Category theory as a foundation for algorithms and programming in computer algebra

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Outline

1. Categories, Algorithms, and Programming

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1. Categories, Algorithms, and Programming
2. Generalized Morphisms

Section 1

Categories, Algorithms, and Programming

Spectral Sequence Algorithm

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Axioms of on Abelian Category

Needed operations: $\circ, +, -$

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its embedding into the object M,

$$\begin{tikzcd}
\ker\varphi \arrow[dr, hook, "\kappa"] & & \\
& M \arrow[r, "\varphi"] & N
\end{tikzcd}$$

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... one has to construct the object $\ker\varphi$,
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and for every test object T

$$
\begin{tikzcd}
\ker\varphi \arrow[dr, hook, "\kappa"] \arrow[drr, bend left, "0"] & & \\
& M \arrow[r, "\varphi"] & N \\
T \arrow[ur, "\tau"] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}
$$

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\end{tikzcd}
$$

Thus a proper implementation of the kernel needs **three** algorithms.

CAP

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$\rightsquigarrow$ CAP

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- CAP serves as a categorical programming language with categorical operations as primitives.

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- CAP has an interface for the interpretation of these categorical operations.

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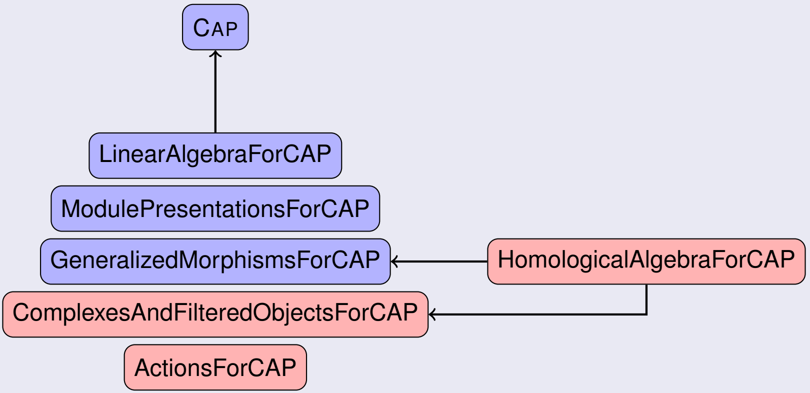
CAP means **Categories, algorithms, and programming** and is a software project implemented in GAP (Groups, algorithms, and programming) (joint work with Sebastian Gutsche, based on ideas of the `homalg` project (Mohamed Barakat).

- CAP serves as a categorical programming language with categorical operations as primitives.
- CAP has an interface for the interpretation of these categorical operations.

Using CAP we are now able to implement the categorical data structures and algorithms which we need.

CAP Packages



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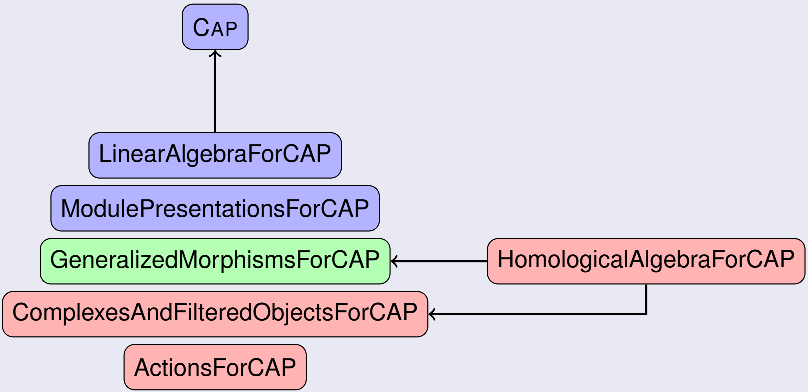

Section 2

Generalized Morphisms

Connecting Homomorphism in the Snake Lemma

$$\begin{tikzcd}
 & & & \ker(\gamma) \arrow[d] & \\
 & A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r, "\epsilon"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "\mu"'] \arrow[d] & B' \arrow[r] & C' & \\
 & \mathrm{coker}(\alpha) & & &
\end{tikzcd}$$

Wanted: $\ker(\gamma) \xrightarrow{\partial} \mathrm{coker}(\alpha)$.

Connecting Homomorphism in the Snake Lemma

$$\begin{tikzcd}
 & & & {\color{blue}c} \in \ker(\gamma) \arrow[d] & \\
 & A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r, "\epsilon"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\
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Start: $c \in \ker(\gamma)$.

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This lies in C.

Connecting Homomorphism in the Snake Lemma

$$
\begin{tikzcd}
 & & & c \in \ker(\gamma) \arrow[d] & \\
 & A \arrow[r] \arrow[d, "\alpha"'] & \textcolor{red}{b} \in B \arrow[r, "\epsilon"] \arrow[d, "\beta"'] & c \in C \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "\mu"'] \arrow[d] & B' \arrow[r] & C' & \\
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$$

Choose: $b \in \epsilon^{-1}(\{c\})$.

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\end{tikzcd}$$

Map: $b \overset{\beta}{\mapsto} b'$.

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\end{tikzcd}
$$

Compute: $a' \in \mu^{-1}(b')$.

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& {\color{blue} a' + \mathrm{im}(\alpha)} \in \mathrm{coker}(\alpha) & & &
\end{tikzcd}$$

Map: $a' \mapsto a' + \mathrm{im}(\alpha)$.

Connecting Homomorphism in the Snake Lemma

$$\begin{tikzcd}
 & & & c \in \ker(\gamma) \arrow[d] \arrow[dddll, rounded corners] & \\
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Result: $c \overset{\partial}{\mapsto} a' + \mathrm{im}(\alpha)$.

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Result: $c \overset{\partial}{\mapsto} a' + \mathrm{im}(\alpha)$. Independent of the **choice**.

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Idea: Use **relations** instead of maps. $c \mapsto \epsilon^{-1}(\{b\})$

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is a relation from C to B, called **pseudo-inverse**.

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Composition of Relations

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If f and g correspond to maps, this describes their usual composition.

Snake Lemma Revisited

$$\begin{tikzcd}
 & & & \ker(\gamma) \arrow[d, "\iota"] & \\
 & A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r, "\epsilon"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "\mu"'] \arrow[d, "\pi"'] & B' \arrow[r] & C' & \\
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ι

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0 \arrow[r] & A' \arrow[r, "\mu"'] \arrow[d, "\pi"'] & B' \arrow[r] & C' & \\
 & \mathrm{coker}(\alpha) & & &
\end{tikzcd}
$$

$$\epsilon^{-1} \circ \iota$$

Snake Lemma Revisited

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& & & \mathrm{ker}(\gamma) \arrow[d, "\iota"] & \\
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$$\beta \circ \epsilon^{-1} \circ \iota$$

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0 \arrow[r] & A' \arrow[r, "\mu"'] \arrow[d, "\pi"'] & B' \arrow[r] \arrow[l, bend left, "\mu^{-1}"] & C' & \\
 & \operatorname{coker}(\alpha) & & &
\end{tikzcd}$$

$$\mu^{-1} \circ \beta \circ \epsilon^{-1} \circ \iota$$

Snake Lemma Revisited

$$
\begin{tikzcd}
 & & & \ker(\gamma) \arrow[d, blue, "\iota"] & \\
 & A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r, "\epsilon"] \arrow[d, blue, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] \arrow[l, red, bend right, "\epsilon^{-1}"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "\mu"'] \arrow[d, blue, "\pi"'] & B' \arrow[r] \arrow[l, red, bend left, "\mu^{-1}"] & C' & \\
 & \operatorname{coker}(\alpha) & & &
\end{tikzcd}
$$

$$\pi \circ \mu^{-1} \circ \beta \circ \epsilon^{-1} \circ \iota$$

Snake Lemma Revisited

$$\begin{tikzcd}
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\end{tikzcd}$$

∂ is a composition of relations!

From Relations to Generalized Morphisms

- **Wanted**: A categorical framework for relations.

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- **Solution**: Generalized Morphisms.

From Relations to Generalized Morphisms

Let A, B be objects in an abelian category $\mathbf{A}$.

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Relation

$$\begin{tikzcd} A \oplus B \\ D \arrow[u, hook] \end{tikzcd}$$

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$$\begin{tikzcd} A & & B \\ & D \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

From Relations to Generalized Morphisms

Let A, B be objects in an abelian category **A**.

Relation $\rightsquigarrow$ Generalized Morphism

$$
\begin{tikzcd}
A \arrow[rr, dashed] & & B \\
& D \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] &
\end{tikzcd}
$$

Composition of Generalized Morphisms

Composition

$$
\begin{tikzcd}
A \arrow[rr, dashed] & & B \\
& D \arrow[ul] \arrow[ur] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
B \arrow[rr, dashed] & & C \\
& E \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

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Composition

$$\begin{tikzcd}
A & & B & & C \\
& D \arrow[ul] \arrow[ur] & & E \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Composition of Generalized Morphisms

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$$
\begin{tikzcd}
A & & B & & C \\
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& & D \times_B E \arrow[ul] \arrow[ur] & &
\end{tikzcd}
$$

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$$\begin{tikzcd}
A & & B & & C \\
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\end{tikzcd}$$

Composition of Generalized Morphisms

Composition

$$\begin{tikzcd}
A \arrow[rr, dashed, bend left] & & B & & C \\
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& & D \times_B E \arrow[ul] \arrow[ur] & &
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\end{tikzcd}
$$

Pullbacks $\rightsquigarrow$ Composition of generalized morphisms

Pseudo-Inverses

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$$\begin{tikzcd} A \arrow[rr, dashed] & & B \\ & D \arrow[ul] \arrow[ur] & \end{tikzcd}$$

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$$\begin{tikzcd}
A \arrow[rr, dashed] & & B \\
& D \arrow[ul, red] \arrow[ur, blue] \arrow[u, phantom, "\leftrightarrow"] &
\end{tikzcd}$$

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$$\begin{tikzcd} A \arrow[rr, dashed] & & B \\ & D \arrow[ul, red] \arrow[ur, blue] \arrow[u, phantom, "\leftrightarrow"] & \end{tikzcd} \qquad \rightsquigarrow$$

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$$
\begin{tikzcd}
A \arrow[rr, dashed] & & B \\
& D \arrow[ul, red] \arrow[ur, blue] \arrow[u, phantom, "\leftrightarrow"] &
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
B \arrow[rr, dashed] & & A \\
& D \arrow[ul, blue] \arrow[ur, red] &
\end{tikzcd}
$$

Generalized Morphisms in CAP

Realization in CAP

- CAP provides generalized morphisms as a data structure.

Generalized Morphisms in CAP

Realization in CAP

- CAP provides generalized morphisms as a data structure.
- 3 different data structures:

Generalized Morphisms in CAP

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- CAP provides generalized morphisms as a data structure.
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  1. Spans: $A \leftarrow D \rightarrow B$

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Download CAP

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http://homalg-project.github.io/CAP_project/