A new approach to formalize real numbers in the UniMath library

Catherine Lelay
Institute for Advanced Study

MAP 2016
Thursday January 14th 2016

Why New Reals Numbers?

UniMath is a new library in Coq which aims to formalize a substantial body of mathematics using the univalent point of view.

Why New Reals Numbers?

UniMath is a new library in Coq which aims to formalize a substantial body of mathematics using the univalent point of view.

- Improve this new library

Why New Reals Numbers?

UniMath is a new library in Coq which aims to formalize a substantial body of mathematics using the univalent point of view.

- Improve this new library
- Real and Complex analysis

Why New Reals Numbers?

UniMath is a new library in Coq which aims to formalize a substantial body of mathematics using the univalent point of view.

- Improve this new library
- Real and Complex analysis
- Metric and normed spaces
- Measure theory, probability

The UniMath library

Axioms of UniMath:

uahp : $\forall$ `P P' : hProp, (P` $\Rightarrow$ `P') and (P'` $\Rightarrow$ `P)` $\Rightarrow$ `P = P'`

funextfunax : $\forall$ `(X Y : UU) (f g : X` $\to$ `Y),` `(`$\forall x$`, f` x `= g` x`)` $\Rightarrow$ `f = g`

funextempty : $\forall$ `(X : UU) (f g : X` $\to \emptyset$`), f = g`

The UniMath library

Axioms of UniMath:

uahp : $\forall$ `P P' : hProp, (P` $\Rightarrow$ `P') and (P'` $\Rightarrow$ `P)` $\Rightarrow$ `P = P'`

funextfunax : $\forall$ `(X Y : UU) (f g : X` $\rightarrow$ `Y),` $(\forall x,$ `f` x `= g` $x)$ $\Rightarrow$ `f = g`

funextempty : $\forall$ `(X : UU) (f g : X` $\rightarrow \varnothing$`), f = g`

A useful consequence:

$$\forall\ \mathtt{A},\ \mathtt{B} \underset{\mathtt{hProp}}{\subseteq} \mathtt{X},\ (\forall x,\ x \in \mathtt{A} \Leftrightarrow x \in \mathtt{B}) \Rightarrow \mathtt{A} = \mathtt{B}$$

Which Real Numbers?

- Axiomatic
  Coq standard library, PVS
- Cauchy sequences
  C-CoRN/MathClasses, Isabelle/HOL, HOL Light
- Dedekind cuts
  Mizar, HOL4

Which Real Numbers?

- ~~Axiomatic~~
  Coq standard library, PVS
- Cauchy sequences
  C-CoRN/MathClasses, Isabelle/HOL, HOL Light
- Dedekind cuts
  Mizar, HOL4

Which Real Numbers?

- ~~Axiomatic~~
  Coq standard library, PVS
- Cauchy sequences
  C-CoRN/MathClasses, Isabelle/HOL, HOL Light
- $\boxed{\text{Dedekind cuts}}$
  Mizar, HOL4

1. Introduction
2. The UniMath library
3. Real Numbers
4. Conclusion

Types in UniMath

Hierarchy of types:

```
isofhlevel   0   X := Σx ∈ X, ∀y ∈ X, x = y
isofhlevel   1   X := ∀x, y ∈ X, isofhlevel 0 (x = y)
isofhlevel   2   X := ∀x, y ∈ X, isofhlevel 1 (x = y)
isofhlevel   3   X := ∀x, y ∈ X, isofhlevel 2 (x = y)
isofhlevel (S n) X := ∀x, y ∈ X, isofhlevel n (x = y)
```

Types in UniMath

Hierarchy of types:

$$
\begin{array}{llllr}
\texttt{isofhlevel} & \texttt{0} & \texttt{X} := & \Sigma x \in \texttt{X},\ \forall y \in \texttt{X},\ x = y & (\texttt{iscontr X}) \\
\texttt{isofhlevel} & \texttt{1} & \texttt{X} := & \forall x, y \in \texttt{X},\ \texttt{isofhlevel 0}\ (x = y) & (\texttt{isaprop X}) \\
\texttt{isofhlevel} & \texttt{2} & \texttt{X} := & \forall x, y \in \texttt{X},\ \texttt{isofhlevel 1}\ (x = y) & (\texttt{isaset X}) \\
\texttt{isofhlevel} & \texttt{3} & \texttt{X} := & \forall x, y \in \texttt{X},\ \texttt{isofhlevel 2}\ (x = y) & \\
\texttt{isofhlevel} & \texttt{(S n)} & \texttt{X} := & \forall x, y \in \texttt{X},\ \texttt{isofhlevel n}\ (x = y) &
\end{array}
$$

Useful types:

$$
\begin{array}{rl}
\texttt{UU} & := \texttt{Type} \\
\texttt{hProp} & := \Sigma\ \texttt{X} : \texttt{UU},\ \texttt{isaprop X} \\
\texttt{hSet} & := \Sigma\ \texttt{X} : \texttt{UU},\ \texttt{isaset X}
\end{array}
$$

Type of Formulas in UniMath

UniMath		Coq	
`empty`	`UU`	`False`	`Prop`
`A` $\times$ `B`	`UU`	`A * B`	`Type`
`A` $\wedge$ `B`	`hProp`	`A /\ B`	`Prop`
`A` $\amalg$ `B`	`UU`	`A + B`	`Type`
`A` $\vee$ `B`	`hProp`	`A \/ B`	`Prop`
$\forall x$`, P` x	`UU`	`forall` x`, P` x	`Prop` or `Type`
Σ x`, P` x	`UU`	`{` x `|` `P` x `}`	`Type`
$\exists x$`, P` x	`hProp`	`exists` x`, P` x	`Prop`

Prove Equality

Let `P : X -> UU`, `A :` Σ a`, P` a and `B :` Σ b`, P` b.

Prove `A = B` ?

Prove Equality

Let `P : X -> UU`, `A : Σ a, P a` and `B : Σ b, P b`.

Prove `A = B` ?

General case:

$$\pi_1 \texttt{ A = } \pi_1 \texttt{ B} \wedge \texttt{"}\pi_2 \texttt{ A = } \pi_2 \texttt{ B"} \Rightarrow \texttt{A = B}$$

Prove Equality

Let `P : X -> UU`, `A : Σ a, P a` and `B : Σ b, P b`.

Prove `A = B` ?

General case:

$$\pi_1 \; \texttt{A} = \pi_1 \; \texttt{B} \wedge \text{"}\pi_2 \; \texttt{A} = \pi_2 \; \texttt{B"} \Rightarrow \texttt{A} = \texttt{B}$$

If $\forall x \in$ `X, isaprop (P` x`)` (or `P : X -> hProp`):

$$\pi_1 \; \texttt{A} = \pi_1 \; \texttt{B} \Rightarrow \texttt{A} = \texttt{B}$$

From UU to hProp – Simpl Solution

$$\|\Box\| \quad : \quad \mathtt{UU} \to \mathtt{hProp}$$

From UU to hProp – Simpl Solution

$$\|\square\| \; : \; \mathtt{UU} \to \mathtt{hProp}$$

- $\mathtt{A} \vee \mathtt{B} := \| \; \mathtt{A} \amalg \mathtt{B} \; \|$
- $\exists \; \mathtt{x}, \; \mathtt{P} := \| \; \Sigma \; \mathtt{x}, \; \mathtt{P} \; \|$

From $\mathtt{UU}$ to $\mathtt{hProp}$ – Simpl Solution

$$\| \square \| \; : \; \mathtt{UU} \to \mathtt{hProp}$$

- $\mathtt{A} \vee \mathtt{B} := \| \mathtt{A} \amalg \mathtt{B} \|$
- $\exists\ \mathtt{x},\ \mathtt{P} := \| \Sigma\ \mathtt{x},\ \mathtt{P} \|$

$\oplus$ Easy to define

$\ominus$ May be hard to use

From `UU` to `hProp` – Alternative Solution

$$\texttt{hProp} := \Sigma\ \texttt{X} : \texttt{UU},\ \texttt{isaprop X}$$

Idea: prove `isaprop X`

From UU to hProp – Alternative Solution

$$\texttt{hProp} := \Sigma\ \texttt{X} : \texttt{UU},\ \texttt{isaprop X}$$

Idea: prove `isaprop X`

- `isaprop empty`
- `isaprop A` and `isaprop B` $\Rightarrow$ `isaprop (A` $\times$ `B)`
- $(\forall t,$ `isaprop (B` t`))` $\Rightarrow$ `isaprop (`$\forall t,$ `B` t`)`
- `isaprop B` $\Rightarrow$ `isaprop (A` $\Rightarrow$ `B)`

From UU to hProp – Alternative Solution

$$\texttt{hProp} := \Sigma\ \texttt{X} : \texttt{UU},\ \texttt{isaprop X}$$

Idea: prove `isaprop X`

- `isaprop empty`
- `isaprop A` and `isaprop B` $\Rightarrow$ `isaprop (A` $\times$ `B)`
- $(\forall t,$ `isaprop (B` t`))` $\Rightarrow$ `isaprop` $(\forall t,$ `B` $t)$
- `isaprop B` $\Rightarrow$ `isaprop (A` $\Rightarrow$ `B)`
- `isaprop A`, `isaprop B`, and (`A` and `B` $\Rightarrow \varnothing$) $\Rightarrow$ `isaprop (A` $\amalg$ `B)`
- $(\forall a, b,$ `P` $a \wedge$ `P` $b \Rightarrow a = b)$ and $(\forall t,$ `isaprop (P` t`))` $\Rightarrow$ `isaprop` $(\Sigma\ t,$ `P` $t)$

From UU to hProp – Alternative Solution

$$\texttt{hProp := } \Sigma \texttt{ X : UU, isaprop X}$$

Idea: prove `isaprop X`

- `isaprop empty`
- `isaprop A` and `isaprop B` $\Rightarrow$ `isaprop (A` $\times$ `B)`
- $(\forall t,$ `isaprop (B` t`))` $\Rightarrow$ `isaprop (`$\forall t,$ `B` t`)`
- `isaprop B` $\Rightarrow$ `isaprop (A` $\Rightarrow$ `B)`
- `isaprop A`, `isaprop B`, and (`A` and `B` $\Rightarrow \varnothing$) $\Rightarrow$ `isaprop (A` $\amalg$ `B)`
- $(\forall a, b,$ `P` $a \wedge$ `P` $b \Rightarrow a = b)$ and $(\forall t,$ `isaprop (P` t`))` $\Rightarrow$ `isaprop (`Σ t, `P` t`)`

⊕ Easy to use

⊖ May be hard to define

Type of Formulas in UniMath

UniMath		Coq	
empty	UU **or** hProp	False	Prop
A × B	UU **or** hProp	A * B	Type
A ∧ B	hProp	A /\ B	Prop
A ⨿ B	UU **or*** hProp	A + B	Type
A ∨ B	hProp	A \/ B	Prop
$\forall x$, P x	UU **or** hProp	forall x, P x	Prop **or** Type
Σ x, P x	UU **or*** hProp	{ x | P x }	Type
$\exists x$, P x	hProp	exists x, P x	Prop

1 Introduction

2 The UniMath library

3 Real Numbers

4 Conclusion

Dedekind Cuts – Historical definition

$(L, U) \subset \mathbb{Q}$ is a Dedekind cut if:

1. $\exists x \in L$
2. $\exists x \in U$
3. $L \cap U = \varnothing$
4. $\forall x \in L,\ \forall y \in \mathbb{Q},\ y \leq x \Rightarrow y \in L$
5. $\forall x \in U,\ \forall y \in \mathbb{Q},\ y \geq x \Rightarrow y \in U$
6. $\forall x \in L,\ \exists y \in \mathbb{Q},\ x < y \wedge y \in L$
7. $\forall x \in U,\ \exists y \in \mathbb{Q},\ x > y \wedge y \in U$
8. $\forall x, y \in \mathbb{Q},\ x < y \Rightarrow x \in L \vee y \in U$

Dedekind Cuts – Historical definition

$(L, U) \subset \mathbb{Q}$ is a Dedekind cut if:

1. $\exists x \in L$
2. $\exists x \in U$
3. $L \cap U = \varnothing$
4. $\forall x \in L,\ \forall y \in \mathbb{Q},\ y \leq x \Rightarrow y \in L$
5. $\forall x \in U,\ \forall y \in \mathbb{Q},\ y \geq x \Rightarrow y \in U$
6. $\forall x \in L,\ \exists y \in \mathbb{Q},\ x < y \wedge y \in L$
7. $\forall x \in U,\ \exists y \in \mathbb{Q},\ x > y \wedge y \in U$
8. $\forall x, y \in \mathbb{Q},\ x < y \Rightarrow x \in L \vee y \in U$

$\oplus$ historical definition

Dedekind Cuts – Historical definition

$(L, U) \subset \mathbb{Q}$ is a Dedekind cut if:

1. $\exists x \in L$
2. $\exists x \in U$
3. $L \cap U = \varnothing$
4. $\forall x \in L,\ \forall y \in \mathbb{Q},\ \ y \leq x \Rightarrow y \in L$
5. $\forall x \in U,\ \forall y \in \mathbb{Q},\ \ y \geq x \Rightarrow y \in U$
6. $\forall x \in L,\ \exists y \in \mathbb{Q},\ \ x < y \wedge y \in L$
7. $\forall x \in U,\ \exists y \in \mathbb{Q},\ \ x > y \wedge y \in U$
8. $\forall x, y \in \mathbb{Q},\ \ x < y \Rightarrow x \in L \vee y \in U$

$\oplus$ historical definition

$\ominus$ many hypotheses

Dedekind Cuts – One-side definition

$L \subset \mathbb{Q}$ is a one-side Dedekind cut if:

1. $\exists x \in L$
2. $\forall x \in L,\ \forall y \in \mathbb{Q},\ y \leq x \Rightarrow y \in L$
3. $\forall x \in L,\ \exists y \in \mathbb{Q},\ x < y \land y \in L$
4. $\forall x, y \in \mathbb{Q},\ x < y \Rightarrow x \in L \lor y \notin L$

Dedekind Cuts – One-side definition

$L \subset \mathbb{Q}$ is a one-side Dedekind cut if:

1. $\exists x \in L$
2. $\forall x \in L,\ \forall y \in \mathbb{Q},\ y \leq x \Rightarrow y \in L$
3. $\forall x \in L,\ \exists y \in \mathbb{Q},\ x < y \wedge y \in L$
4. $\forall x, y \in \mathbb{Q},\ x < y \Rightarrow x \in L \vee y \notin L$

$\oplus$ 4 hypotheses (can be reduced to 3)

Dedekind Cuts – One-side definition

$L \subset \mathbb{Q}$ is a one-side Dedekind cut if:

1. $\exists x \in L$
2. $\forall x \in L,\ \forall y \in \mathbb{Q},\ y \leq x \Rightarrow y \in L$
3. $\forall x \in L,\ \exists y \in \mathbb{Q},\ x < y \wedge y \in L$
4. $\forall x, y \in \mathbb{Q},\ x < y \Rightarrow x \in L \vee y \notin L$

$\oplus$ 4 hypotheses (can be reduced to 3)

$\ominus$ difficulties to define multiplication

Dedekind Cuts – Non-negative definition

$L \subset \mathbb{NQ} (= \{q \in \mathbb{Q} \mid 0 \leq q\})$ is a one-side non-negative Dedekind cut if:

1. $\forall x \in L,\ \forall y \in \mathbb{NQ},\ y \leq x \Rightarrow y \in L$
2. $\forall x \in L,\ \exists y \in \mathbb{NQ},\ x < y \wedge y \in L$
3. $\forall c \in \mathbb{NQ},\ 0 < c \Rightarrow (\neg c \in L) \vee \exists x \in L,\ (x + c) \notin L$

Dedekind Cuts – Non-negative definition

$L \subset \mathbb{NQ} (= \{q \in \mathbb{Q} \mid 0 \leq q\})$ is a one-side non-negative Dedekind cut if:

1. $\forall x \in L,\ \forall y \in \mathbb{NQ},\ y \leq x \Rightarrow y \in L$
2. $\forall x \in L,\ \exists y \in \mathbb{NQ},\ x < y \wedge y \in L$
3. $\forall c \in \mathbb{NQ},\ 0 < c \Rightarrow (\neg c \in L) \vee \exists x \in L,\ (x + c) \notin L$

- $\oplus$ 3 hypotheses
- $\oplus$ simpl definitions for arithmetic operations
- $\ominus$ cannot prove Dedekind completness without excluded middle

Dedekind Cuts – Non-negative definition

$L \subset \mathbb{NQ} (= \{q \in \mathbb{Q} \mid 0 \leq q\})$ is a one-side non-negative Dedekind cut if:

1. $\forall x \in L,\ \forall y \in \mathbb{NQ},\ y \leq x \Rightarrow y \in L$
2. $\forall x \in L,\ \exists y \in \mathbb{NQ},\ x < y \land y \in L$
3. $\forall c \in \mathbb{NQ},\ 0 < c \Rightarrow (\neg c \in L) \lor \exists x \in L,\ (x + c) \notin L$

- $\oplus$ 3 hypotheses
- $\oplus$ simpl definitions for arithmetic operations
- $\ominus$ cannot prove Dedekind completeness without excluded middle
  - $\rightarrow$ Cauchy completeness

Order and Apartness

Let $X, Y \in \mathbb{NR}$

$$
\begin{aligned}
X < Y &:= \exists x \in Y,\ x \notin X \\
X \leq Y &:= \forall x \in X,\ x \in Y \\
X \neq Y &:= (X < Y) \amalg (Y < X) \\
X \approx Y &:= \forall x \in \mathbb{NQ},\ x \in X \Leftrightarrow x \in Y
\end{aligned}
$$

Order and Apartness

Let $X, Y \in \mathbb{NR}$

$$
\begin{aligned}
X < Y &:= \exists x \in Y,\ x \notin X \\
X \leq Y &:= \forall x \in X,\ x \in Y \\
X \neq Y &:= (X < Y) \amalg (Y < X) \\
X \approx Y &:= \forall x \in \mathbb{NQ},\ x \in X \Leftrightarrow x \in Y
\end{aligned}
$$

Proved:

- `isaset` $\mathbb{NR}$
- Order properties of $\leq$ and $<$
- $\neq$ is a tight apartness relation

Algebraic Structure

Let $X, Y \in \mathbb{NR}$ and $q \in \mathbb{NQ}$

$$\begin{array}{rcl} \texttt{NQ\_to\_NR}(q) & := & \{r \in \mathbb{NQ} | r < q\} \\ X + Y & := & X \cup Y \cup \{x + y \ ; \ x \in X \wedge y \in Y\} \\ X \times Y & := & \{x \times y \ ; \ x \in X \wedge y \in Y\} \\ (X, , Hx0 : X \neq 0)^{-1} & := & \{q \in \mathbb{NQ} \mid \exists l \in (0;1), \ \forall x \in X, \ x \times q \leq l\} \end{array}$$

Algebraic Structure

Let $X, Y \in \mathbb{NR}$ and $q \in \mathbb{NQ}$

$$\begin{aligned}
\mathtt{NQ_to_NR}(q) &:= \{r \in \mathbb{NQ} | r < q\} \\
X + Y &:= X \cup Y \cup \{x + y \; ; \; x \in X \wedge y \in Y\} \\
X \times Y &:= \{x \times y \; ; \; x \in X \wedge y \in Y\} \\
(X, , Hx0 : X \neq 0)^{-1} &:= \{q \in \mathbb{NQ} \mid \exists l \in (0;1), \; \forall x \in X, \; x \times q \leq l\}
\end{aligned}$$

Proved:

- $\mathbb{NR}$ is a commutative rig
- $\forall X \in \mathbb{NR}, \; X \neq 0 \Rightarrow X \times X^{-1} = 1$

Algebraic Structure

Let $X, Y \in \mathbb{NR}$ and $q \in \mathbb{NQ}$

$$
\begin{array}{rcl}
\mathtt{NQ\_to\_NR}(q) & := & \{r \in \mathbb{NQ} | r < q\} \\
X + Y & := & X \cup Y \cup \{x + y \; ; \; x \in X \wedge y \in Y\} \\
X \times Y & := & \{x \times y \; ; \; x \in X \wedge y \in Y\} \\
(X, , Hx0 : X \neq 0)^{-1} & := & \{q \in \mathbb{NQ} \mid \exists l \in (0;1), \; \forall x \in X, \; x \times q \leq l\}
\end{array}
$$

Proved:

- $\mathbb{NR}$ is a commutative rig
- $\forall X \in \mathbb{NR}, \; X \neq 0 \Rightarrow X \times X^{-1} = 1$

Remark: $X \neq 0 \Leftrightarrow 0 < X \Leftrightarrow 0 \in X \Leftrightarrow \exists q \in X, \; 0 < q$

Completness

Let $u \in \mathbb{NR}^{\mathbb{N}}$.

$$\begin{aligned}
\texttt{Cauchy}(u) &:= \forall \varepsilon > 0,\ \exists N,\ \forall n, m \geq N, \\
&\qquad u_n \leq u_m + \varepsilon \wedge u_m \leq u_n + \varepsilon \\
\texttt{is_lim_seq}(u, \ell) &:= \forall \varepsilon > 0,\ \exists N,\ \forall n \geq N, \\
&\qquad u_n \leq \ell + \varepsilon \wedge \ell \leq u_n + \varepsilon \\
\texttt{Cauchy_lim_seq}(u, Cu) &:= \{q \in \mathbb{NQ} \mid \exists c \in \mathbb{NQ},\ (0 < c) \\
&\qquad \times \Sigma N \in \mathbb{N}, \forall n \geq N,\ (q + c) \in u_n)\}
\end{aligned}$$

Completness

Let $u \in \mathbb{NR}^{\mathbb{N}}$.

$$
\begin{aligned}
\mathtt{Cauchy}(u) &:= \forall \varepsilon > 0,\ \exists N,\ \forall n, m \geq N, \\
&\qquad u_n \leq u_m + \varepsilon \wedge u_m \leq u_n + \varepsilon \\
\mathtt{is\_lim\_seq}(u, \ell) &:= \forall \varepsilon > 0,\ \exists N,\ \forall n \geq N, \\
&\qquad u_n \leq \ell + \varepsilon \wedge \ell \leq u_n + \varepsilon \\
\mathtt{Cauchy\_lim\_seq}(u, Cu) &:= \{q \in \mathbb{NQ} \mid \exists c \in \mathbb{NQ},\ (0 < c) \\
&\qquad \times\ \Sigma N \in \mathbb{N}, \forall n \geq N,\ (q + c) \in u_n)\}
\end{aligned}
$$

Proved: $\mathtt{is\_lim\_seq}(u, \mathtt{Cauchy\_lim\_seq}(u, Cu))$

Useful functions

$$
\begin{aligned}
X - Y &:= \{q \in \mathbb{NQ} \mid \exists x \in X,\ \forall y \in Y,\ y + q \leq x\} \\
\max(X, Y) &:= X \cup Y
\end{aligned}
$$

Proved:

- $\forall X, Y \in \mathbb{NR},\ X - Y \leq X$
- $\forall X, Y \in \mathbb{NR},\ X < Y \Leftrightarrow 0 < Y - X$

Useful functions

$$\begin{aligned} X - Y &:= \{q \in \mathbb{NQ} \mid \exists x \in X,\ \forall y \in Y,\ y + q \le x\} \\ \max(X, Y) &:= X \cup Y \end{aligned}$$

Proved:

- $\forall X, Y \in \mathbb{NR},\ X - Y \le X$
- $\forall X, Y \in \mathbb{NR},\ X < Y \Leftrightarrow 0 < Y - X$
- $\forall X, Y \in \mathbb{NR},\ (X - Y) + Y =$

Useful functions

$$\begin{aligned} X - Y &:= \{q \in \mathbb{NQ} \mid \exists x \in X,\ \forall y \in Y,\ y + q \leq x\} \\ \max(X, Y) &:= X \cup Y \end{aligned}$$

Proved:

- $\forall X, Y \in \mathbb{NR},\ X - Y \leq X$
- $\forall X, Y \in \mathbb{NR},\ X < Y \Leftrightarrow 0 < Y - X$
- $\forall X, Y \in \mathbb{NR},\ (X - Y) + Y = \max(X, Y)$

Real Numbers

$$\mathbb{R} \;:=\; \texttt{commrigtocommrng}\ \mathbb{NR}$$

In other words:

- $\mathbb{R} \subseteq \mathbb{NR} \times \mathbb{NR}$
- $\forall X \in \mathbb{R},\ \exists x \in X$
- $\forall X \in \mathbb{R},\ \forall x \in X,\ \forall y \in \mathbb{NR} \times \mathbb{NR},$
  $\pi_1(x) + \pi_2(y) = \pi_1(y) + \pi_2(x) \Leftrightarrow y \in X$

Real Numbers

$$\mathbb{R} \texttt{ := commrigtocommrng } \mathbb{NR}$$

In other words:

- $\mathbb{R} \subseteq \mathbb{NR} \times \mathbb{NR}$
- $\forall X \in \mathbb{R},\ \exists x \in X$
- $\forall X \in \mathbb{R},\ \forall x \in X,\ \forall y \in \mathbb{NR} \times \mathbb{NR},$
  $\pi_1(x) + \pi_2(y) = \pi_1(y) + \pi_2(x) \Leftrightarrow y \in X$

Free: $0,\ 1,\ +,\ -,\ \times,$ ring properties
Almost free: $<,\ \leq,\ \neq$

Real Numbers

$$\mathbb{R} \ \texttt{:=}\ \texttt{commrigtocommrng}\ \mathbb{NR}$$

In other words:

- $\mathbb{R} \subseteq \mathbb{NR} \times \mathbb{NR}$
- $\forall X \in \mathbb{R},\ \exists x \in X$
- $\forall X \in \mathbb{R},\ \forall x \in X,\ \forall y \in \mathbb{NR} \times \mathbb{NR},$
  $\pi_1(x) + \pi_2(y) = \pi_1(y) + \pi_2(x) \Leftrightarrow y \in X$

Free: 0, 1, $+$, $-$, $\times$, ring properties
Almost free: $<$, $\leq$, $\neq$

Missing: Multiplicative inverse, Completeness

From Reals to Non-negative Reals

Let $X \in \mathbb{R}$,

- I have $\exists x \in X \;\; := \;\; \|\Sigma x \in X\|$
- I need $\Sigma x \in X$

From Reals to Non-negative Reals

Let $X \in \mathbb{R}$,

- I have $\exists x \in X \;\; := \;\; \|\Sigma x \in X\|$
- I need $\Sigma x \in X$

Remind: $(\forall a, b,\ \mathtt{P}\ a \wedge \mathtt{P}\ b \Rightarrow a = b)$
$\Rightarrow \mathtt{isaprop}\ (\Sigma\ t,\ \mathtt{P}\ t)$

From Reals to Non-negative Reals

Let $X \in \mathbb{R}$,

- I have $\exists x \in X \;\; := \;\; \|\Sigma x \in X\|$
- I need $\Sigma x \in X$

Remind: $(\forall a, b, \;\mathtt{P}\;\; a \;\wedge\; \mathtt{P}\;\; b \;\Rightarrow\; a \;\mathtt{=}\; b)$
$\;\;\Rightarrow\; \mathtt{isaprop}\;\; (\Sigma\;\; t,\;\; \mathtt{P}\;\; t)$

Idea: Find $\mathtt{P} : \mathbb{NR} \times \mathbb{NR} \to \mathtt{hProp}$ such that $(\forall a, b, \;\mathtt{P}\;\; a \;\wedge\; \mathtt{P}\;\; b \;\Rightarrow\; a \;\mathtt{=}\; b)$ and prove $\mathtt{R\_to\_NR\_NR}(X) \;:\; \Sigma y \in X, \;\mathtt{P}\;\; y$.

From Reals to Non-negative Reals

Let $X \in \mathbb{R}$,

- I have $\exists x \in X \;\; := \;\; \|\Sigma x \in X\|$
- I need $\Sigma x \in X$

Remind: $(\forall a, b,\ \mathtt{P}\ a \land \mathtt{P}\ b \Rightarrow a = b)$
$\Rightarrow \mathtt{isaprop}\ (\Sigma\ t,\ \mathtt{P}\ t)$

Idea: Find $\mathtt{P} : \mathbb{NR} \times \mathbb{NR} \to \mathtt{hProp}$ such that $(\forall a, b,\ \mathtt{P}\ a \land \mathtt{P}\ b \Rightarrow a = b)$ and prove $\mathtt{R\_to\_NR\_NR}(X)\ :\ \Sigma y \in X,\ \mathtt{P}\ y$.

Solution:

- Chose $\mathtt{P} = \{y \in X \mid \forall x \in X, \pi_1(y) \leq \pi_1(x) \land \pi_2(y) \leq \pi_2(x)\}$
- Prove $(\pi_1(x) - \pi_2(x), \pi_2(x) - \pi_1(x)) \in P$

Multiplicative Inverse

Let $X \in \mathbb{R}$ such that $X \neq 0$

Multiplicative Inverse

Let $X \in \mathbb{R}$ such that $X \neq 0 \Leftrightarrow (0 < X \amalg X < 0)$,

- If $0 < X$ then $\texttt{R\_to\_NR\_NR}(X) = (x,,0)$ with $0 < x$,
  $\Rightarrow X^{-1} = \texttt{RN\_RN\_to\_R}(x^{-1},,0)$
- If $0 > X$ then $\texttt{R\_to\_NR\_NR}(X) = (0,,x)$ with $0 < x$,
  $\Rightarrow X^{-1} = \texttt{RN\_RN\_to\_R}(0,,x^{-1})$

Multiplicative Inverse

Let $X \in \mathbb{R}$ such that $X \neq 0 \Leftrightarrow (0 < X \amalg X < 0)$,

- If $0 < X$ then $\texttt{R\_to\_NR\_NR}(X) = (x,,0)$ with $0 < x$,
  $\Rightarrow X^{-1} = \texttt{RN\_RN\_to\_R}(x^{-1},,0)$
- If $0 > X$ then $\texttt{R\_to\_NR\_NR}(X) = (0,,x)$ with $0 < x$,
  $\Rightarrow X^{-1} = \texttt{RN\_RN\_to\_R}(0,,x^{-1})$

Proved: X^{-1} is the multiplicative inverse of X.

Completness

Let $u \in \mathbb{R}^{\mathbb{N}}$.

$$\begin{aligned} \texttt{Cauchy}(u) \quad &:= \quad \forall \varepsilon > 0,\ \exists N,\ \forall n, m \geq N,\ |u_n - u_m| \leq \varepsilon \\ \texttt{is\_lim\_seq}(u, \ell) \quad &:= \quad \forall \varepsilon > 0,\ \exists N,\ \forall n \geq N,\ |u_n - \ell| \leq \varepsilon \end{aligned}$$

Proved: $\Sigma \ell \in \mathbb{R},\ \texttt{is\_lim\_seq}(u, \ell)$

Completeness

Let $u \in \mathbb{R}^{\mathbb{N}}$.

$$\begin{aligned} \mathtt{Cauchy}(u) &:= \forall \varepsilon > 0,\ \exists N,\ \forall n, m \geq N,\ |u_n - u_m| \leq \varepsilon \\ \mathtt{is\_lim\_seq}(u, \ell) &:= \forall \varepsilon > 0,\ \exists N,\ \forall n \geq N,\ |u_n - \ell| \leq \varepsilon \end{aligned}$$

Proved: $\Sigma \ell \in \mathbb{R},\ \mathtt{is\_lim\_seq}(u, \ell)$

Idea of proof: Let $\forall n \in \mathbb{N},\ \mathtt{R\_to\_NR\_NR}(u_n) = (x_n, y_n)$.

Completness

Let $u \in \mathbb{R}^{\mathbb{N}}$.

$$\begin{aligned} \texttt{Cauchy}(u) &:= \forall \varepsilon > 0,\ \exists N,\ \forall n, m \geq N,\ |u_n - u_m| \leq \varepsilon \\ \texttt{is\_lim\_seq}(u, \ell) &:= \forall \varepsilon > 0,\ \exists N,\ \forall n \geq N,\ |u_n - \ell| \leq \varepsilon \end{aligned}$$

Proved: $\Sigma \ell \in \mathbb{R},\ \texttt{is\_lim\_seq}(u, \ell)$

Idea of proof: Let $\forall n \in \mathbb{N},\ \texttt{R\_to\_NR\_NR}(u_n) = (x_n, y_n)$.

- If $\texttt{Cauchy}(u)$, then $\texttt{Cauchy}(x)$ and $\texttt{Cauchy}(y)$

Completness

Let $u \in \mathbb{R}^{\mathbb{N}}$.

$$\begin{aligned}\texttt{Cauchy}(u) &:= \forall \varepsilon > 0,\ \exists N,\ \forall n, m \geq N,\ |u_n - u_m| \leq \varepsilon \\ \texttt{is\_lim\_seq}(u, \ell) &:= \forall \varepsilon > 0,\ \exists N,\ \forall n \geq N,\ |u_n - \ell| \leq \varepsilon\end{aligned}$$

Proved: $\Sigma \ell \in \mathbb{R},\ \texttt{is\_lim\_seq}(u, \ell)$

Idea of proof: Let $\forall n \in \mathbb{N},\ \texttt{R\_to\_NR\_NR}(u_n) = (x_n, y_n)$.

- If $\texttt{Cauchy}(u)$, then $\texttt{Cauchy}(x)$ and $\texttt{Cauchy}(y)$
- ℓ := $\texttt{RN\_RN\_to\_R (Cauchy\_lim\_seq}(x, Cx)$, , $\texttt{Cauchy\_lim\_seq}(y, Cy))$

Completness

Let $u \in \mathbb{R}^{\mathbb{N}}$.

$$\begin{aligned} \mathtt{Cauchy}(u) &:= \forall \varepsilon > 0,\ \exists N,\ \forall n, m \geq N,\ |u_n - u_m| \leq \varepsilon \\ \mathtt{is\_lim\_seq}(u, \ell) &:= \forall \varepsilon > 0,\ \exists N,\ \forall n \geq N,\ |u_n - \ell| \leq \varepsilon \end{aligned}$$

Proved: $\Sigma \ell \in \mathbb{R},\ \mathtt{is\_lim\_seq}(u, \ell)$

Idea of proof: Let $\forall n \in \mathbb{N},\ \mathtt{R\_to\_NR\_NR}(u_n) = (x_n, y_n)$.

- If $\mathtt{Cauchy}(u)$, then $\mathtt{Cauchy}(x)$ and $\mathtt{Cauchy}(y)$
- ℓ := RN_RN_to_R (Cauchy_lim_seq(x, Cx) ,, Cauchy_lim_seq(y, Cy))
- $\mathtt{is\_lim\_seq}(u, \ell)$

The Dedekind Directory in UniMath

$\mathbb{NQ}$ $0,\ 1,\ +,\ \times,\ \square^{-1}$	non-negative commutative rig decidable equality and orders totally ordered $\forall x, x \neq 0 \Rightarrow \Sigma y,\ \ x \times y = 1$
$\downarrow$	
$\mathbb{NR}$ $0,\ 1,\ +,\ \times,\ \square^{-1}$ `Cauchy_lim_seq`	non-negative constructive commutative rig totally ordered $\forall x, x \neq 0 \Rightarrow \Sigma y,\ \ x \times y = 1$ completeness
$\downarrow$	
$\mathbb{R}$ $0,\ 1,\ +,\ -,\ \times,\ \square^{-1}$ `Cauchy_lim_seq`	constructive field totally ordered completeness

Future Work

About real numbers:

- Archimedean property
- "user-friendlize" the library
- Identify and define missing generic structures (non-negative rigs, complete spaces, ...)

To go further:

- Extended non-negative reals: $\overline{\mathbb{NR}} = \mathbb{NR} \cup \{+\infty\}$
- Bases of analysis (limits, series, derivatives, Riemann integral, usual functions, ...)
- Measure theory, Lebesgue integral
- Topology


Questions?