Shortest path embeddings of graphs on surfaces

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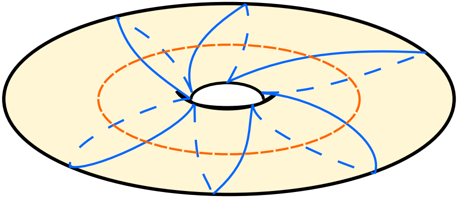
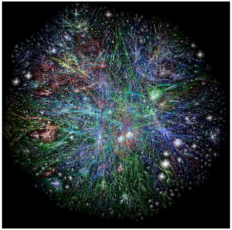


First, some motivations

Question

How can one represent a graph nicely?



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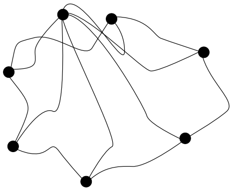

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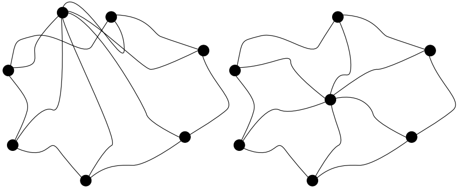

- It is easier if the graph is planar.

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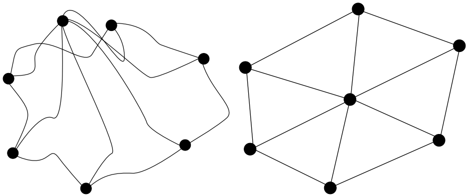


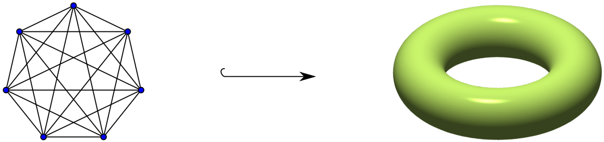
- It is easier if the graph is planar.
- In this case, the edges can even be straightened.

Theorem (Fàry-Wagner)

Any simple planar graph can be embedded in the plane with straight lines.

Surfaces

- But what about non-planar graphs?
- They can always be embedded on some surface.



The “number of holes” of a surface is called its genus, denoted by g.

Fàry's Theorem on surfaces

Theorem (Fàry-Wagner)

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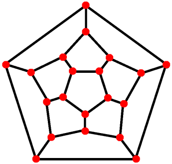


Plane	$\rightarrow$	Surface
Euclidean metric	$\rightarrow$	Some metric?
Straight line	$\rightarrow$	Shortest path

Question

For each surface S, does there exist a metric such that any simple graph embeddable on S can be embedded with the edges drawn as shortest paths?

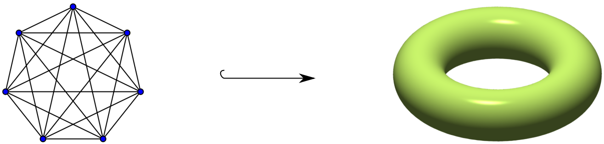
Motivation 1: Graph Drawing

Question

How to draw "nicely" a surface-embedded graph?

1. Draw the surface
2. Draw the graph "nicely" with respect to the metric of the surface

$\rightarrow$ Is there a universal metric to draw all the graphs nicely?



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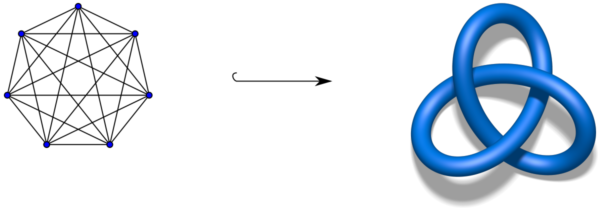

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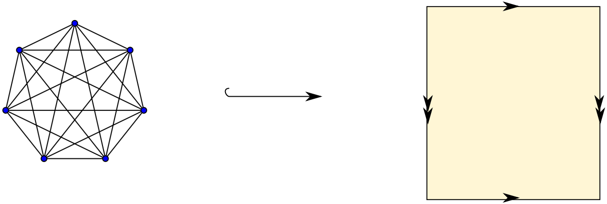

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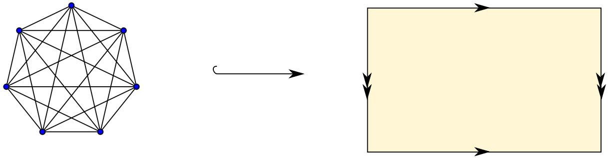

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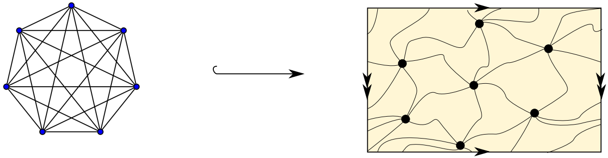

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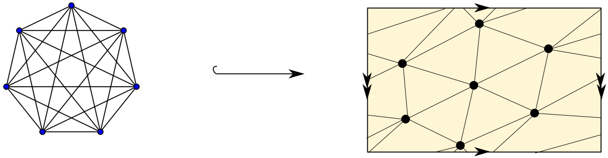
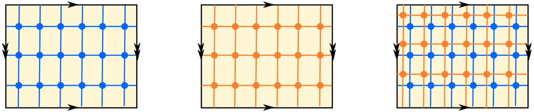


Motivation 2: Joint crossing numbers

- Let G_1 and G_2 be two graphs embeddable on a surface S of genus g.
- The *joint crossing number* $jcr(G_1, G_2)$ is the minimal number of crossings in two simultaneous and transverse embeddings of G_1 and G_2.



Theorem (Negami)

There is a constant c *such that* $jcr(G_1, G_2) \leq cg|E(G_1)||E(G_2)|$.

Conjecture (Negami)

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Motivation 2: Many different occurences

- [Negami '01]: Study of flips in triangulations.
- Related to topological decompositions, e.g., canonical systems of loops [Lazarus, Pocchiola, Vegter, Verroust '01], pants decompositions [É. Colin de Verdière, Lazarus '07], octagonal decompositions [É. Colin de Verdière, Erickson '10].

Very related problems:

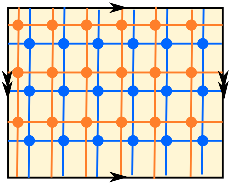
- [Matoušek, Sedgwick, Tancer, Wagner '14]: Complexity bound for embeddability of 2-complexes into $\mathbb{R}^3$.
- [Geelen, Huynh, Richter '13]: Explicit bounds for graph minors.
- [Mohar '09]: Value of the genus crossing number.

Motivation 2: Negami's conjecture

Conjecture (Negami)

There is a constant c *such that* $jcr(G_1, G_2) \leq c|E(G_1)||E(G_2)|$.

- [Archdeacon, Bonnington '01]: "The authors conjecture the opposite"
- [Richter, Salazar '05]: "On the one hand, this seems eminently reasonable: why should two edges be forced to cross more than once?"



- Shortest paths cross at most once!

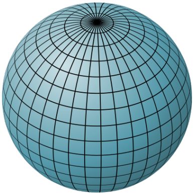
→ If the answer to our question is affirmative, Negami's conjecture is true.

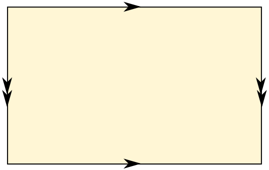
Main question

Question

For each surface *S*, does there exist a metric such that any simple graph embeddable on *S* can be embedded with the edges drawn as shortest paths?

- We call such a metric a *universal shortest path metric*.
- We focus on Riemannian constant-curvature metrics: spherical, flat or hyperbolic.





Our results

Theorem (The good)

The sphere S^2, the projective plane $\mathbb{R}P^2$, the torus T^2, and the Klein bottle K can be endowed with a universal shortest path metric.

Theorem (The bad)

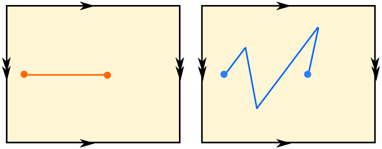
There exists a flat metric on K and a graph embeddable into K which cannot be embedded into K so that the edges are shortest paths.

Theorem (The ugly)

Let S denote a surface of genus $g \geq 2$, with probability tending to 1 as g goes to infinity, a random hyperbolic metric is not a universal shortest path metric.

Our results II

- Relax the problem: look for embeddings with concatenations of k shortest paths $\rightarrow$ *k-universal shortest path metrics*.



Theorem

For every $g > 1$, there exists an $O(g)$-universal shortest path hyperbolic metric m on the orientable surface S of genus g.

Related work I

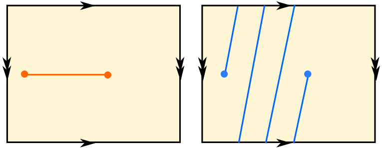
Question

For each surface S, does there exist a metric such that any simple graph embeddable on S can be embedded with the edges drawn as shortest paths?

Related work I

Question

For each surface S, does there exist a metric such that any simple graph embeddable on S can be embedded with the edges drawn as ~~shortest paths~~ *geodesics*?



Theorem (Y. Colin de Verdière)

Let S be a surface endowed with a metric of nonpositive curvature. Then any simple graph G embeddable on S can be embedded such that the edges are drawn as geodesics.

- The proof is à la Tutte: put springs on the edges and relax.
- But geodesics may wind an arbitrary number of times.

Related work II

Question

For a surface S , does there exist a metric such that any simple graph embeddable on S can be embedded with the edges drawn as shortest paths?

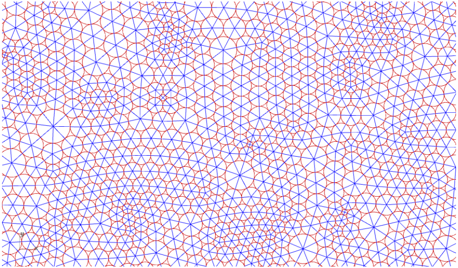
Related work II

Question

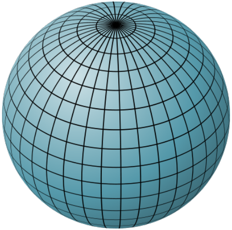
For a surface S and a simple graph G embeddable on S, does there exist a metric such that G can be embedded with the edges drawn as shortest paths?

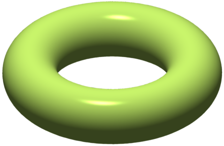
Theorem (Circle packing theorem, Koebe-Andreev-Thurston)

For any triangulated graph G on a surface S, there exists a metric on S such that G can be represented as the contact graph of circles.



The good





The sphere and the projective plane

Theorem

The usual round metrics for the sphere and the projective plane are universal shortest path metrics.

- For the sphere: Circle-packing actually works.
- For the projective plane: Circle pack the spherical cover and quotient it.

The torus

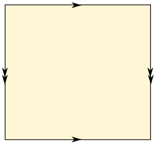
Theorem

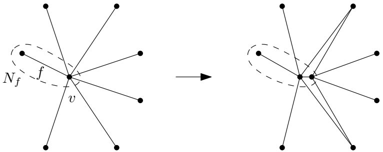
The unit square flat metric on the torus is a universal shortest path metric.

- A triangulation of the torus is *reducible* if there is an edge e such that the contraction T/e is still a triangulation.

Lemma

If T/e admits a shortest path embedding with some ε-slack, then so does T.





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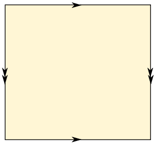
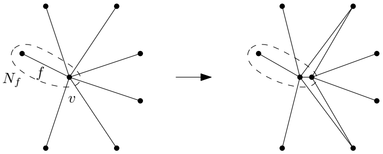
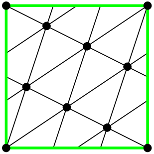




$\rightarrow$ It suffices to find shortest path embeddings for the 21 irreducible triangulations of the torus [Lawrencenko '87].

Irreducible triangulations of the torus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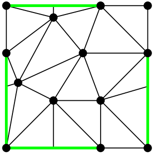
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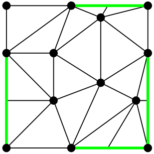
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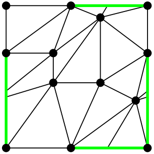
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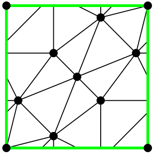
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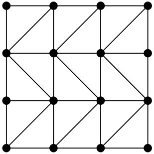
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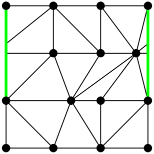
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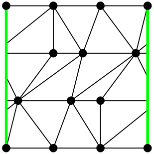
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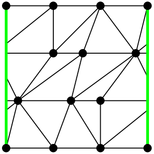
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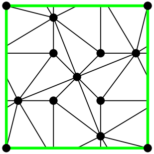
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The bad

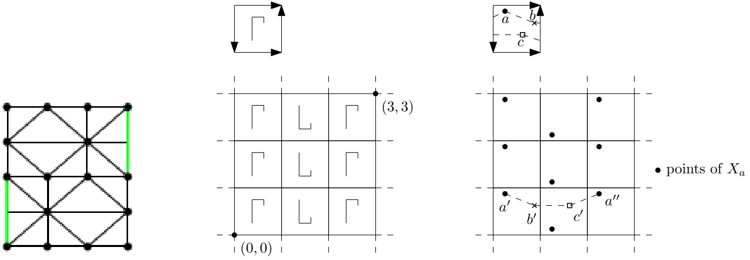
First try on the Klein bottle...

Theorem

The 1×1 flat metric with scheme $aba^{-1}b$ is not a universal shortest path metric.

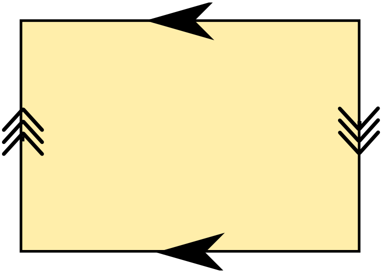
<u>Proof</u>:

- We look at the following graph.
- This embedding contains a non-trivial separating cycle of length 3. Such a cycle needs to have "horizontal" length at least 2.



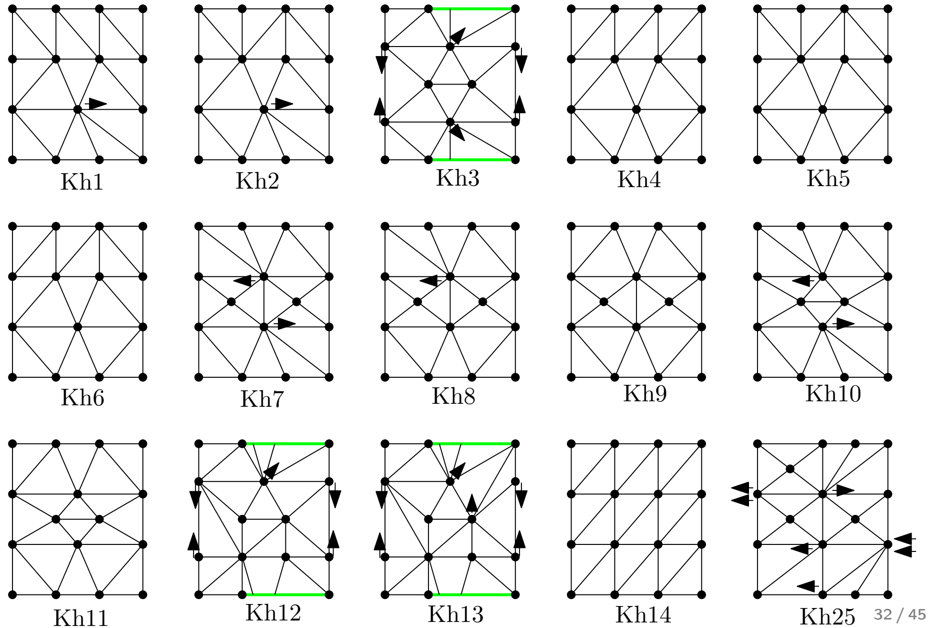
Theorem

The $1 \times \sqrt{4/3} + \varepsilon$ *flat metric with scheme* $aba^{-1}b$ *is a universal shortest path metric.*

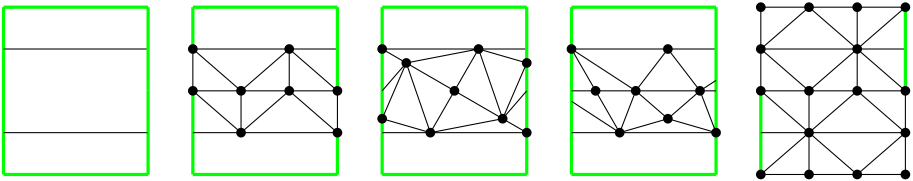


Let us check the 29 irreducible triangulations of the Klein bottle.

Some irreducible triangulations of the Klein bottle...



...and some more



Mb1 Mb2 Mb3 Kc1

The ugly

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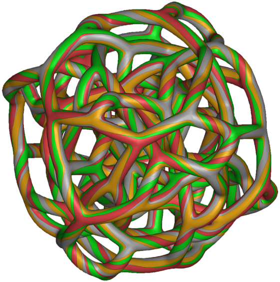


Positive genus

- Irreducible triangulations become non-tractable. (396784 for S_2)
- Hard to find any metric for which it does not work, but...

Theorem

For a surface S of genus g, with probability tending to 1 as $g \to \infty$, a random hyperbolic metric is not a universal shortest path metric.

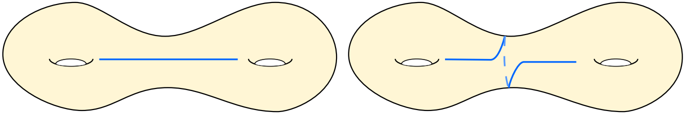
Actually, the result is stronger:

Theorem

For any $\varepsilon > 0$, with probability tending to 1 as $g \to \infty$, a random hyperbolic metric is not a $O(g^{1/3-\varepsilon})$-shortest path metric.

Random metric?

- The space of hyperbolic metrics up to isotopy on a surface of genus g is the *Teichmüller space* $\mathcal{T}_g$ of the surface.
- For our problem, two hyperbolic metrics related by an isometric homeomorphism are equivalent.

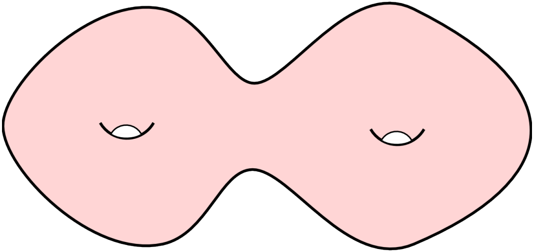


$\rightarrow$ We quotient by the action of the group of homeomorphisms (the *Mapping class group*).

- We obtain the *Moduli space* $\mathcal{M}_g$.
- This moduli space can be endowed with the *Weil-Petersson* metric, for which $\mathcal{M}_g$ has finite volume. $\rightarrow$ Probability space.

Pants decomposition

A *pants decomposition* is a family of disjoint closed curves on a surface cutting it into *pairs of pants*.



Properties of random metrics

Theorem (Mirzakhani)

With probability tending to 1, the diameter of a surface with a random hyperbolic metric is $O(\log g)$.

Theorem (Guth, Parlier and Young)

For any $\varepsilon > 0$, a pants decomposition of a surface with a random hyperbolic metric has length $\Omega(g^{7/6-\varepsilon})$ with probability tending to 1.

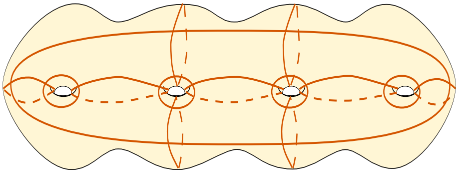
- We build a graph G with $O(g)$ edges containing a pants decomposition (in all of its possible embeddings) and a $O(g^{1/6-\varepsilon})$ lower bound follows.
- We get to $O(g^{1/3-\varepsilon})$ with a bit more work.

A relaxed upper bound

Theorem

For every $g > 1$, there exists a $O(g)$-universal shortest path hyperbolic metric on the orientable surface of genus g.

- Starting tool: *hexagonal decompositions*.



Theorem (Follows easily from É. Colin de Verdière, Erickson)

Let G be a graph embedded on S_g, there exists a hexagonal decomposition Δ such that each edge of G crosses the curves of Δ at most $O(g)$ times.

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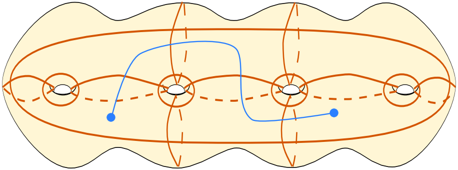

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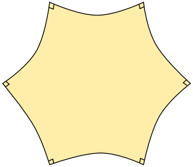
The hyperbolic metric

- We endow each hexagon with the hyperbolic metric m_H of equilateral right-angled hexagons.



- We reembed G separately in each hexagon with shortest paths.
  $\rightarrow$ We need a hyperbolic Tutte theorem with a non-strictly convex boundary.

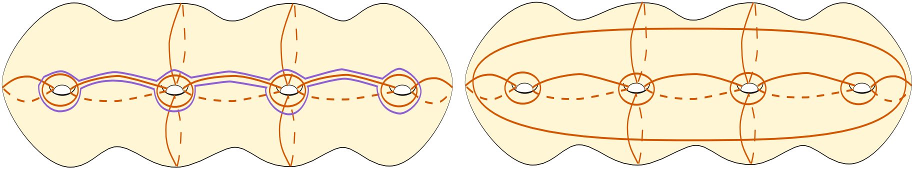
Theorem (Almost Y. Colin de Verdière)

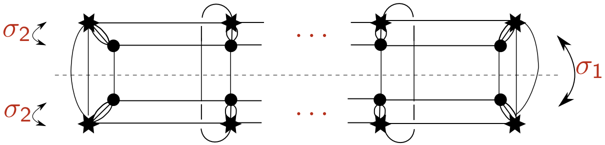
Let G be a graph embedded as a triangulation in a hyperbolic hexagon H endowed with the metric m_H. If there are no dividing edges in G, then G can be embedded with geodesics, with the vertices on the boundary of H in the same positions as in the initial embedding.

The exchange argument

Lemma

Geodesics in the hexagons are shortest paths in the surface.





We mirror shortest paths until they stay in a single hexagon.

Concluding words

Remarks:

- Our results also hold for graphs with a fixed rotation system (i.e., *maps*).

Questions:

- For positive genus, do there exist universal shortest path metrics?
- If not, can we improve the upper bound on the concatenation of shortest paths?

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Thank you! Questions?