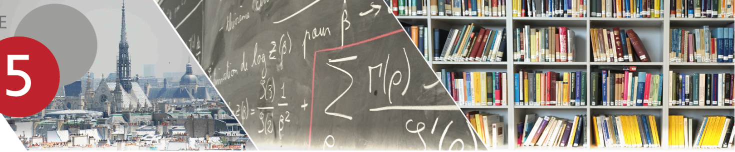
LABORATOIRE MAP5



The Shannon Total Variation

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joint work with Rémy Abergel

Total Variation in image processing

Let Ω be an open subset of $\mathbb{R}^2$ and $U : \Omega \to \mathbb{R}$ an intensity image ($U(x, y)$ is the light intensity at point (x, y) of the plane).

If $U \in L^1_{\mathrm{loc}}(\Omega)$, one can define the Total Variation of U by

$$\mathrm{TV}(U) = \sup \left\{ -\int_\Omega U \,\mathrm{div}\phi, \quad \phi \in C_c^\infty(\Omega, \mathbb{R}^2), |\phi(x, y)| \leq 1 \ \forall (x, y) \in \Omega \right\}.$$

If $U \in \mathcal{W}^{1,1}(\Omega)$ this definition simplifies into

$$\mathrm{TV}(U) = \int_\Omega |\mathrm{D}U(x, y)| \, dxdy \,.$$

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First proposed for image restoration by Rudin, Osher and Fatemi in 1992, TV is still a very popular choice for image regularization

The L^1 norm promotes sparsity, hence minimizing $TV(U)$ tend to produce images U with sparse gradients (“cartoon” images)

Applications: image deblurring, inpainting, spectrum extrapolation, image decomposition, super-resolution, stereovision, etc.

The discrete TV model

Definition (discrete total variation)

Let Ω a bounded subset of $\mathbb{Z}^2$, and let $u : \Omega \to \mathbb{R}$ a discrete (grayscaled) image. The discrete total variation of u is defined by

$$\mathrm{TV}^{\mathrm{d}}(u) = \|\nabla u\|_{1,2} := \sum_{(x,y)\in\Omega} |\nabla u(x,y)|,$$

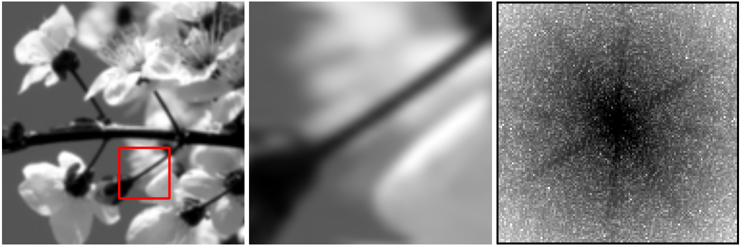
where ∇ denotes a finite differences scheme, typically

$$\nabla u(x,y) = \begin{pmatrix} u(x+1,y) - u(x,y) \\ u(x,y+1) - u(x,y) \end{pmatrix}.$$

Such discretizations produce images that cannot be easily interpolated

Interpolating TV^{d} processed images

Given u_0 compute a minimizer of $E(u) := \|u - u_0\|_2^2 + \lambda \mathrm{TV}^{\mathrm{d}}(u)$.



(a) reference image | bicubic resampling of (a) | spectrum of (a)

(b) TV^{d} processed | bicubic resampling of (b) | spectrum of (b)

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Aim of the present work: propose a new discretization of TV that reconciliates Total Variation minimization with linear interpolation (and in particular Shannon interpolation)

Shannon sampling theory

The Shannon Sampling Theorem states that a band-limited function can be exactly reconstructed from a discrete (but infinite) set of samples.

Theorem (Shannon)

Consider an absolutely integrable function $U : \mathbb{R}^d \to \mathbb{R}$ whose Fourier Transform

$$\forall \xi \in \mathbb{R}^d, \quad \widehat{U}(\xi) = \int_{\mathbb{R}^d} U(x) e^{-i\langle \xi, x \rangle} \, dx \, ,$$

satisfies $\widehat{U}(\xi) = 0$ if $\xi \notin [-\pi, \pi]^d$. Then we have

$$\forall x \in \mathbb{R}^d, \quad U(x) = \sum_{k \in \mathbb{Z}^d} U(k) \, sinc(x - k)$$

noting $sinc((x_1, \ldots, x_d)) = \prod_{j=1}^{d} \frac{\sin(\pi x_j)}{\pi x_j}$, and setting $\frac{sinc(0)}{0} = 1$.

The 2D discrete Shannon interpolation (odd case)

Definition (Shannon interpolate of a 2D image)

Given a discrete domain $\Omega = \{0, \ldots, M-1\} \times \{0, \ldots, N-1\}$, and a signal $u : \Omega \to \mathbb{R}$, we define the discrete Shannon interpolation of u as the (M, N)-periodic trigonometric polynomial $U : \mathbb{R}^2 \to \mathbb{R}$,

$$U(x, y) = \frac{1}{MN} \sum_{\substack{-\frac{M}{2} < \alpha < \frac{M}{2} \\ -\frac{N}{2} < \beta < \frac{N}{2}}} \widehat{u}(\alpha, \beta)\, e^{2i\pi\left(\frac{\alpha x}{M} + \frac{\beta y}{N}\right)}$$

if M and N are odd integers.

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where ε_M and ε_N are defined by

$$\varepsilon_M(\alpha) = \begin{cases} 1 & \text{if } |\alpha| < M/2 \\ 1/2 & \text{if } |\alpha| = M/2 \end{cases} \qquad \varepsilon_N(\beta) = \begin{cases} 1 & \text{if } |\alpha| < N/2 \\ 1/2 & \text{if } |\alpha| = N/2 \end{cases}$$

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This interpolation can be used to efficiently compute subpixellic geometrical transforms (rotations, translations, zoom, etc.)

The Shannon total variation

We call Shannon total variation of the discrete image u the exact continuous total variation of U.

Definition (Shannon total variation)

$$\mathrm{STV}_\infty(u) := \mathrm{TV}(U) = \int_{[0,M]\times[0,N]} |\mathrm{D}U(x,y)|\,dxdy\,.$$

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For practical implementation, we can approximate $\mathrm{STV}_\infty(u)$ using a Riemann sum (in practice we use an oversampling factor $n = 2$ or 3).

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Definition (STV_n)

For any integer $n \geq 1$, set

$$\mathrm{STV}_n(u) = \frac{1}{n^2} \sum_{(k,l)\in\Omega_n} \left|\mathrm{D}U\left(\tfrac{k}{n}, \tfrac{l}{n}\right)\right| = \frac{1}{n^2} \sum_{(k,l)\in\Omega_n} |\mathrm{D}_n u(k,l)|\ ,$$

where $\mathrm{D}_n u(k,l) = \mathrm{D}U(\frac{k}{n}, \frac{l}{n})$, and $\Omega_n = \{0, \ldots, nM-1\} \times \{0, \ldots, nN-1\}$.

Numerical computation of $\mathrm{STV}_n(u)$

The following proposition shows how $\mathrm{D}_n u$ can be efficiently computed in the Fourier domain.

Proposition (fast computation of $\mathrm{D}_n u$)

Let $n > 1$ and $\widehat{\Omega_n} := \left[-\frac{nM}{2}, \frac{nM}{2}\right) \times \left[-\frac{nM}{2}, \frac{nM}{2}\right) \cap \mathbb{Z}^2$ denote the frequency domain associated to Ω_n. For any $(\alpha, \beta) \in \widehat{\Omega_n}$, we have

$$\widehat{\mathrm{D}_n u}(\alpha, \beta) = n^2\, \varepsilon_M(\alpha)\varepsilon_N(\beta)\, Z_n\widehat{u}(\alpha, \beta)\, 2i\pi \begin{pmatrix} \alpha/M \\ \beta/N \end{pmatrix},$$

where

$$Z_n\widehat{u}(\alpha, \beta) = \begin{cases} \widehat{u}(\alpha, \beta) & \text{if } |\alpha| \leq \frac{M}{2}, \text{and } |\beta| \leq \frac{N}{2} \\ 0 & \text{otherwise.} \end{cases}$$

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Besides, we have the upper bound $|||\mathrm{D}_n||| \le \pi n\sqrt{2}$.

Numerical computation of $\mathrm{STV}_n(u)$

We set $\mathrm{div}_n = -\mathrm{D}_n^*$, by analogy with the continuous setting.

Proposition (fast computation of $\mathrm{div}_n(p)$)

For any $n > 1$ and any $p = (p_x, p_y) : \Omega_n \to \mathbb{R}^2$, we have

$$\forall (\alpha,\beta) \in \widehat{\Omega}, \quad \widehat{\mathrm{div}_n(p)}(\alpha,\beta) = 2i\pi \left(\frac{\alpha}{M} h_{\widehat{p_x}}(\alpha,\beta) + \frac{\beta}{N} h_{\widehat{p_y}}(\alpha,\beta) \right),$$

where

$$h_{\widehat{p_x}}(\alpha,\beta) = \begin{cases} \widehat{p_x}(\alpha,\beta) & \text{if } |\alpha| < \frac{M}{2},\ |\beta| < \frac{N}{2} \\ \frac{1}{2}\left(\widehat{p_x}(\alpha,\beta) - \widehat{p_x}(-\alpha,\beta)\right) & \text{if } \alpha = -\frac{M}{2},\ |\beta| < \frac{N}{2} \\ \frac{1}{2}\left(\widehat{p_x}(\alpha,\beta) + \widehat{p_x}(\alpha,-\beta)\right) & \text{if } |\alpha| < \frac{M}{2},\ \beta = -\frac{N}{2} \\ \frac{1}{4} \sum\limits_{\substack{s_1=\pm 1 \\ s_2=\pm 1}} s_1\, \widehat{p_x}(s_1\alpha, s_2\beta) & \text{if } (\alpha,\beta) = (-\frac{M}{2}, -\frac{N}{2}) \end{cases}$$

and a similar definition stands for $h_{\widehat{p_y}}(\alpha,\beta)$.

Dual formulation

As in the discrete setting, a dual formulation of STV_n can be easily derived.

Proposition (dual formulation of STV_n)

$$\mathrm{STV}_n(u) = \max_{p:\Omega_n \to \mathbb{R}^2} \ \langle \tfrac{1}{n^2} \mathrm{D}_n u, p \rangle - \delta_{\mathscr{B}_*}(p)$$

where

$$\delta_{\mathscr{B}_*}(p) = \begin{cases} 0 & \text{if } \displaystyle\max_{(x,y)\in\Omega_n} |p(x,y)| \leq 1\,, \\ +\infty & \text{otherwise}\,. \end{cases}$$

Sketch of proof.

1. The Legendre-Fenchel transform of $\|\cdot\|_{1,2}$ is $\|\cdot\|_{1,2}^{\star} = \delta_{\mathscr{B}_*}$,
2. thus $\mathrm{STV}_n(u) = \|\frac{1}{n^2}\mathrm{D}_n u\|_{1,2} = \|\frac{1}{n^2}\mathrm{D}_n u\|_{1,2}^{\star\star} = \delta_{\mathscr{B}_*}^{\star}(\frac{1}{n^2}\mathrm{D}_n u)$,
3. besides, the supremum involved in $\delta_{\mathscr{B}_*}^{\star}$ is a maximum.

Image denoising

Given a noisy image u_0, we consider the STV_n variant of the ROF model

$$\operatorname*{argmin}_{u:\Omega\to\mathbb{R}} \|u-u_0\|_2^2 + \lambda \mathrm{STV}_n(u)\,,$$

with primal-dual reformulation

$$\operatorname*{argmin}_{u:\Omega\to\mathbb{R}} \max_{p:\Omega_n\to\mathbb{R}^2} \|u-u_0\|_2^2 + \langle \tfrac{\lambda}{n^2}\mathrm{D}_n u, p\rangle - \delta_{\mathscr{B}_*}(p)\,,$$

for which a solution can be numerically computed using the Chambolle-Pock algorithm[1].

[1] **A. Chambolle, T. Pock**: “A first-order primal-dual algorithm for convex problems with applications to imaging”, Journal of Mathematical Imaging and Vision, 2011

Image denoising

Primal-dual saddle-point problem:

$$\underset{u:\Omega\to\mathbb{R}}{\operatorname{argmin}} \max_{p:\Omega_n\to\mathbb{R}^2} \|u-u_0\|_2^2 + \langle \tfrac{\lambda}{n^2}\mathrm{D}_n u, p\rangle - \delta_{\mathscr{B}_*}(p)\,,$$

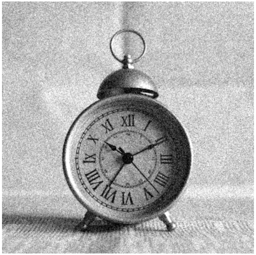
Chambolle-Pock Algorithm

Initialization: Choose $\tau,\sigma>0$ such as $\tau\sigma|||\frac{\lambda}{n^2}\mathrm{D}_n|||^2<1$, $p^0\in\mathbb{R}^{2|\Omega_n|}$ and $u^0\in\mathbb{R}^\Omega$ (for instance $p^0=0$ and $u^0=u_0$). Set $\bar{u}^0=u^0$.

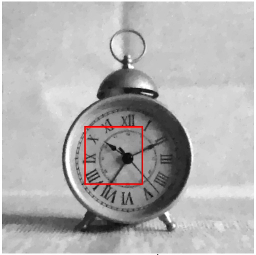
Iterations: For $k\geq 1$, update p^k, u^k and $\bar{u}^k$ as follows,

- $p^{k+1}(x,y)=\dfrac{p^k(x,y)+\frac{\sigma\lambda}{n^2}\mathrm{D}_n\bar{u}^k(x,y)}{\max\left(1,\left|p^k(x,y)+\frac{\sigma\lambda}{n^2}\mathrm{D}_n\bar{u}^k(x,y)\right|\right)}$
- $u^{k+1}(x,y)=\dfrac{u^k(x,y)+\frac{\tau\lambda}{n^2}\mathrm{div}_n(p^{k+1})(x,y)+2\tau u_0(x,y)}{1+2\tau}$
- $\bar{u}^{k+1}(x,y)=2\,u^{k+1}(x,y)-u^k(x,y)$

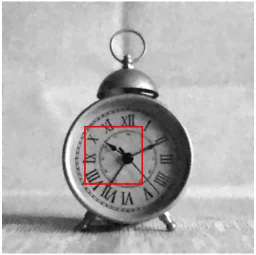
Image denoising



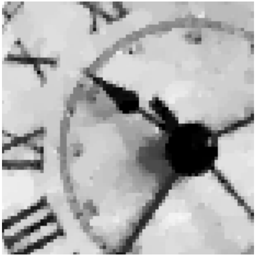
(a) noisy image ($\sigma = 20$)



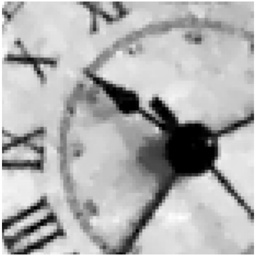
(b) TV^d



(c) STV_3

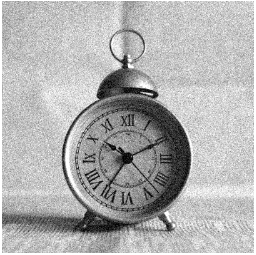


details of (b)



details of (c)

Image denoising



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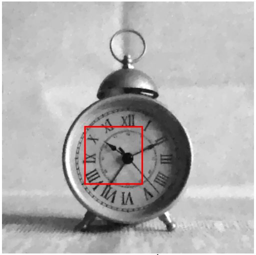
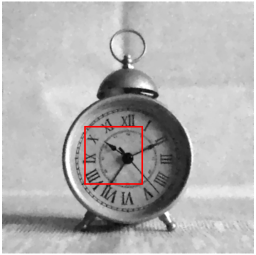


(b) TV^d

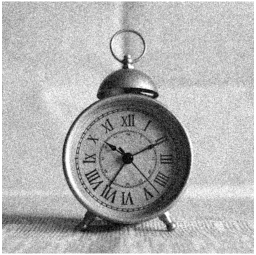


(c) STV_3

bicubic resampling of (b)

bicubic resampling of (c)

Image denoising



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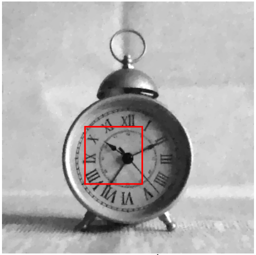
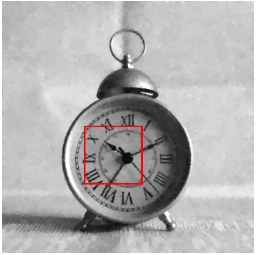


(b) TV^d



(c) STV_3

Shannon resampling of (b)

Shannon resampling of (c)

Inverse problems

We can also use STV_n as a regularizer for inverse problem. Given a linear operator $A : \mathbb{R}^\Omega \to \mathbb{R}^\omega$, and $u_0 : \omega \to \mathbb{R}$,

$$\underset{u:\Omega\to\mathbb{R}}{\operatorname{argmin}} \ \underbrace{\|Au - u_0\|_2^2}_{f(Au)} + \lambda \mathrm{STV}_n(u)\,,$$

with primal-dual reformulation (use $f(Au) = f^{\star\star}(Au)$)

$$\underset{u:\Omega\to\mathbb{R}}{\operatorname{argmin}} \ \max_{\substack{p:\Omega_n\to\mathbb{R}^2 \\ q:\omega\to\mathbb{R}}} \ \left\langle \left(\tfrac{\lambda}{n^2} \mathrm{D}_n u, Au \right), (p, q) \right\rangle - \left(\delta_{\mathscr{B}_*}(p) + \| \tfrac{q}{2} + u_0 \|_2^2 \right).$$

and the Chambolle-Pock algorithm can be used again.

Inverse problems

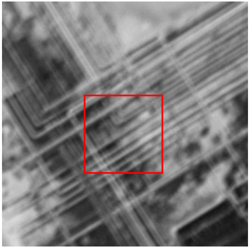
Chambolle-Pock Algorithm

Initialization: Choose $\tau, \sigma > 0$ such as $\tau\sigma\left(|||\frac{\lambda}{n^2}\mathrm{D}_n|||^2 + |||A|||^2\right) < 1$, $(p^0, q^0) \in \mathbb{R}^{2|\Omega_n|} \times \mathbb{R}^{\omega}$ and $u^0 \in \mathbb{R}^{\Omega}$ (for instance $p^0 = 0$, $q^0 = 0$ and $u^0 = u_0$). Set $\bar{u}^0 = u^0$.

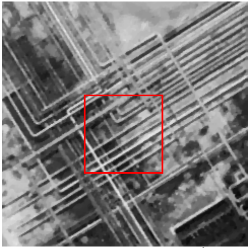
Iterations: For $k \geq 1$, update p^k, q^k u^k and $\bar{u}^k$ as follows,

- $p^{k+1}(x,y) = \dfrac{p^k(x,y) + \frac{\sigma\lambda}{n^2}\mathrm{D}_n\bar{u}^k(x,y)}{\max\left(1, \left|p^k(x,y) + \frac{\sigma\lambda}{n^2}\mathrm{D}_n\bar{u}^k(x,y)\right|\right)}$
- $q^{k+1}(x,y) = \dfrac{2\,q^k(x,y) + 2\sigma\left(A\bar{u}^k - u_0\right)}{2+\sigma}$
- $u^{k+1}(x,y) = u^k(x,y) + \frac{\tau\lambda}{n^2}\mathrm{div}_n(p^{k+1})(x,y) - \tau A^* q^{k+1}(x,y)$
- $\bar{u}^{k+1}(x,y) = 2\,u^{k+1}(x,y) - u^k(x,y)$

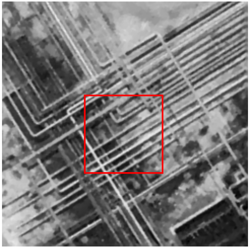
Consider that $Au = k * u$ is the convolution between u and a given motion blur kernel k.



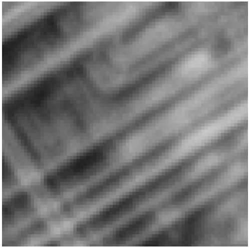
(a) blurry and noisy ($\sigma = 2$)



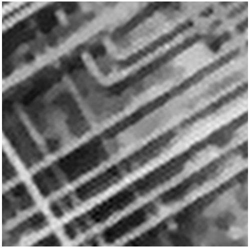
(b) discrete TV (TV^{d})



(c) Shannon TV (STV_3)



details of (a)

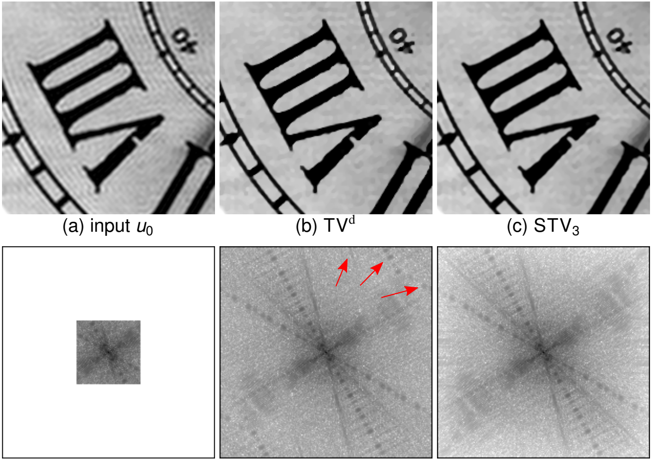


Shannon resampling of (b)

Shannon resampling of (c)

Now A is a frequency masking operator, of the type

$$\widehat{Au}(\alpha,\beta) = \begin{cases} \widehat{u}(\alpha,\beta) & \text{if } (\alpha,\beta) \in \widehat{\omega_0}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$



(a) input u_0 (b) TV^{d} (c) STV_3

spectrum of (a) spectrum of (b) spectrum of (c)

Spectrum extrapolation

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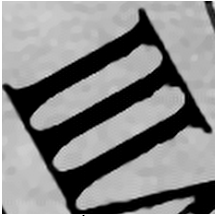
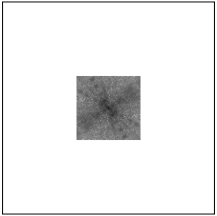
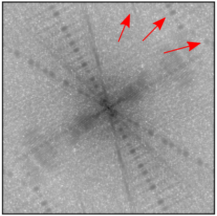


(b) TV^{d} (resampled)

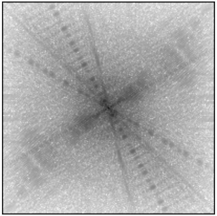
(c) STV_3 (resampled)



spectrum of (a)



spectrum of (b)



spectrum of (c)

Regularization with weighted frequencies

Given an input image $u_0 : \Omega \to \mathbb{R}$ and a weight mapping $\xi : \Omega \to \mathbb{R}_+$, we compute

$$\underset{u:\Omega\to\mathbb{R}}{\operatorname{argmin}} \quad \|\widehat{u} - \widehat{u_0}\|_\xi^2 + \lambda \mathrm{STV}_n(u)\,,$$

where

$$\|\widehat{u} - \widehat{u_0}\|_\xi^2 = \sum_{(\alpha,\beta)\in\widehat{\Omega}} \xi(\alpha,\beta) \cdot |\widehat{u}(\alpha,\beta) - \widehat{u_0}(\alpha,\beta)|^2\,,$$

is a weighted ℓ^2 square distance between u and u_0, which makes the regularization adaptative with respect to the frequency.

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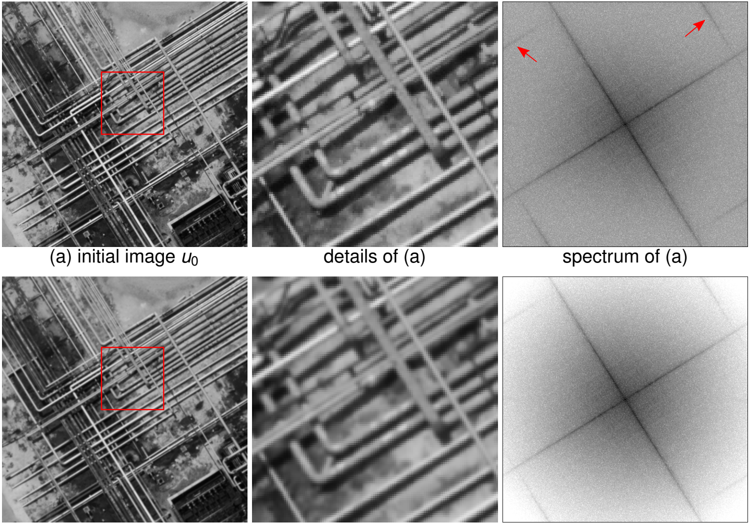
An interesting choice of weighting:

$$\forall(\alpha,\beta)\in\widehat{\Omega},\quad \xi(\alpha,\beta) = e^{-\pi^2\sigma^2\left(\frac{\alpha^2}{M^2}+\frac{\beta^2}{N^2}\right)}$$

- for low frequencies, $\xi(\alpha,\beta)$ is high, it enforces $\widehat{u}(\alpha,\beta)\approx\widehat{u_0}(\alpha,\beta)$;
- for high frequencies, $\widehat{u}(\alpha,\beta)$ is driven by the regularity term STV_n.

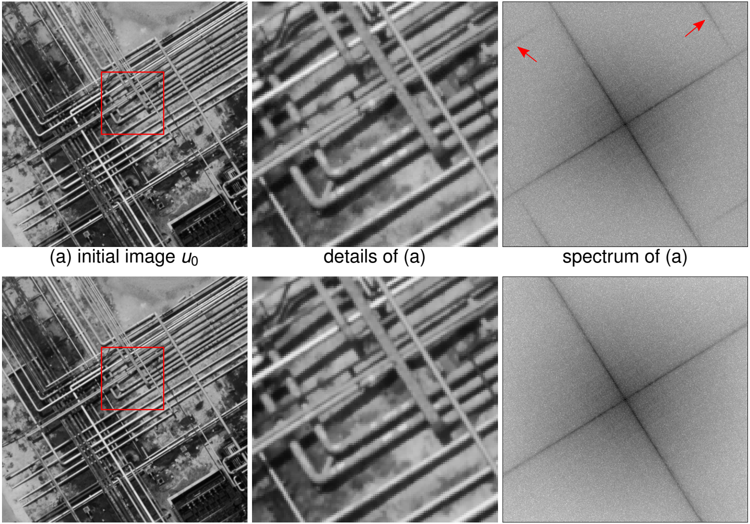
Regularization with weighted frequencies



(a) initial image u_0 | details of (a) | spectrum of (a)

(b) frequency attenuation | details of (b) | spectrum of (b)

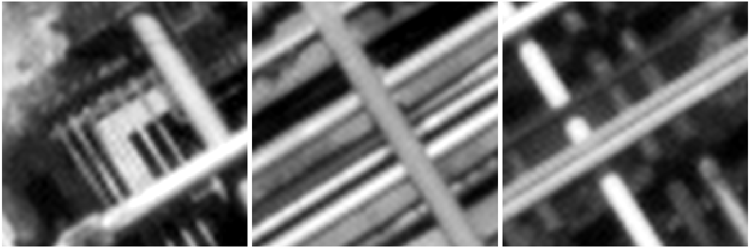
Regularization with weighted frequencies



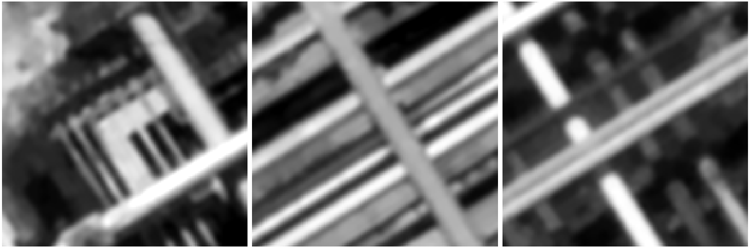
(a) initial image u_0 — details of (a) — spectrum of (a)

(c) restored with STV_3 — details of (c) — spectrum of (c)

Regularization with weighted frequencies



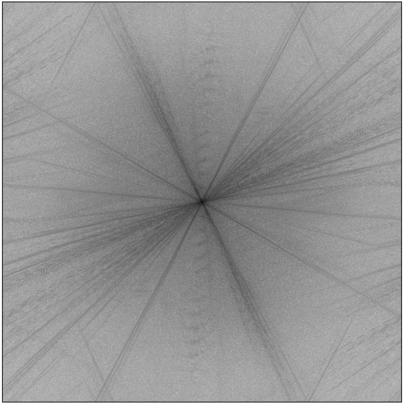
Shannon resamplings of the initial image



Shannon resamplings STV-processed image

Regularization with weighted frequencies

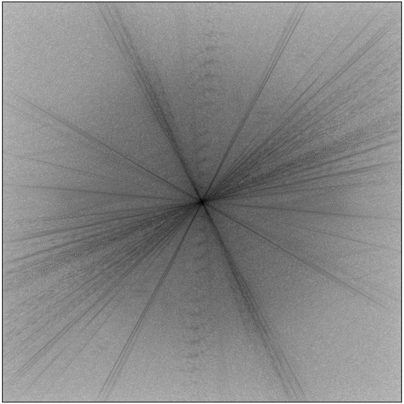
(a) initial image



spectrum of (a)

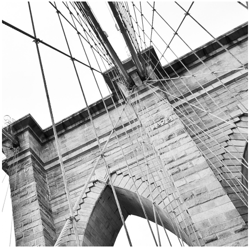
Regularization with weighted frequencies

(b) restored with STV_3



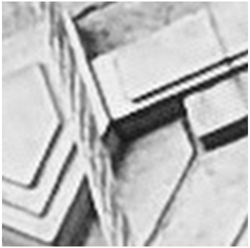
spectrum of (b)

Regularization with weighted frequencies

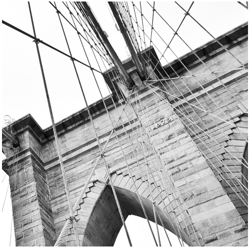


(a) initial image

details of (a)



Shannon resampling of (a)



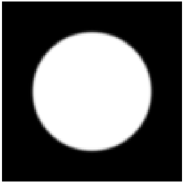
(b) restored with STV_3

details of (b)

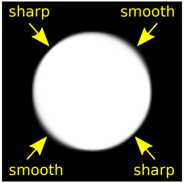
Shannon resampling of (b)

Denoising a rotationally invariant image

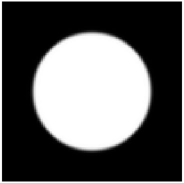
Use ROF to denoise a rotationally invariant image.



(a) reference

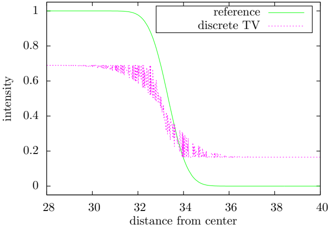


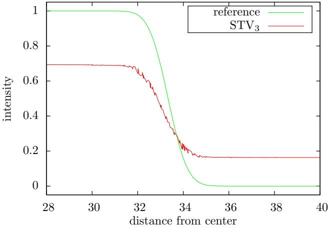
(b) discrete TV



(c) Shannon TV

Control isotropy level by displaying the gray-levels as a function of the distance from the center of the image.





Improvement of the isotropy

We can force the isotropy of the frequency support by adding the constraint $\mathrm{Supp}(\widehat{u}) \subset \mathcal{D}_{\widehat{\Omega}}$, where

$$\mathcal{D}_{\widehat{\Omega}} = \left\{ (\alpha, \beta) \in \widehat{\Omega},\ \left(\frac{\alpha}{M/2}\right)^2 + \left(\frac{\beta}{N/2}\right)^2 \leq 1 \right\} .$$

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We consider the constrained problem

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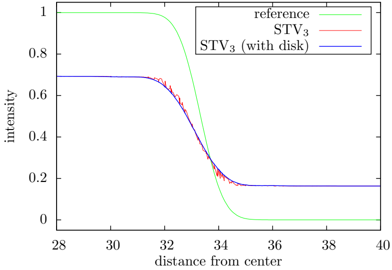

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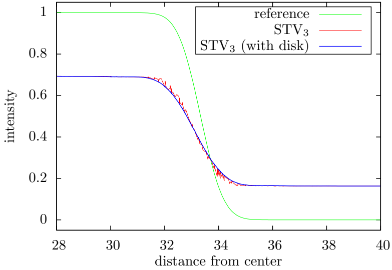

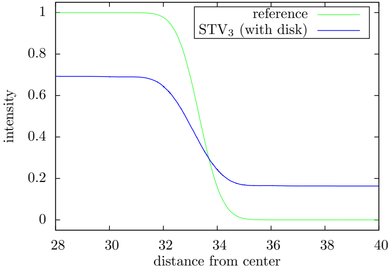

Conclusion

We studied a **Fourier-based TV model** called **STV**.

- STV reconciliates TV regularization with Shannon interpolation.
- STV-based minimization problems can be handled using classical duality tools.
- The STV model comes at the expense of a few Fourier Transforms at each iteration of the optimization process, which is an affordable cost considering the strong benefits in terms of image quality.
- Preliminary results indicate an excellent level of isotropy offered by the STV model.
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Thank you!