Algebra and Computation in the Lean Theorem Prover

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Two Goals For This Talk

- Introduce (and advertise!) Lean: a proof assistant based on dependent type theory
- Introduce Polya: a system for verifying real nonlinear inequalities

Credit To...

- Leonardo de Moura
- Soonho Kong, Floris van Doorn, Daniel Selsam
- Jeremy Avigad, Cody Roux

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- Jeremy Avigad, Cody Roux

- Leonardo de Moura: lead developer of Lean
- Soonho Kong, Floris van Doorn, Daniel Selsam, and others: contributors to library and system
- Jeremy Avigad: Lean standard library and Polya
- Cody Roux: contributions to Polya

Lean Details

- Constructive dependent type theory (CIC)
- Designed with automation in mind
  - Interactive theorem prover with strong automation
  - Automated theorem prover with verified mathematical library
- "Standard" and "HoTT" flavors
  - Standard: proof-irrelevant, impredicative Prop, classical logic available, quotient types
  - HoTT: proof-relevant, no impredicative Prop, univalence, HIT
- Seamlessly integrate classical reasoning
- May see similarities to Coq: not surprising!

Lean Details

- Small kernel
  - No termination checker, pattern matching, etc.
- Reference type checker
- Mixed tactic and declarative proof styles
- Powerful elaborator with strong type class inference mechanism

Example

```
definition infinite_primes (n : nat) : {p | p ≥ n ∧ prime p} :=
let m := fact (n + 1) in
have m ≥ 1,     from le_of_lt_succ (succ_lt_succ (fact_pos _)),
have m + 1 ≥ 2, from succ_le_succ this,
obtain p `prime p` `p | m + 1`, from sub_prime_and_dvd this,
have p ≥ 2,     from ge_two_of_prime `prime p`,
have p > 0,     from lt_of_succ_lt (lt_of_succ_le `p ≥ 2`),
have p ≥ n,     from by_contradiction
 (suppose ¬ p ≥ n,
   have p < n,   from lt_of_not_ge this,
   have p ≤ n + 1, from le_of_lt (lt.step this),
   have p | m,   from dvd_fact `p > 0` this,
   have p | 1,   from
         dvd_of_dvd_add_right (!add.comm ▸ `p | m + 1`) this,
   have p ≤ 1,   from le_of_dvd zero_lt_one this,
   absurd (le.trans `2 ≤ p` `p ≤ 1`) dec_trivial),
subtype.tag p (and.intro this `prime p`)
```

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    absurd (le.trans `2 ≤ p` `p ≤ 1`) dec_trivial),
subtype.tag p (and.intro this `prime p`)
```

- This slide shows off the declarative style
- Note that `by contradiction` is inferring the decidability of $\geq$ on $\mathbb{N}$.

Type Class Inference

- Can declare classes and instances
- Variables marked with [ ] are inferred by searching for instances of the correct types
- Search is recursive and backtracking and caches aggressively

Type Class Inference

```
inductive inhabited [class] (A : Type) : Type :=
mk : A → inhabited A

definition default (A : Type) [h : inhabited A] : A :=
inhabited.rec (λ a, a) h

definition prop_inhabited [instance] : inhabited Prop :=
inhabited.mk true

definition fun_inhabited [instance]
    (A B : Type) [h : inhabited B] : inhabited (A → B) :=
inhabited.mk (λ x : A, default B)

definition prod_inhabited [instance]
    (A B : Type) [ha : inhabited A] [hb : inhabited B] : inhabited (A × B)
    :=
inhabited.mk (default A, default B)

eval default (nat → nat × Prop)
-- λ (a : nat), (0, true)
```

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```

- We define a class `inhabited` and atomic instances `prop inhabited` and `nat inhabited`.
- Then compound instances are defined.
- Evaluating `default` traces through the instances to synthesize a term of the correct type.

Algebraic Hierarchy

Type class inference lets us construct the algebraic hierarchy in a uniform way.

```
structure semigroup [class] (A : Type) extends has_mul A :=
(mul_assoc : ∀ a b c, mul (mul a b) c = mul a (mul b c))

structure monoid [class] (A : Type) extends semigroup A, has_one A :=
(one_mul : ∀ a, mul one a = a) (mul_one : ∀ a, mul a one = a)

structure group [class] (A : Type) extends monoid A, has_inv A :=
(mul_left_inv : ∀ a, mul (inv a) a = one)

theorem inv_mul_cancel_left {A : Type} [H : group A] (a b : A) :
          a⁻¹ · (a · b) = b :=
 by rewrite [←mul.assoc, mul.left_inv, one_mul]

structure linear_ordered_field [class] (A : Type) extends
    linear_ordered_ring A, field A
```

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structure linear_ordered_field [class] (A : Type) extends
    linear_ordered_ring A, field A
```

- Lots of structures left out of this list, for room
- `structure` is shorthand for an inductive type with one constructor.
- Defining a structure with `extends` automatically defines instances to the smaller class.
- So using `mul.assoc` in `inv mul cancel left` finds the `mul.assoc` defined on semigroups.

Concrete Number Structures

When types instantiate these algebraic structures, all theorems proved in the general case are immediately available in the concrete setting.

```
definition real_ord_ring [reducible] [instance] : ordered_ring ℝ :=
{ ordered_ring, real.comm_ring,
  le_refl := real.le_refl,
  le_trans := @real.le_trans,
  mul_pos := real.mul_pos,
  mul_nonneg := real.mul_nonneg,
  zero_ne_one := real.zero_ne_one,
  add_le_add_left := real.add_le_add_left,
  le_antisymm := @real.eq_of_le_of_ge,
  lt_irrefl := real.lt_irrefl,
  lt_of_le_of_lt := @real.lt_of_le_of_lt,
  lt_of_lt_of_le := @real.lt_of_lt_of_le,
  le_of_lt := @real.le_of_lt,
  add_lt_add_left := real.add_lt_add_left
}
```

Concrete Number Structures

When types instantiate these algebraic structures, all theorems proved in the general case are immediately available in the concrete setting.

```
theorem translate_cts {f : ℝ → ℝ} (Hcon : continuous f) (a : ℝ) :
    continuous (λ x, (f x) + a) :=
 begin
  intros x ϵ Hϵ,
  cases Hcon x Hϵ with δ Hδ,
  cases Hδ with Hδ₁ Hδ₂,
  existsi δ,
  split,
  assumption,
  intros x' Hx',
  rewrite [add_sub_comm, sub_self, add_zero],
  apply Hδ₂,
  assumption
 end
```

Upshots for Automation

This uniform development is helpful for automation.

- Term simplifier
- Fourier-Motzkin linear inequality solver
- Simplex solver
- Blast (general purpose auto proof search)
- Blast (machine learning)
- Polya

Upshots for Automation

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- Theorems are not duplicated, so learned strategies apply across structures.
- Numerals behave identically, and easy to identify when the necessary properties are present.
- Term simplifier: present
- FM solver: waiting to be integrated
- Simplex solver: in progress
- Blast: partially implemented
- Machine learning: coming eventually

Polya

Polya: a tool for heuristically verifying real-valued inequalities over extensions of RCF

- Lightweight
- Flexible, extensible
- "Reasonably" constructive
- NOT a decision procedure
- Avigad, Lewis, Roux. *A heuristic prover for real inequalities* (2014)

A motivating example

$$0 < x < y, \; u < v$$

$$\Longrightarrow$$

$$2u + \exp(1 + x + x^4) < 2v + \exp(1 + y + y^4)$$

- This inference is not contained in linear arithmetic or real closed fields.
- This inference is tight: symbolic or numeric approximations to exp are not useful.
- Backchaining using monotonicity properties suggests many equally plausible subgoals.
- But, the inference is completely straightforward.

A new method

We propose and implement a method based on this type of heuristically guided forward reasoning. Our method:

- Verifies inequalities on which other procedures fail.
- Is relatively easy to implement in Lean.
- Captures natural, human-like inferences.
- Performs well on real-life problems.

- Is not complete.
- Is not guaranteed to terminate.

We envision it as a complement, not a replacement, to other verification procedures.

Implementations

- Python prototype: not proof-producing, but can experiment
- Lean version: on the way!

Terms and normal forms

The inequality

$$15 < 3(3y + 5x + 4xy)^2 f(u + v)^{-1}$$

is expressed canonically as

$$\underbrace{\underbrace{\mathbf{1}}_{t_0} < 5 \cdot \underbrace{(\underbrace{\underbrace{x}_{t_1} + \frac{3}{5} \cdot \underbrace{y}_{t_2} + \frac{4}{5} \cdot \underbrace{xy}_{t_3 = t_1 t_2}}_{t_6 = t_1 + \frac{3}{5} t_2 + \frac{4}{5} t_3})^2 \underbrace{f(\underbrace{\underbrace{u}_{t_4} + \underbrace{v}_{t_5}}_{t_7 = t_4 + t_5})}_{t_8 = f(t_7)}{}^{-1}}_{t_9 = t_6^2 t_8^{-1}}}_{t_0 \leq 5 t_9}$$

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- First the term is put in a normal form, with terms arranged alphabetically and constants factored.
- Then subterms are named. Names are given to problem variables, sums of terms, products of terms, and functions applied to terms.

Modules and database

Any comparison between canonical terms can be expressed as $t_i \bowtie 0$ or $t_i \bowtie c \cdot t_j$, where $\bowtie \in \{=, \neq, <, \leq, >, \geq\}$. This is in the common language of addition and multiplication.

A central database (the blackboard) stores term definitions and comparisons of this form.

Modules use this information to learn and assert new comparisons.

The procedure has succeeded in verifying an implication when modules assert contradictory information.

Arithmetic modules

If we restrict out attention to atomic comparisons and only additive (or only multiplicative) definitions, then linear methods can be used.

Given additive equations $\{t_i = \sum_j c_j \cdot t_{k_j}\}$ and atomic comparisons $\{t_i \bowtie c \cdot t_j\}$ and $\{t_i \bowtie 0\}$, want to saturate the blackboard with the strongest implied atomic comparisons. Two methods:

- Fourier-Motzkin elimination
- Geometric techniques

Modulo some concerns about sign information and irrational numbers, we can do the same with multiplicative equations.

Fourier-Motzkin additive module

To find comparisons between t_1 and t_2, eliminate t_3:

$$\begin{aligned}
3t_1 + 2t_2 - t_3 &> 0 \\
4t_1 + t_2 + t_3 &\geq 0 \\
2t_1 - t_2 - 2t_3 &\geq 0 \\
-2t_2 - t_3 &> 0
\end{aligned}$$

Fourier-Motzkin additive module

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Fourier-Motzkin additive module

To find comparisons between t_1 and t_2, find the strongest pair:

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Fourier-Motzkin additive module

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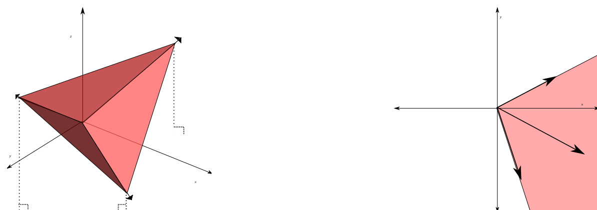
- Fourier Motzkin steps are simple. It's easy to see how this method can be made proof-producing.

Geometric additive module

The equalities and inequalities in the blackboard describe a polyhedron with vertex at the origin, in halfplane representation.

By projecting this polyhedron to the $t_i t_j$ plane, one can find the strongest implied comparisons between t_i and t_j.



We use the computational geometry packages cdd and lrs for the conversion from half-plane representation to vertex representation.

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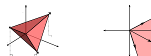

We use the computational geometry packages cdd and lrs for the conversion from half-plane representation to vertex representation.

- This approach scales better than the FM method.
- It's not as clear how to formalize this process.
- Depends on external packages, which can be buggy.

Axiom instantiation module

- Users can define function terms and axiomatize their behavior.
  - f increasing, positive, etc.
- This module instantiates these axioms heuristically.
- Built in handling for min/max, sin/cos, exp/log

Axiom instantiation module

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  - f increasing, positive, etc.
- This module instantiates these axioms heuristically.
- Built in handling for min/max, sin/cos, exp/log

- There's a balance to the instantiation. Too much will slow the process down, too little will miss results.
- Choices are heuristic, based on unifying "trigger terms" from the axioms with terms in the problem.
- Unification has to happen modulo AC equality (either additively or multiplicatively, not both).

Successes

Our implementation in Python successfully proves many theorems, some of which are not proved by other systems.

$$0 < x < 1 \implies 1/(1-x) > 1/(1-x^2) \tag{1}$$

$$0 < u,\ u < v,\ 0 < z,\ z+1 < w \implies (u+v+z)^3 < (u+v+w)^5 \tag{2}$$

$$\begin{aligned}(\forall x, y.\ x \leq y \to f(x) \leq f(y)),\ u < v,\ 1 < v,\ x \leq y \implies \\ u + f(x) \leq v^2 + f(y)\end{aligned} \tag{3}$$

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$$(\forall x, y.\ x \le y \to f(x) \le f(y)),\ u < v,\ 1 < v,\ x \le y \implies u + f(x) \le v^2 + f(y) \tag{3}$$

- Z3 will solve the first two. But in the second, if the exponents are increased, Z3 will slow down and then fail. Polya finds the “same” proof no matter what the exponents are.

Successes

$$(\forall x, y.\ f(x+y) = f(x)f(y)),\ f(a+b) > 2,\ f(c+d) > 2 \implies f(a+c+b+d) > 4 \tag{4}$$

$$0 \leq n,\ n < (K/2)x,\ 0 < c,\ 0 < \epsilon < 1 \implies \left(1 + \frac{\epsilon}{3(C+3)} \cdot n < Kx\right) \tag{5}$$

$$x < y,\ u \leq v \implies u + \min(x + 2u, y + 2v) \leq x + 3v \tag{6}$$

$$y > \max(2, 3x),\ x > 0 \implies \exp(4y - 3x) > \exp(6) \tag{7}$$

Successes

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- The first problem involves complicated matching in the unifier.
- The second comes from Avigad's formalization of the central limit theorem.

Limitations

Since our method is incomplete, it fails on a wide class of problems where other methods succeed.

$$x > 0,\ xyz < 0,\ xw > 0 \implies w > yz \tag{8}$$

$$x^2 + 2x + 1 \geq 0 \tag{9}$$

$$\begin{aligned} 4 \leq x_i \leq 6.3504 \implies & \\ & x_1 x_4(-x_1 + x_2 + x_3 - x_4 + x_5 + x_6) \\ & + x_2 x_5(x_1 - x_2 + x_3 + x_4 - x_5 + x_6) \\ & + x_3 x_6(x_1 + x_2 - x_3 + x_4 + x_5 + -x_6) \\ & - x_2 x_3 x_4 - x_1 x_3 x_5 - x_1 x_2 x_6 - x_4 x_5 x_6 > 0 \end{aligned} \tag{10}$$

KeYmaera examples

On a collection of 4442 problems generated automatically by KeYmaera, we solve 4130 (93%) with a 10-second timeout.

- 10 minutes using geometric packages
- 18 using Fourier-Motzkin

Example

```
Hypothesis:  ru10**2 == (1/3)*x1u0**2
Hypothesis:  x1u0 <= 0
Hypothesis:  ru10 > 0
Hypothesis:  d1 == -1 * om * (h2 + -1*x2)
Hypothesis:  d2 == om * (h1 + -1*x1)
Hypothesis:  (h1 + -1*x1)**2 + (h2 + -1*x2)**2 == r**2
Hypothesis:  1 != ru10**-1 * ru10
Conclusion:  False
```

Lean implementation

- Work in progress!
- Question: how to formalize geometric version?

Thanks for listening!

Lean:

- `http://leanprover.github.io` (interactive tutorial)
- de Moura, Kong, Roux. *Elaboration in dependent type theory.* (Available online)

Polya:

- `http://github.com/avigad/polya` (source code and directions)
- Avigad, Lewis, Roux. *A heuristic prover for real inequalities.* (JAR, forthcoming)