varepsilon\big\|U_\varepsilon(t)\big\|_{L^\infty} \leq \frac{1}{\sqrt{2}},$$

as well as the energy estimate

$$\begin{aligned}\big\|U_\varepsilon(t)\big\|_{H^k} + \varepsilon\big\|\nabla U_\varepsilon(t)\big\|_{H^k} + \big\|\nabla\Phi_\varepsilon(t)\big\|_{H^k} + \big\|\sin(\Phi_\varepsilon(t))\big\|_{H^k} \\ \leq C_*\Big(\big\|U_\varepsilon^0\big\|_{H^k} + \varepsilon\big\|\nabla U_\varepsilon^0\big\|_{H^k} + \big\|\nabla\Phi_\varepsilon^0\big\|_{H^k} + \big\|\sin(\Phi_\varepsilon^0)\big\|_{H^k}\Big),\end{aligned}$$

for any $0 \leq t \leq T_\varepsilon$.

We suppose first that initial condition is regular enough. By the Sobolev embedding theorem, there is K_1 s.t.

$$\varepsilon\left\|U_\varepsilon^0\right\|_{L^\infty} \leq \varepsilon K_1\left\|U_\varepsilon^0\right\|_{H^k} \leq \frac{K_1}{C} < \frac{1}{\sqrt{2}}, \quad \textit{when } C > \sqrt{2}K_1.$$

From the continuity of the solution, the following stopping time is positive:

$$T_* := \sup\left\{t \in [0, T_{\max}) : \frac{1}{2} \leq \inf_{x\in\mathbb{R}^N} \rho_\varepsilon(x,\tau) \text{ and } \Sigma_k(\tau) \leq 2\Sigma_k(0) \text{ for any } \tau \in [0,t]\right\}.$$

Since $k > N/2 + 2$, there is K_2 s.t.

$$\begin{aligned} &\|\sin(\Phi_\varepsilon(t))\|_{L^\infty}^2 + \|U_\varepsilon(t)\|_{L^\infty}^2 + \|\nabla\Phi_\varepsilon(t)\|_{L^\infty}^2 \\ &\qquad + \|\nabla U_\varepsilon(t)\|_{L^\infty}^2 + \|D^2\Phi_\varepsilon(t)\|_{L^\infty}^2 + \varepsilon^2\|D^2 U_\varepsilon(t)\|_{L^\infty}^2 \leq K_3\Sigma_k(t). \end{aligned} \tag{1}$$

Therefore

$$\Sigma_k'(t) \leq K\left(\Sigma_k(t)^2 + \varepsilon\Sigma_k(t)^{\frac{5}{2}}\right), \quad 0 \leq t < T_*$$

so we can simplify this inequality as

$$\Sigma_k'(t) \leq K\left(1 + \varepsilon\sqrt{\Sigma_k(0)}\right)\Sigma_k(t)^2,$$

and using the hypothesis on the bound on the initial condition:

$$\Sigma_k'(t) \leq K\,\Sigma_k(t)^2.$$

At this stage, we set

$$T_\varepsilon := \frac{1}{4K\Sigma_k(0)},$$

and we deduce from the previous differential inequality that

$$\Sigma_k(t) \leq \frac{\Sigma_k(0)}{1 - 2K\Sigma_k(0)t} \leq 2\Sigma_k(0), \quad \text{for } t \leq T_\varepsilon, t < T_*.$$

In particular

$$\varepsilon \| U_\varepsilon(t) \|_{L^\infty} \leq \varepsilon K_3^{\frac{1}{2}} \Sigma_k(0)^{\frac{1}{2}} < \frac{1}{\sqrt{2}},$$

so that $\inf_{x \in \mathbb{R}^N} \rho_\varepsilon(x,t) \geq 1/2$, for $t \leq T_\varepsilon$, $t < T_*$.

Finally, we derive as before from the Sobolev embedding theorem that

$$\int_0^t \left(\varepsilon^2 \left\| \frac{\nabla U_\varepsilon(s)}{\rho_\varepsilon(s)^{\frac{1}{2}}} \right\|_{L^\infty}^2 + \left\| \rho_\varepsilon(s)^{\frac{1}{2}} \nabla \Phi_\varepsilon(s) \right\|_{L^\infty}^2 \right) ds \leq K_6 \int_0^t \Sigma_k(s)\, ds \leq K < \infty.$$

In view of the characterization for the maximal time $T_{\max}$, this guarantees that the stopping time T_* is at least equal to T_ε.

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Wave regime

Consider

$$\lambda_1 = \sigma\varepsilon, \quad \text{and} \quad \lambda_3 := \frac{1}{\varepsilon},$$

with

$$\sigma, \varepsilon \to 0$$

From the equation

$$\begin{cases} \partial_t U_\varepsilon = \operatorname{div}\left((1-\varepsilon^2 U_\varepsilon^2)\nabla\Phi_\varepsilon\right) - \frac{\sigma}{2}(1-\varepsilon^2 U_\varepsilon^2)\sin(2\Phi_\varepsilon), \\ \partial_t \Phi_\varepsilon = U_\varepsilon\left(1-\varepsilon^2\sigma\sin^2(\Phi_\varepsilon)\right) - \varepsilon^2 \operatorname{div}\left(\frac{\nabla U_\varepsilon}{1-\varepsilon^2 U_\varepsilon^2}\right) + \varepsilon^4 U_\varepsilon \frac{|\nabla U_\varepsilon|^2}{(1-\varepsilon^2 U_\varepsilon^2)^2} - \varepsilon^2 U_\varepsilon |\nabla\Phi_\varepsilon|^2 \end{cases}$$

we formally get the free wave system

$$\begin{cases} \partial_t U = \Delta\Phi, \\ \partial_t \Phi = U, \end{cases}$$

i.e. Φ is solution to the free wave equation

$$\partial_{tt}\Phi - \Delta\Phi = 0.$$

Similar arguments allow to establish this quantified convergence.

Work in progress: NLS regime

Consider

$$\lambda_1 = \frac{1}{\varepsilon}, \quad \text{and} \quad \lambda_3 := \frac{1}{\varepsilon},$$

with $\varepsilon \to 0$. Then, after a good change of variables (also known in the physical literature), we have *formally* convergence to

$$i\partial\psi + \Delta\psi + \psi|\psi|^2 = 0.$$

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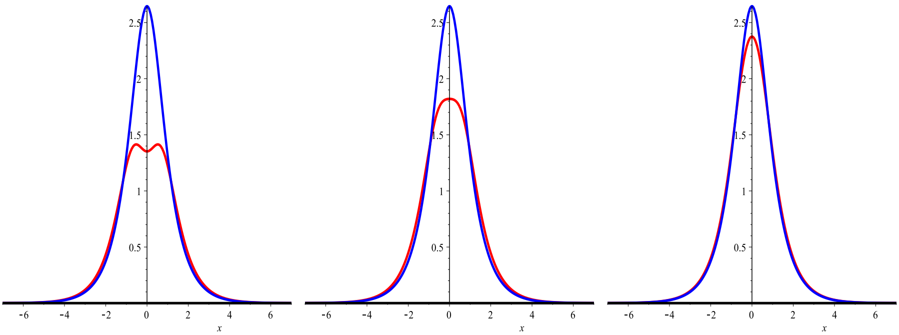
A good test: solitons

$\varepsilon = 0.5$ $\varepsilon = 0.3$ $\varepsilon = 0.1$



Thank you for your attention!