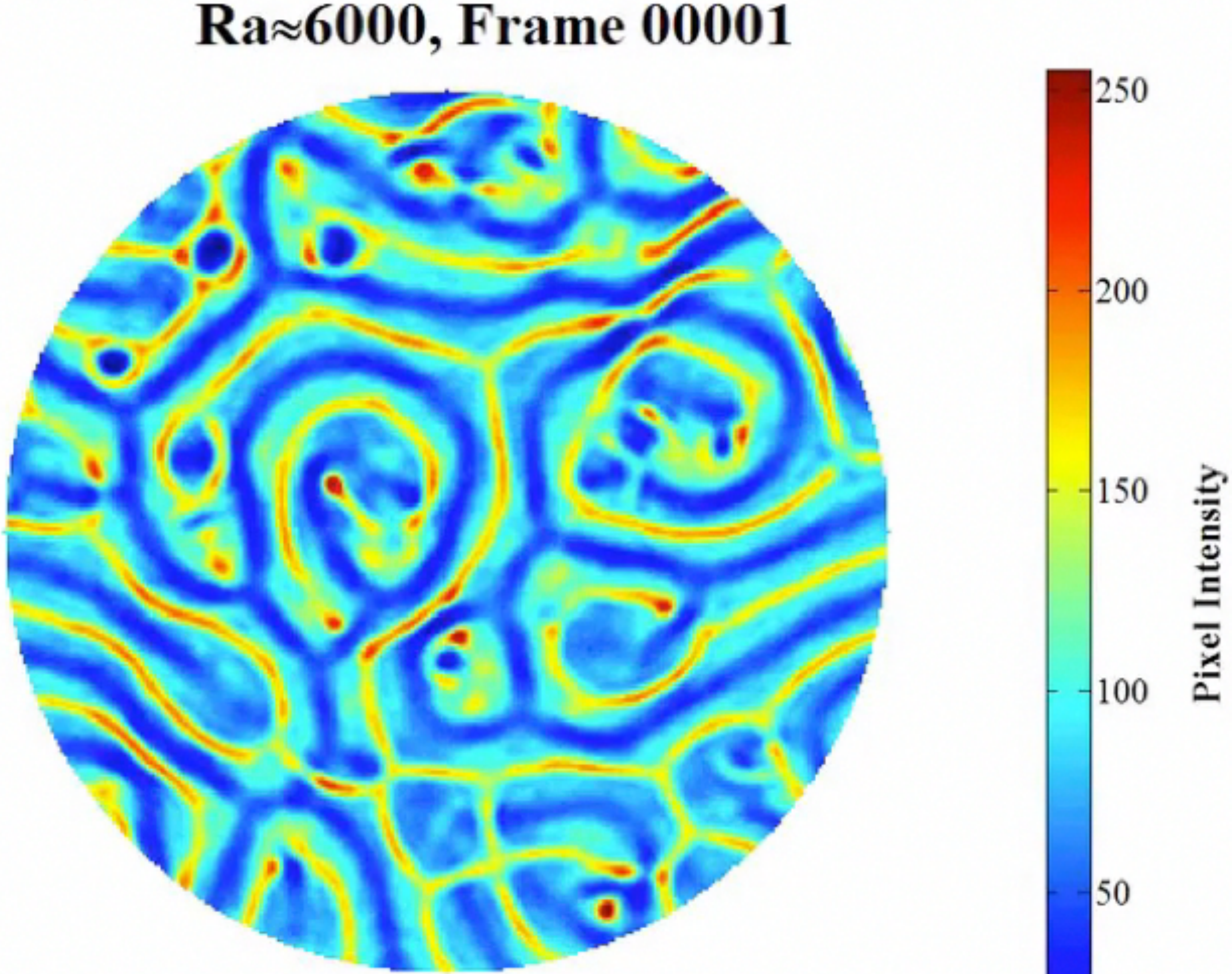
Nonlinear Dynamics, High Dimensional Data, and Persistent Homology

Konstantin Mischaikow

Dept. of Mathematics, Rutgers
mischaik@math.rutgers.edu

CIRM, Nov. 2016

Rayleigh-Benard Convection - Experiments



M. Schatz, GaTech

How can we characterize the dynamics in a comprehensible manner?

Rayleigh-Benard Convection - Numerics

$$Pr^{-1}\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \nabla^2\mathbf{u} + RaT\hat{\mathbf{z}}$$

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \nabla^2 T$$

$$\nabla\cdot\mathbf{u} = 0$$

$\mathbf{u}(x,y,z,t)$ is velocity field

$p(x,y,z,t)$ is pressure field

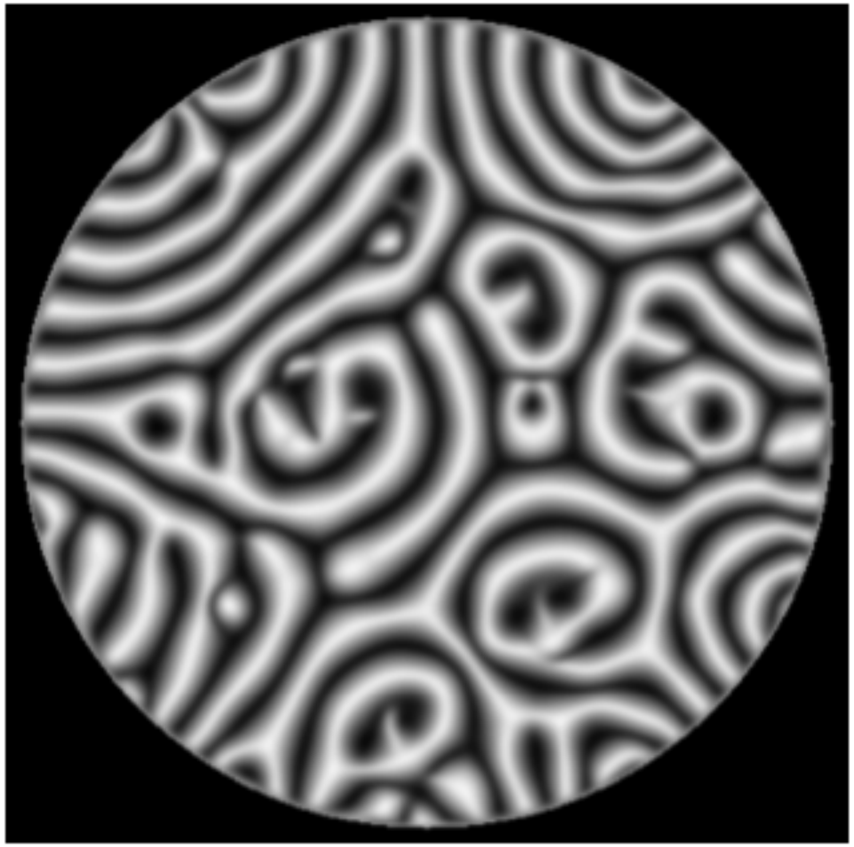
$T(x,y,z,t)$ is temperature field

Boussinesq Equations

Galerkin Projection

$$\dot{x} = F(x), \quad x \in \mathbb{R}^d$$

d>>1



M. Paul, VaTech

Do numerics and experiment agree?

Simple characterization????

Dense Granular Media

Dense Granular Media

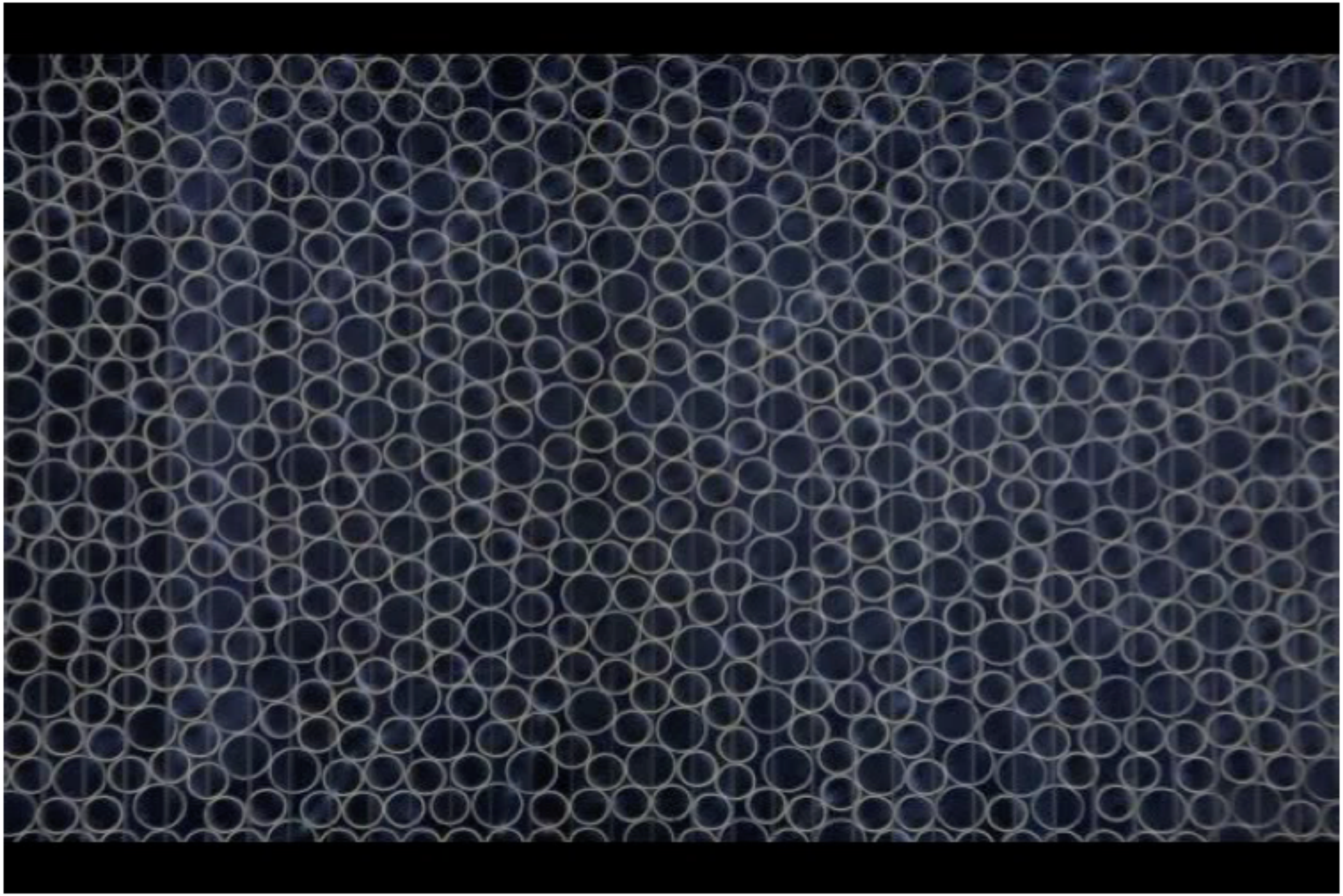
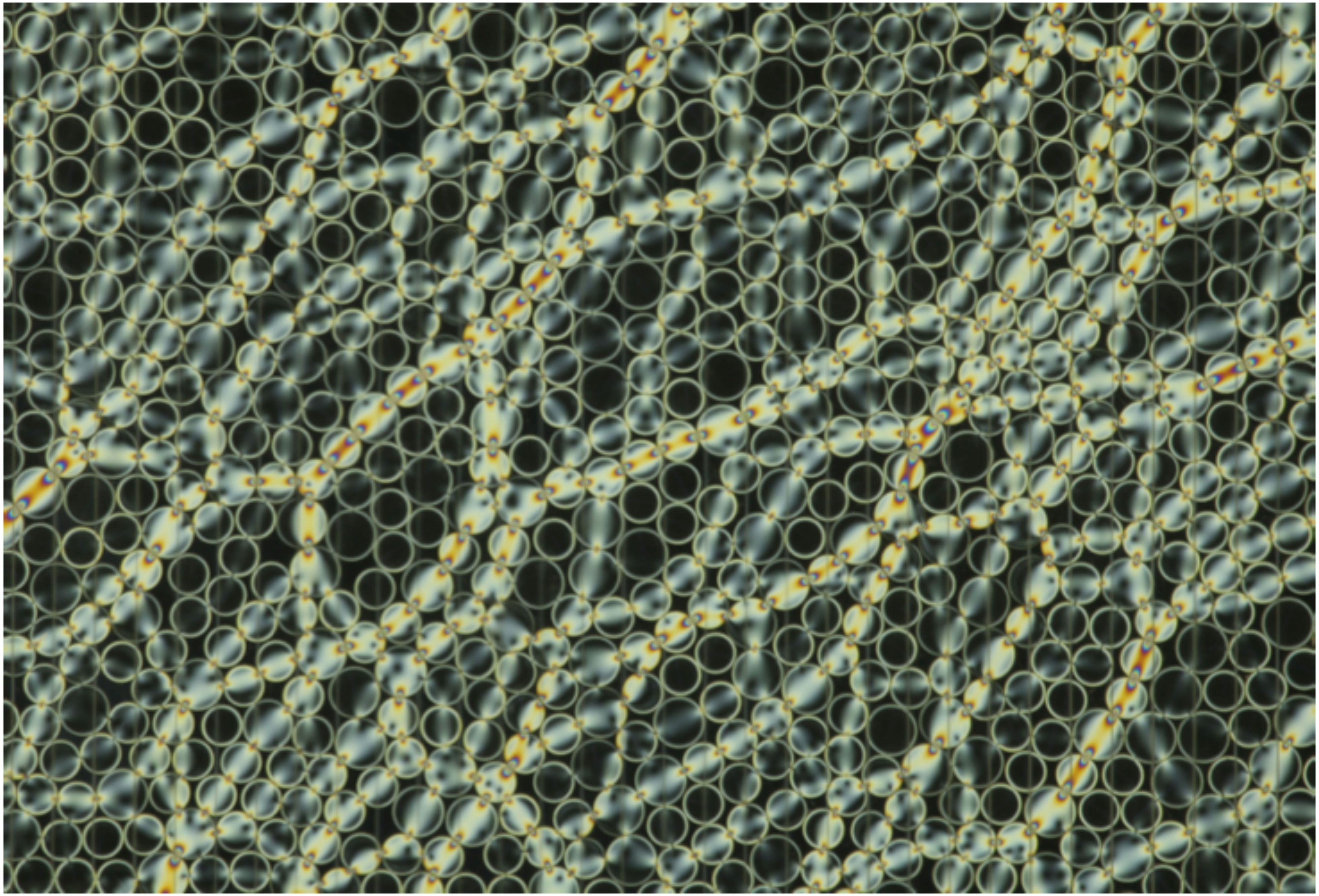


Photo elastic particles placed between two cross polarizers.

R. P. Behringer's lab

Dense Granular Media

Dense Granular Media



Summary: (1) The natural phase space for these systems is infinite dimensional.

(2) Approximations (both numerical and experimental) are high dimensional.

Goal: (1) Because I am trained in dynamical systems I would like to describe this dynamics in terms of low dimensional invariant structures (fixed points, periodic orbits, heteroclinic orbits, symbolic dynamics).

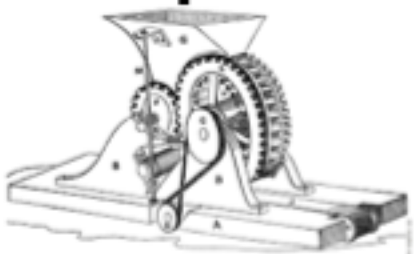
(2) I would like this know that, at least theoretically, the description is mathematically rigorous.

Is this a realistic goal?

Infinite dimensional map

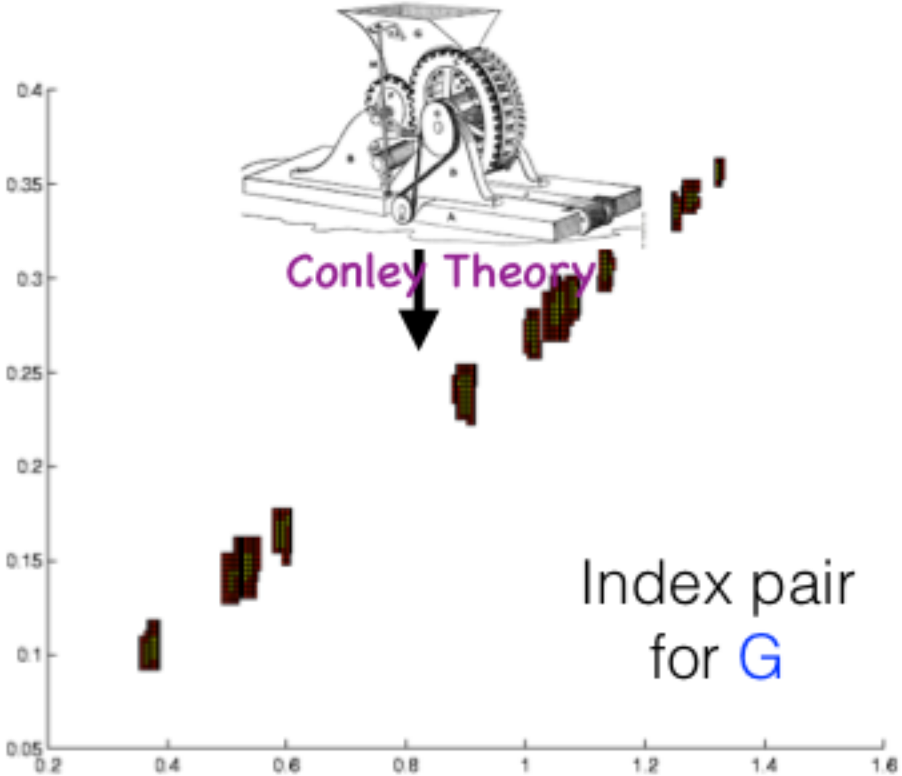
Infinite dimensional map

$$u \mapsto F[u](\bullet) = \int_{-\pi}^{\pi} D(|\bullet - y|)\mu u(y)\left[1 - \frac{u(y)}{c(y)}\right] dy \qquad u\colon [-\pi,\pi] \to \mathbb{R}$$



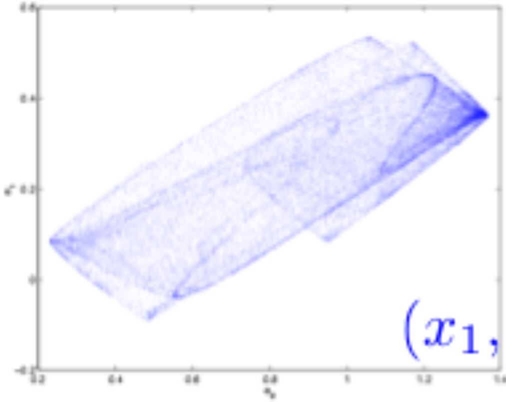
Galerkin Projection

$G\colon \mathbb{R}^6 \to \mathbb{R}^6$

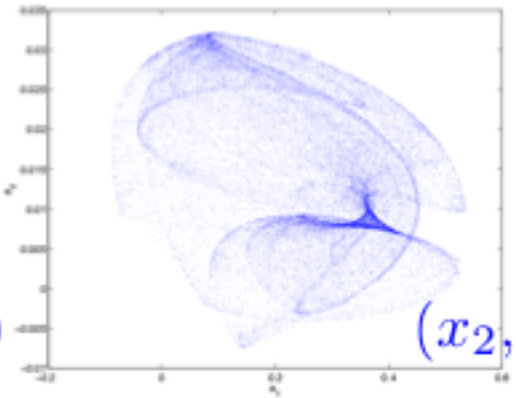


Conley Theory

Index pair for G

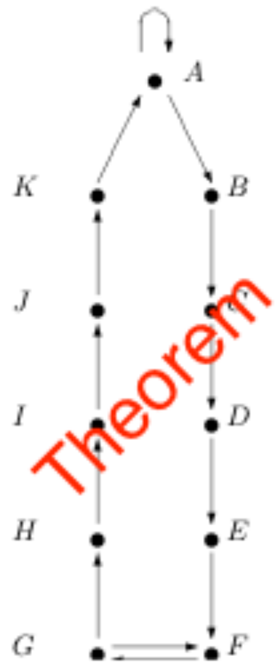


(x_1, x_2)



(x_2, x_3)

"time series data for a infinite dimensional map"



Theorem

S. Day, O. Junge, K.M.
SIADS (2004)

Summary: (1) The natural phase space for these systems is infinite dimensional.

(2) Approximations (both numerical and experimental) are high dimensional.

Long Term Goal: Because I am trained in dynamical systems I would like to describe this dynamics in terms of low dimensional invariant structures (fixed points, periodic orbits, heteroclinic orbits, symbolic dynamics).

Observation: Evolution of the patterns is of central interest.

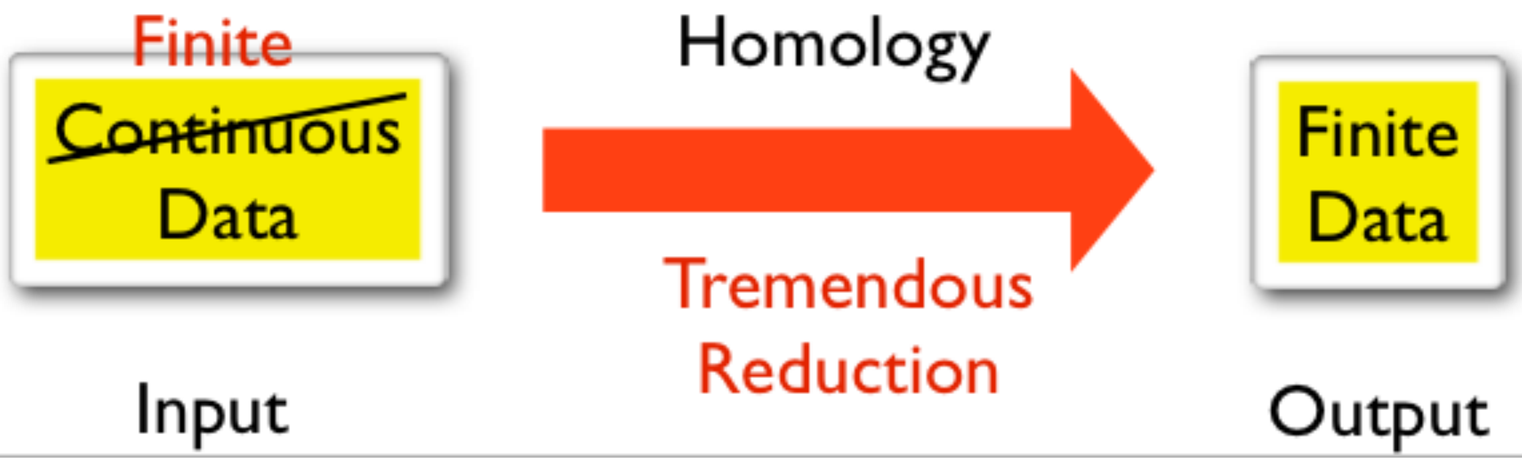
Immediate Goals: (1) Perform dimension reductions that preserve geometric information.

(2) Reconstruct dynamics from reduced system.

Persistent Homology

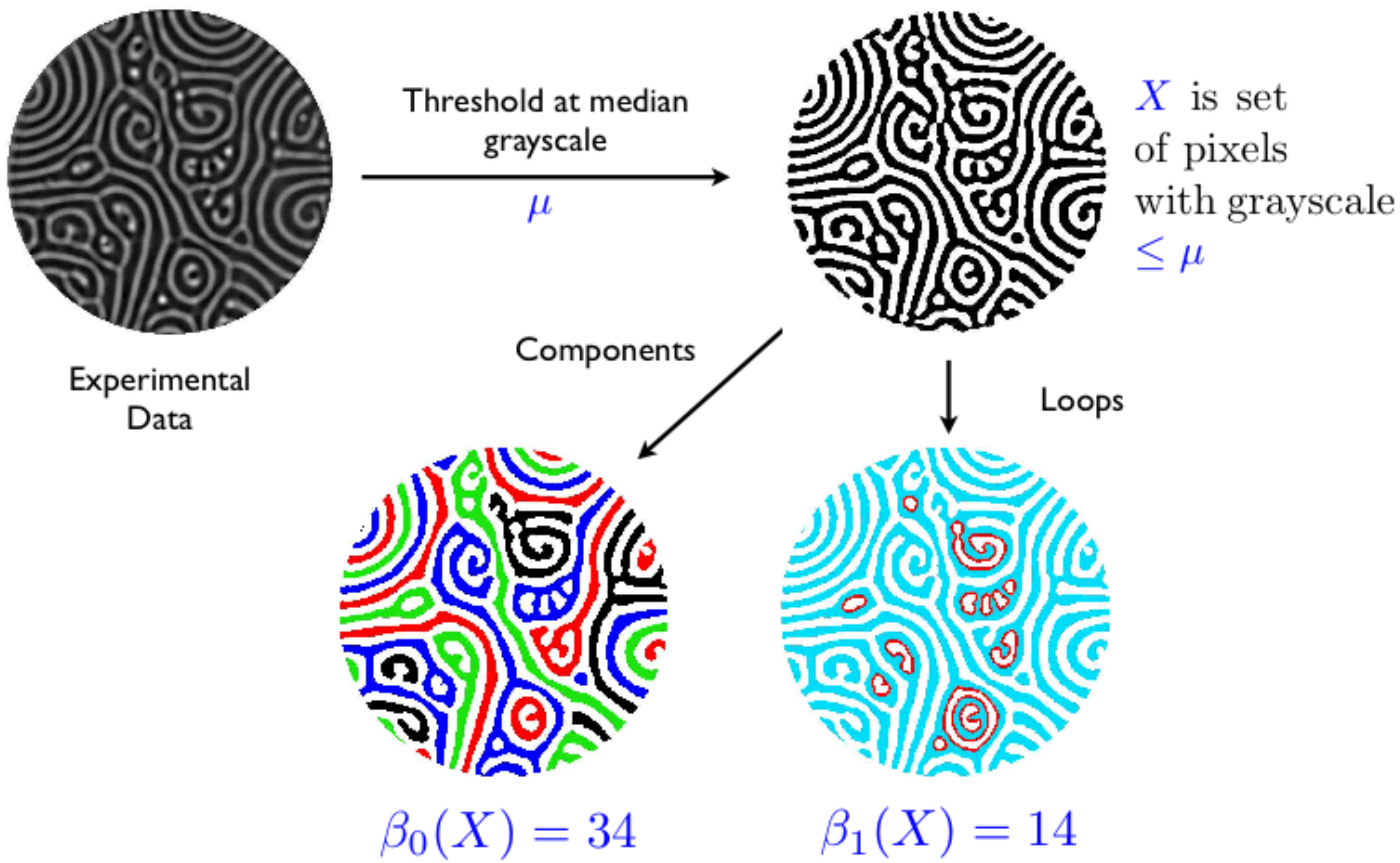
What do we need to know about Homology?

Topological Space		~~Homology Groups~~ Vector Spaces	
X	→	$H_k(X)$	$\beta_k(X) = \dim H_k(X)$
			Counts number of k dimensional holes
Continuous Map		Linear Maps	
$f : X \to Y$	→	$f_k : H_k(X) \to H_k(Y)$	Indicates how holes map to holes



Shape of a Density Function via Homology

Shape of a Density Function via Homology

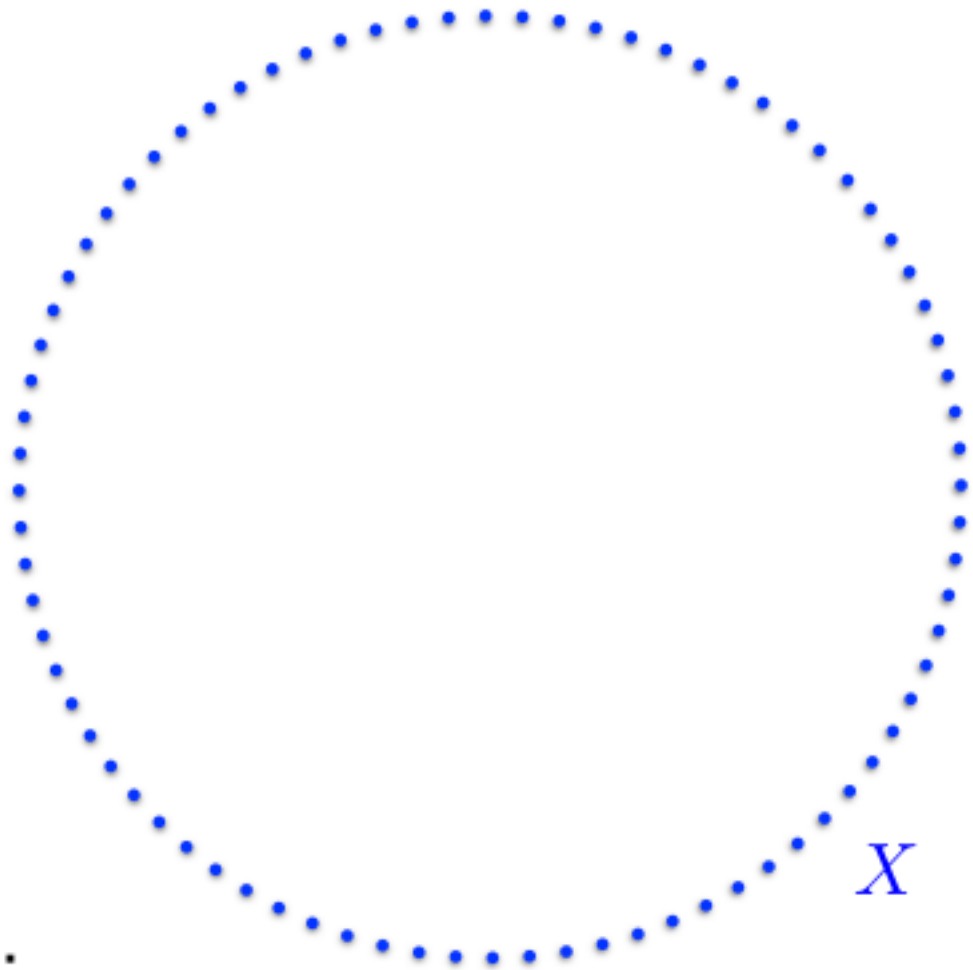


Choice of a unique threshold is Artificial

Topology of a Time Series

Time series from

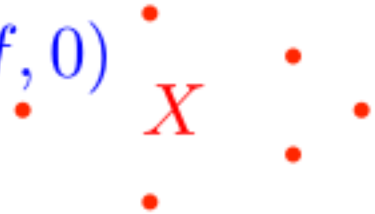
$$\dot{x} = y$$
$$\dot{y} = -x$$



Homology of time series data:

$$\beta_0(X) \approx 100$$
$$\beta_1(X) = 0$$

$\mathsf{Fn}(f, 0)$ X



$f : \mathbb{R}^2 \to \mathbb{R}$

$f(y) := \mathsf{dist}(y, X)$

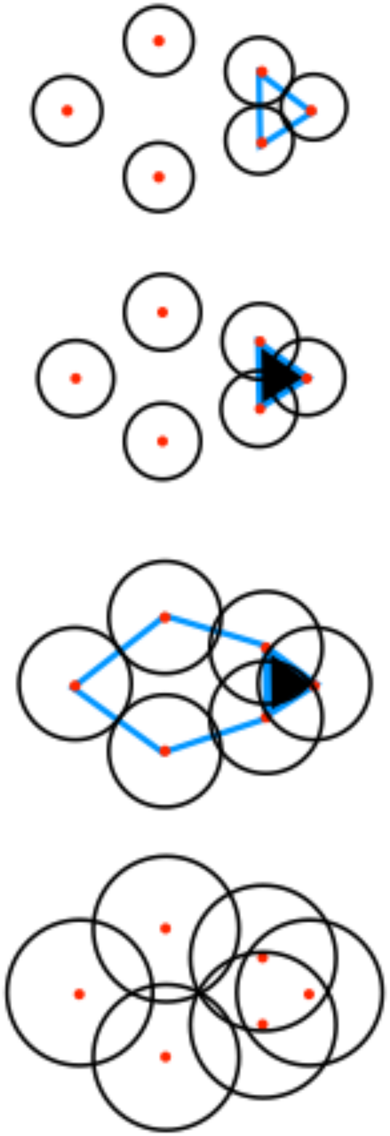
$\mathsf{Fn}(f, \theta) := \{y \mid f(y) \leq \theta\}$

$\mathsf{Fn}(f, 60)$

$\mathsf{Fn}(f, 70)$

$\mathsf{Fn}(f, 100)$

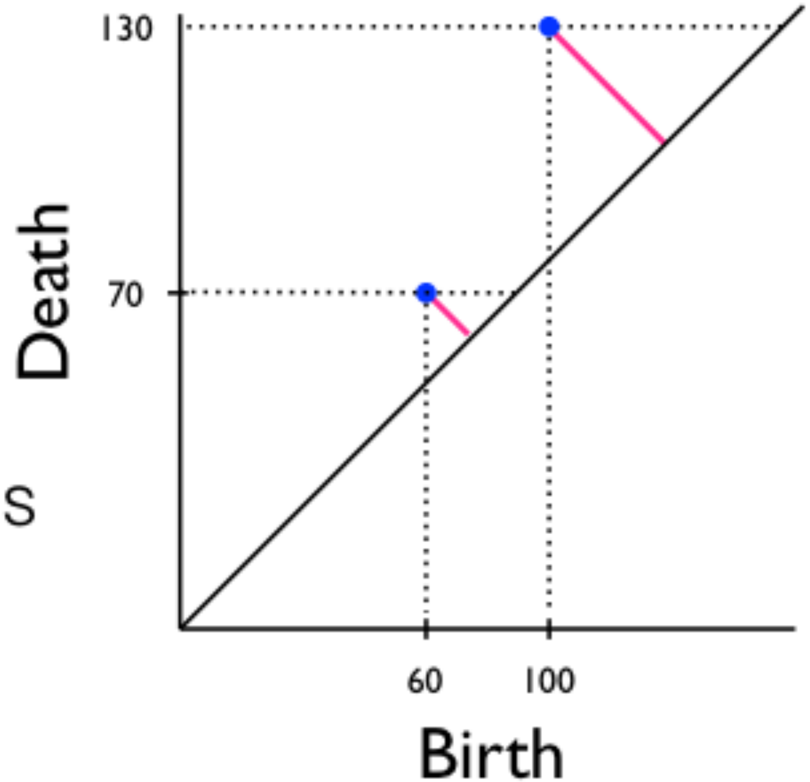
$\mathsf{Fn}(f, 130)$



Cech Complex

Vietoris-Rips Complex

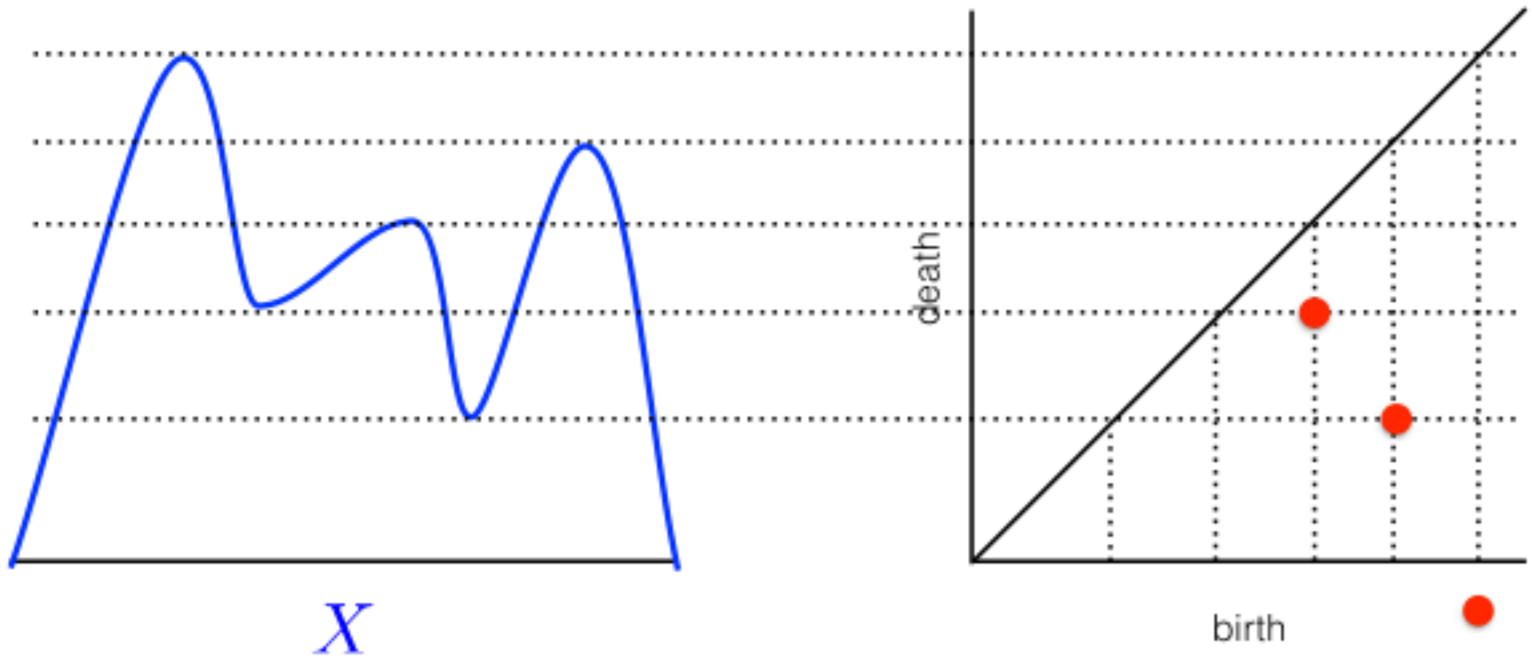
Significance of a feature



β_1 Persistence Diagram

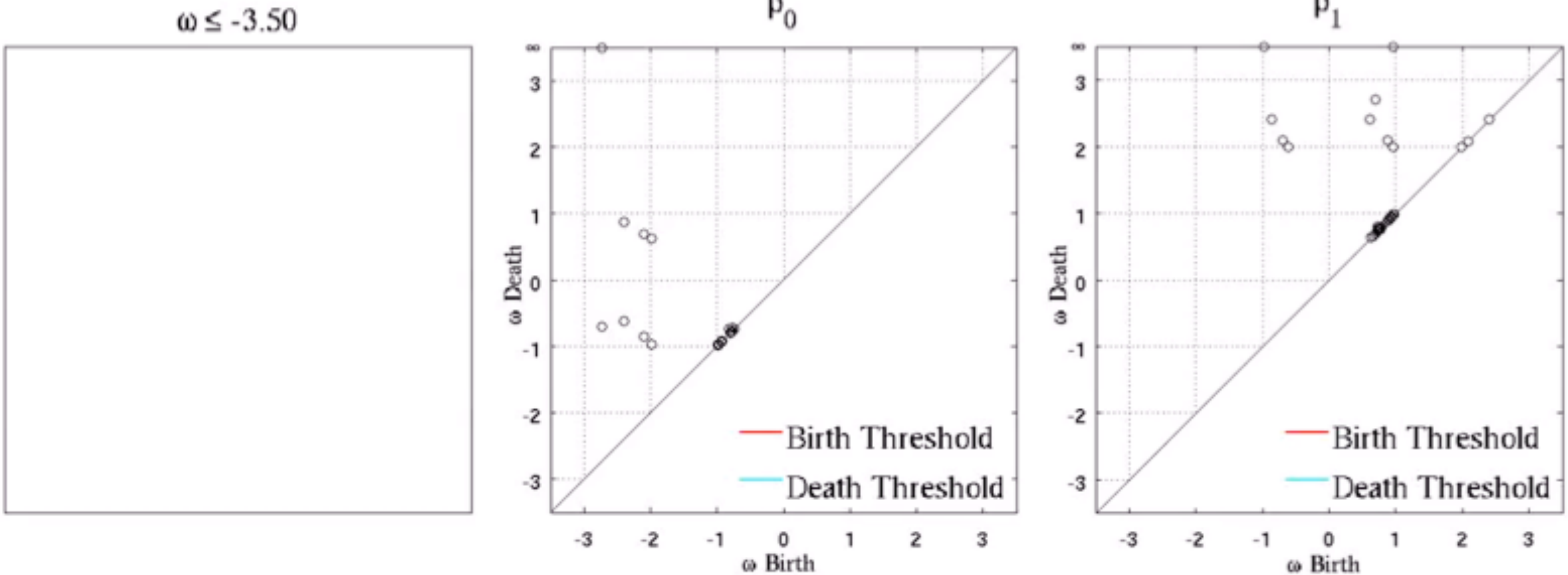
Persistent Homology (II)

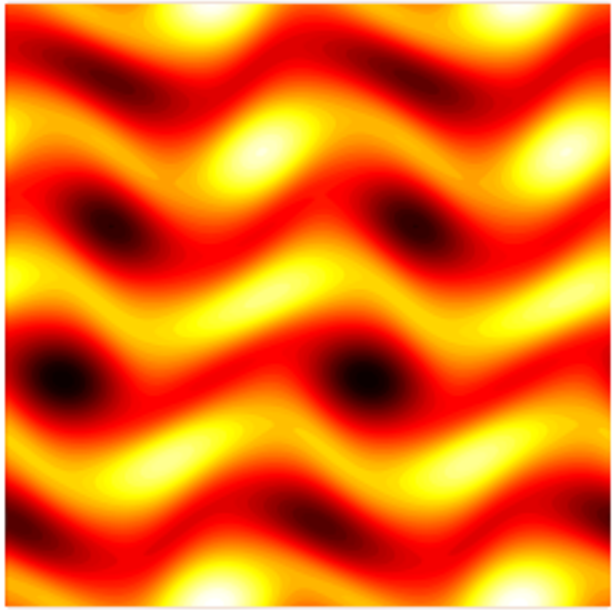
$f : X \to \mathbb{R}$



β_0 persistence diagram

$\mathsf{Fn}(f, 0) := \{x \mid f(x) \geq \theta\}$

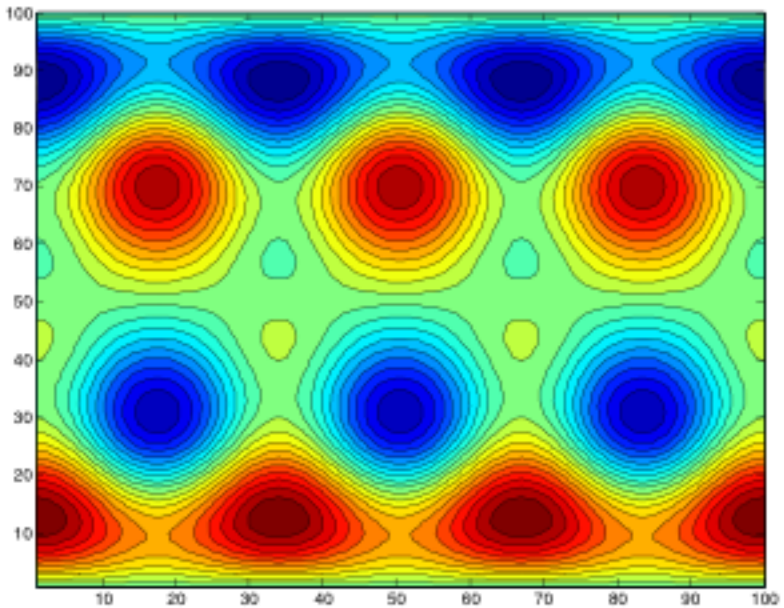


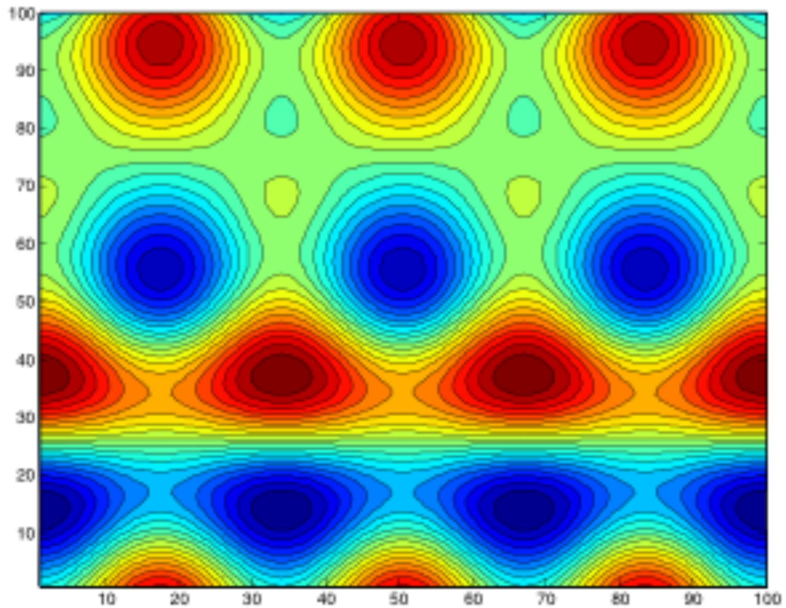


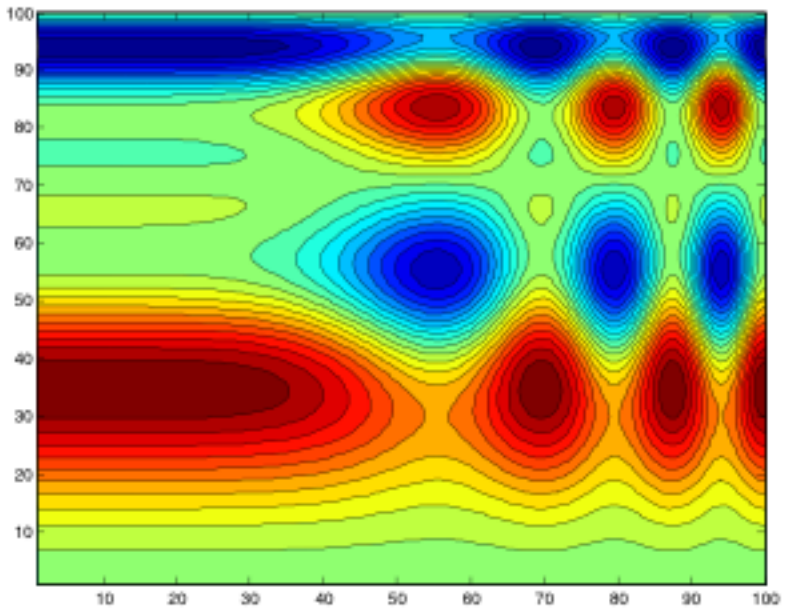
Kolmogorov Flow

$$\frac{\partial \omega}{\partial t} + \beta \mathbf{u} \cdot \nabla \omega = \bar{\nu} \nabla^2 \omega - \alpha \omega + A \cos(2\pi y / \lambda)$$

Persistent homology is invariant under homeomorphisms of the domain homeomorphisms of the domain





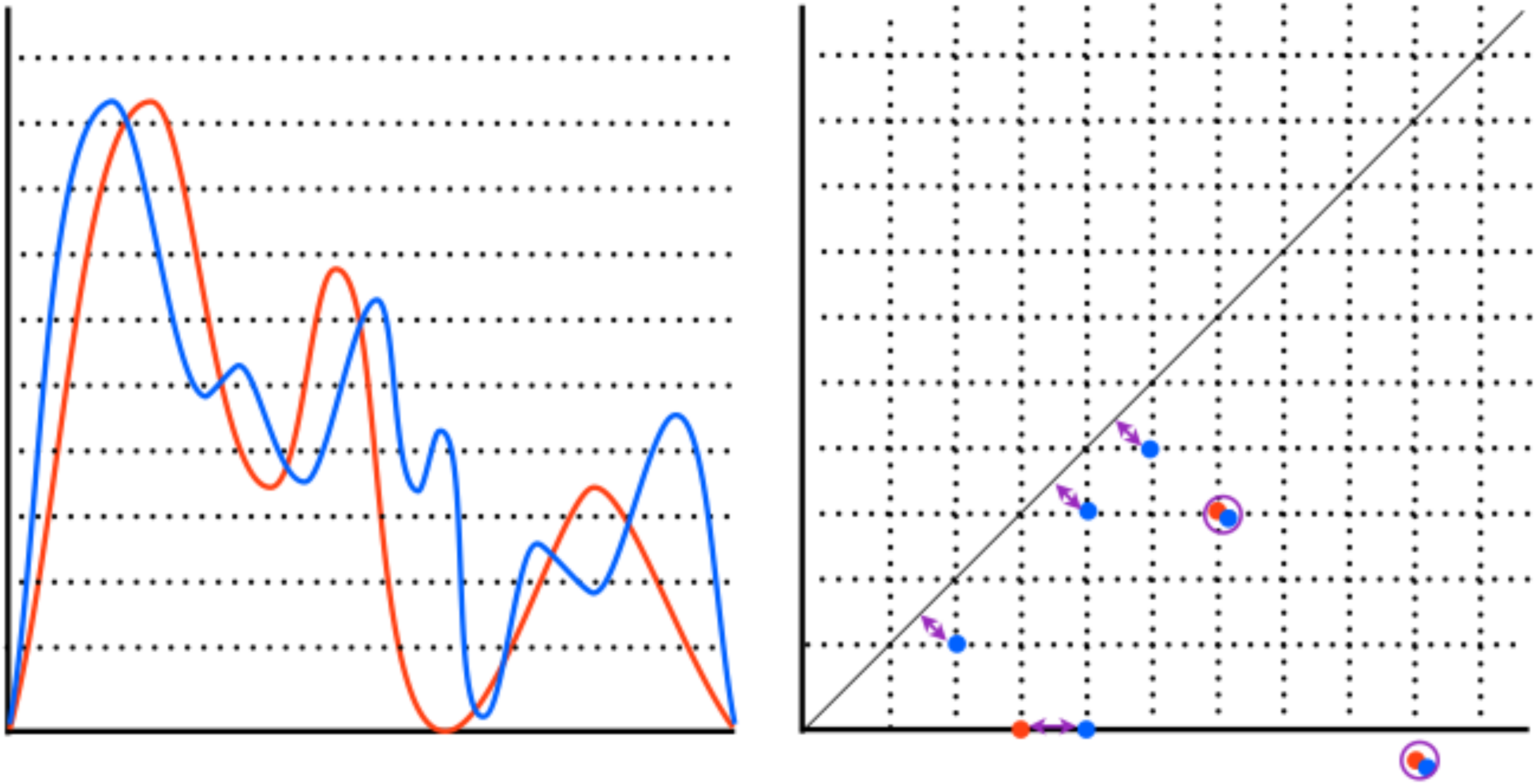


Definition of Persistence Diagram

Given a scalar field $f: X \to \mathbb{R}$ and a filtration

Given a scalar field $f: X \to \mathbb{R}$ and a filtration $\Theta = \{\theta_j \mid \theta_0 < \theta_1 < \cdots < \theta_N\}$ the associated persistence diagram $\mathsf{PD_i}(f, \theta_1)$ is the multi set consisting of

- one point for each i-th persistence pair (θ_j, θ_k),
- infinitely many copies of points on the diagonal (θ_j, θ_j)



Metrics For Persistence Diagram

Given two persistence diagrams PD_k and PD'_k let

$$W_p(\mathsf{PD}_k, \mathsf{PD}'_k) = \left[\inf_{\gamma} \sum_{z \in \mathsf{PD}_i} ||z - \gamma(z)||_{\infty}^{p}\right]^{\frac{1}{p}}$$

Wasserstein distance

where γ ranges over all bijections from PD_k to PD'_k.

Let per denote the space of all persistence diagrams.

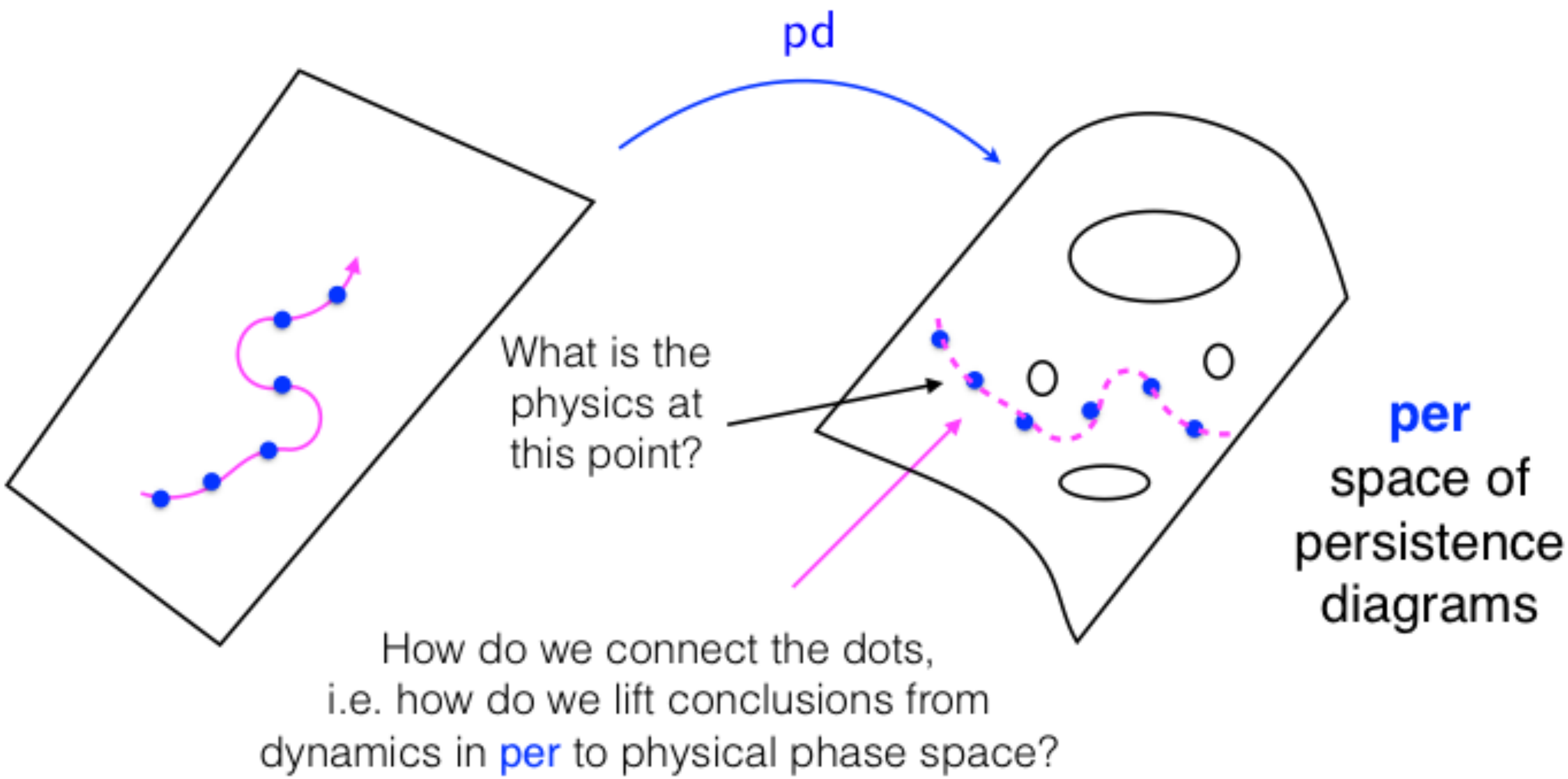
We have defined a function

$$\mathsf{pd}\colon L^{\infty}(X, \mathbb{R}) \to \mathsf{per}$$
$$f \mapsto \mathsf{pd}(f)$$

Persistence diagram of *f*.

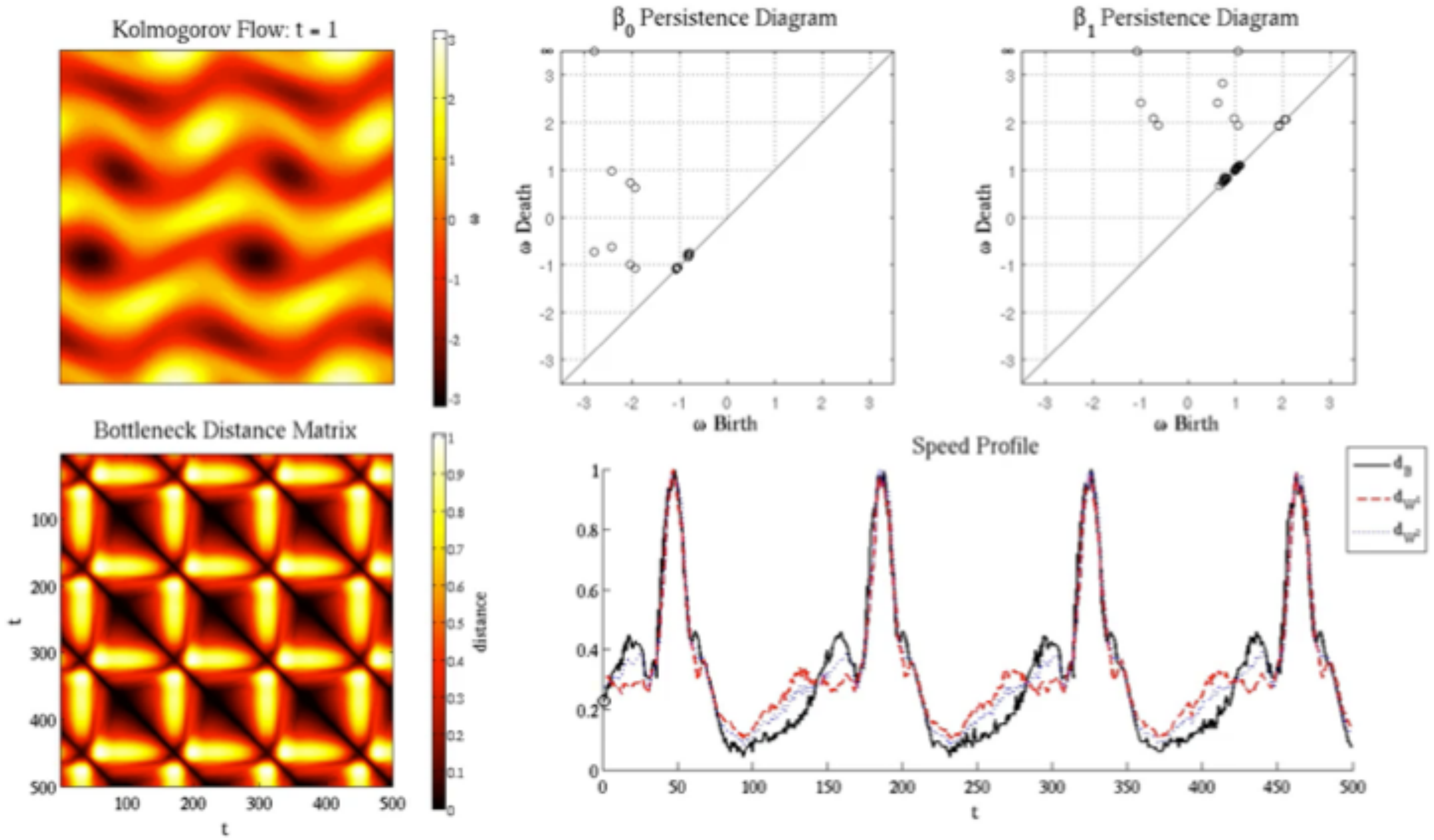
Theorem (Cohen-Steiner, Edelsbrunner, Harer) pd is a Lipschitz continuous function

Strategy: Study dynamics in space of persistence diagrams

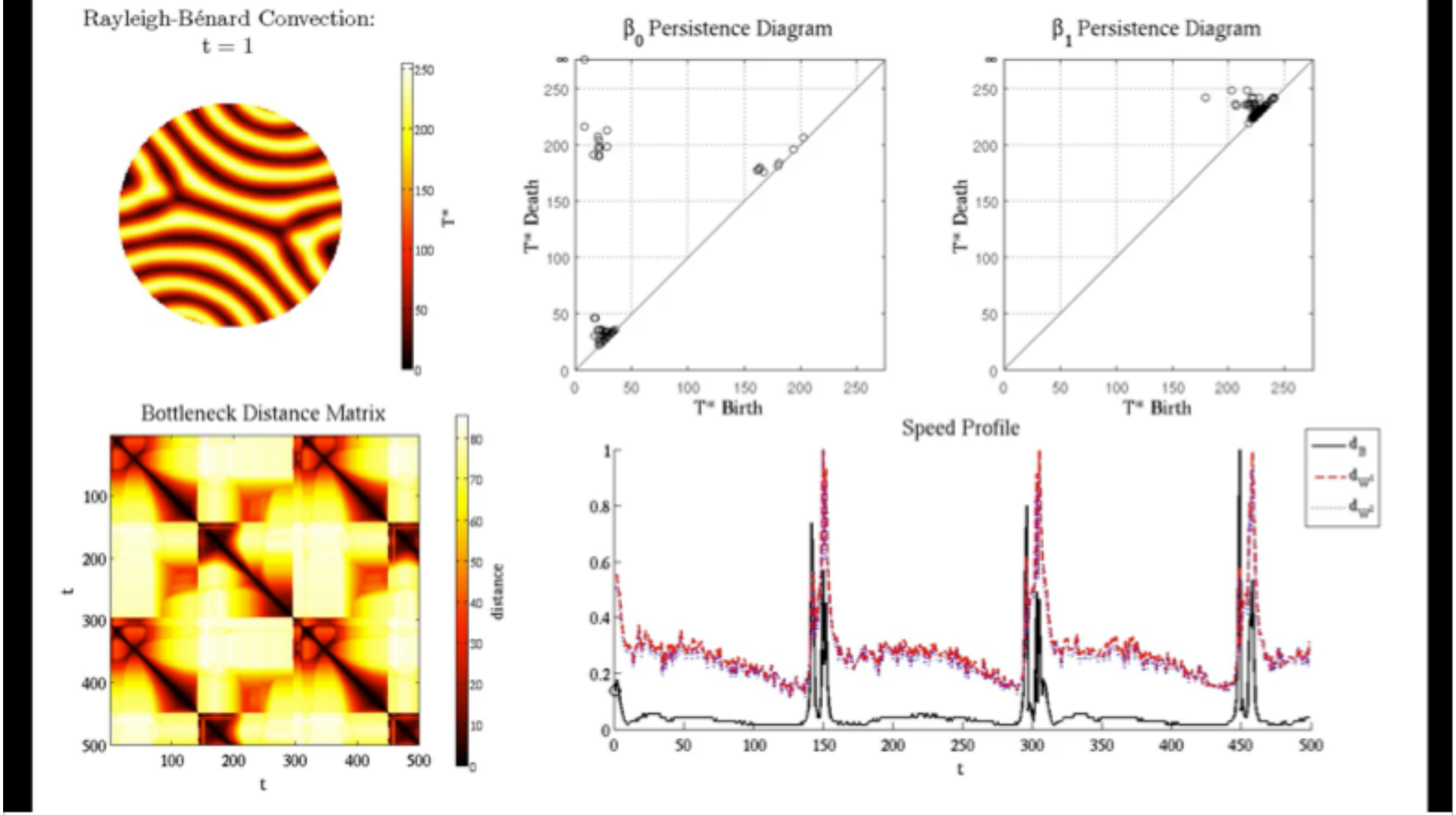


Trajectory in space
of temperature fields,
normal forces, etc.
(high dimensional)

Want to do dynamics
in this low dimensional
space.



Kolmogorov Flow: t = 1
β0 Persistence Diagram
β1 Persistence Diagram
ω Death
ω Birth
Bottleneck Distance Matrix
distance
Speed Profile
t



Persistence Module

(Making the map pd precise)

Function: $f : \Omega \to \mathbb{R}$

Filtration: $F(\theta) := \{x \in \Omega \mid f(x) \leq \theta\}$

$$\theta_s \leq \theta_t \quad \Rightarrow \quad F(\theta_s) \subset F(\theta_t) \quad \Rightarrow \quad H_* F(\theta_s) \xrightarrow{\iota_{t,s}} H_*(F(\theta_t))$$

pd(f)

$$H_*(F(\theta_r)) \xrightarrow{\iota_{\theta_s,\theta_r}} H_*(F(\theta_s)) \quad \xrightarrow{\iota_{\theta_t,\theta_s}} \quad H_*(F(\theta_t))$$

θ

Continuum of Vector Spaces (V, ι_V)

Error Bounds

Theory: Function: $f : \Omega \to \mathbb{R}$ (V, ι_V)

PD(f)

$$H_*(F(\theta_s)) \xrightarrow{\iota_{\theta_t,\theta_s}} H_*(F(\theta_t))$$

θ

Application:

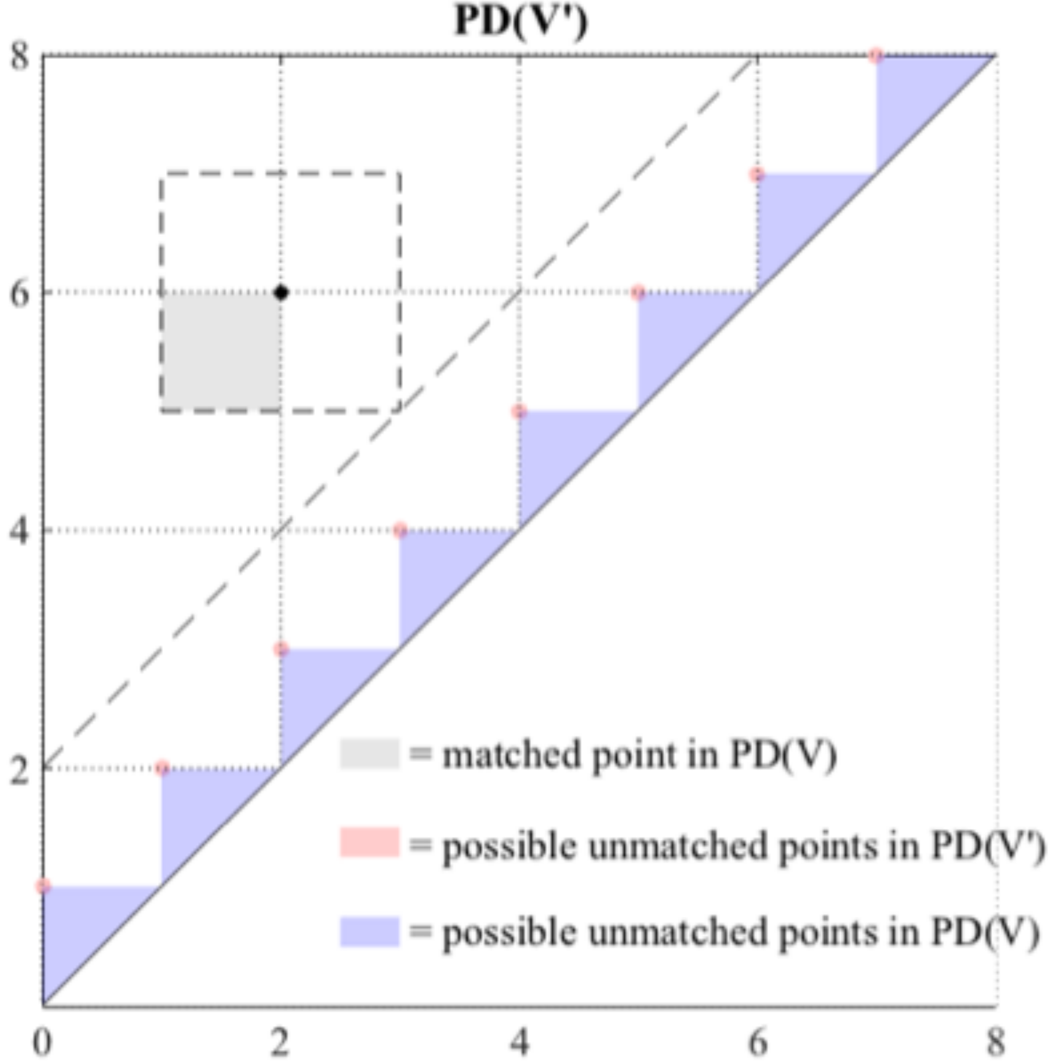
$(V', \iota_{V'})$

PD(f)

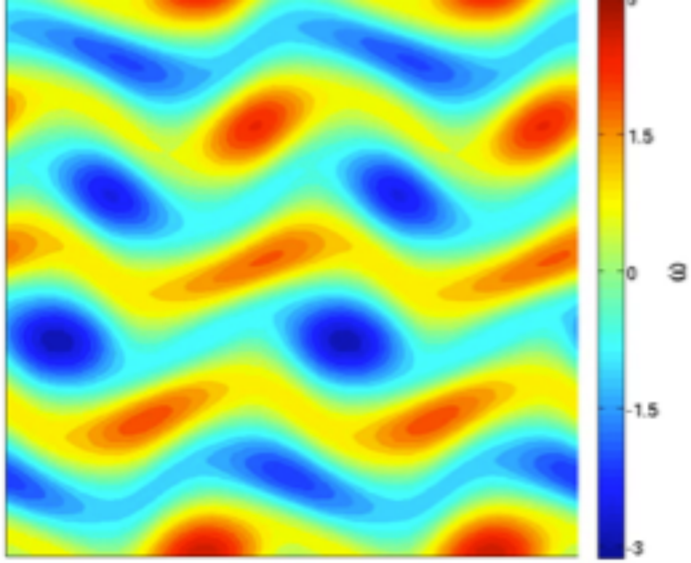
$$H_*(F(\theta_n)) \xrightarrow{\iota} H_*(F(\theta_{n+1}))$$

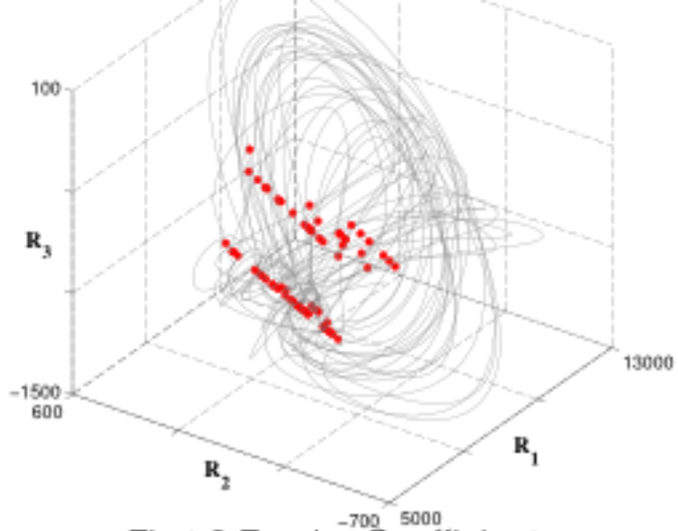
θ

What is the relationship between the persistence diagrams?



An Application





First 3 Fourier Coefficients

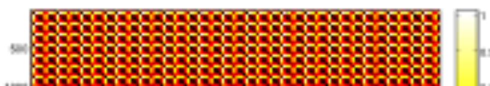
Equilibria identified using Newton's method.

Hypothesis: Chaotic Dynamics is generated by heteroclinic connections between unstable invariant sets, e.g. equilibria and periodic orbits.

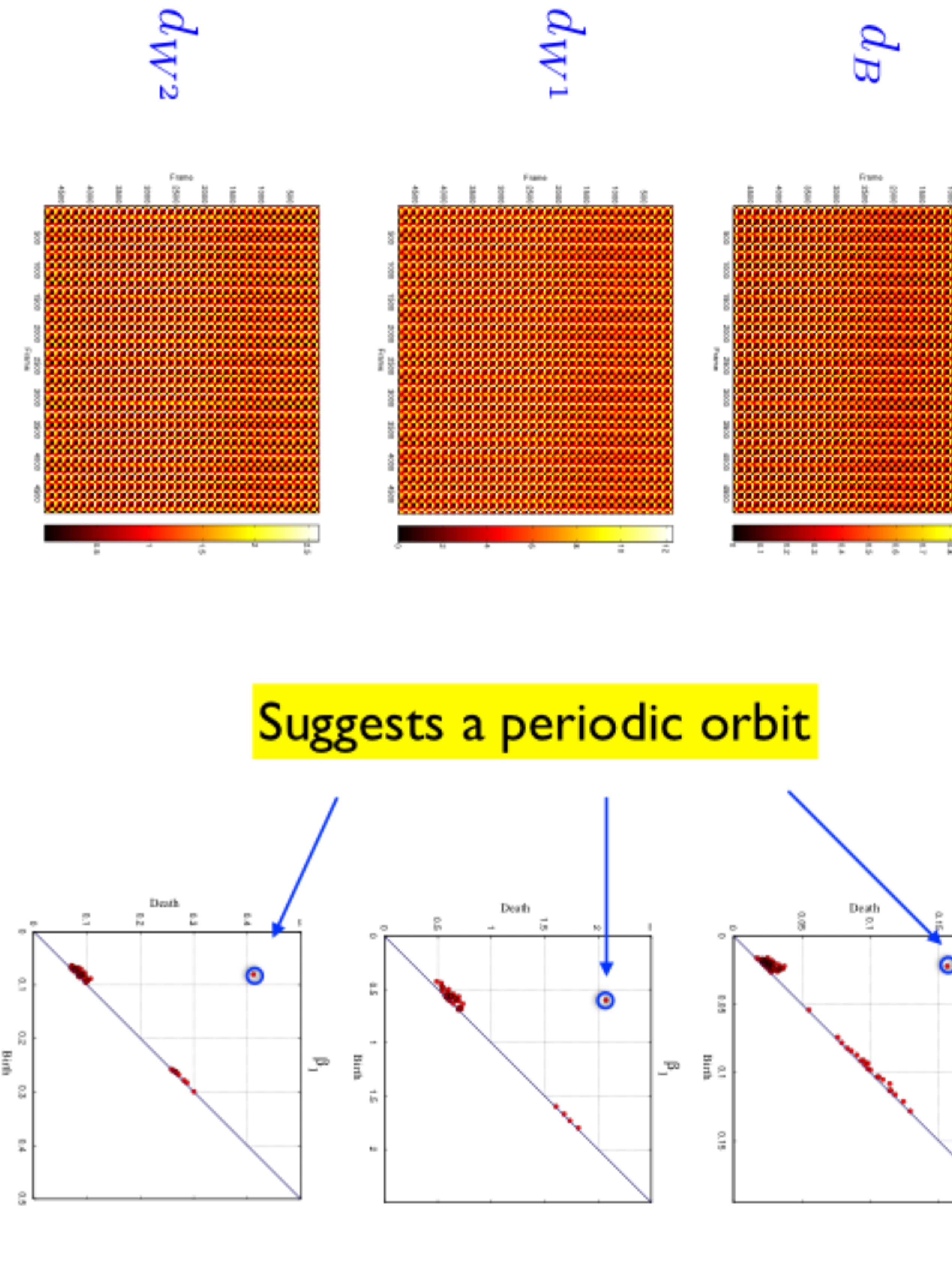
Question: Can we identify periodic orbits in the data?

Solution: A periodic orbit is a circle, use PD to identify nontrivial first homology in Per.

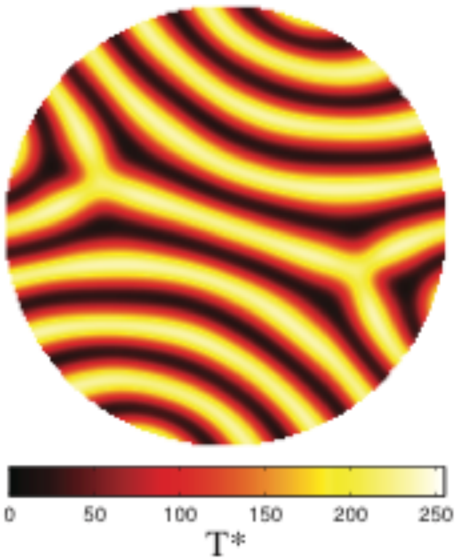
Numerical Simulations of Kolmogorov flow

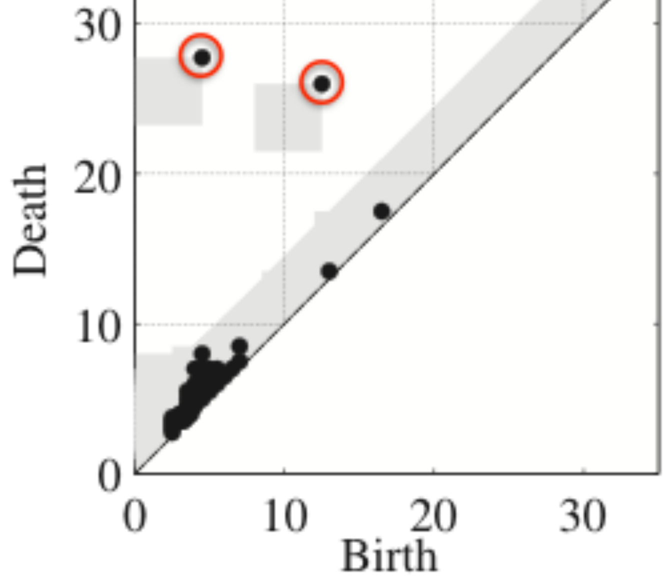


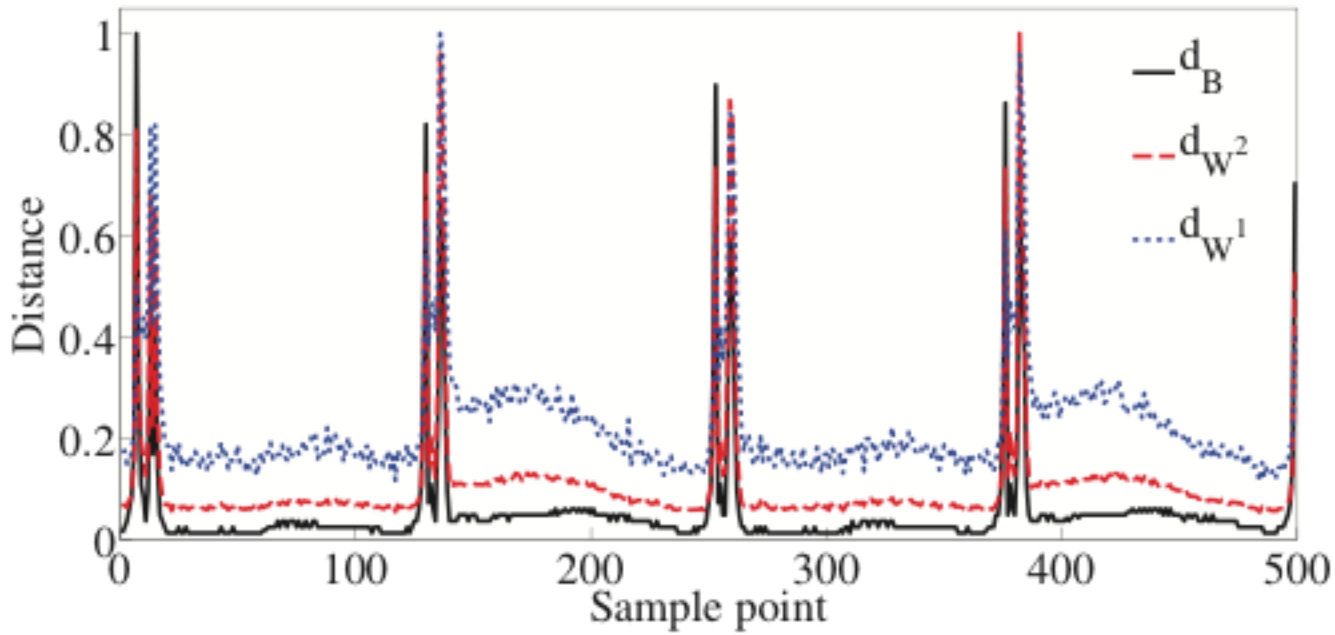
Numerical Simulations of Boussinesq



β_1 persistence diagrams of point cloud m
up of persistence diagrams of vorticity fi





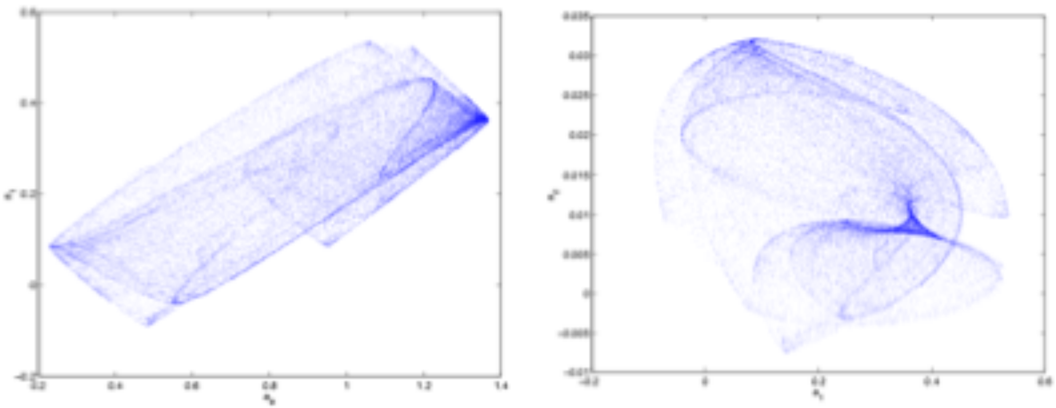


β_1 persistence diagram of point cloud made u
persistence diagrams for temperature field.

Conceptually Realistic Long Term Goals:

1. Compare the geometry of attractors of experimental data against numerical simulations.

2. Characterize complex dynamics on attractors.



Infinite dimensional map (Kot-Schaffer)
S. Day, O. Junge, K.M. SIADS (2004).

Collaborators

Rutgers
M. Kramar
R. Levanger

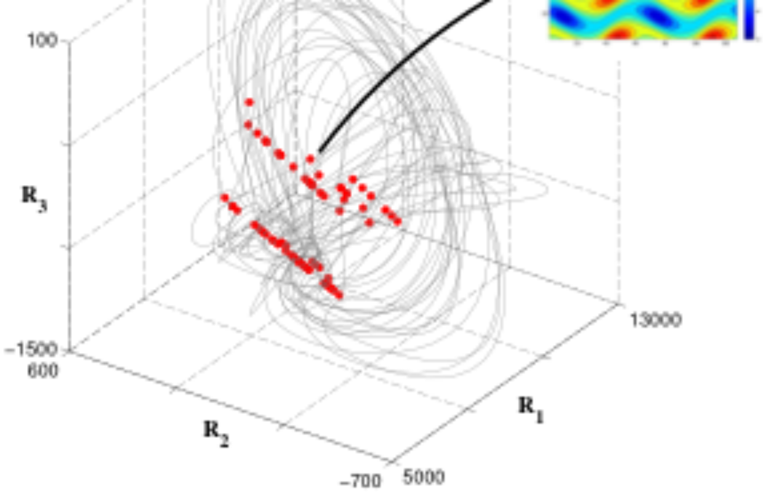
Georgia Tech
M. Schatz
J. Tithof
B. Suri

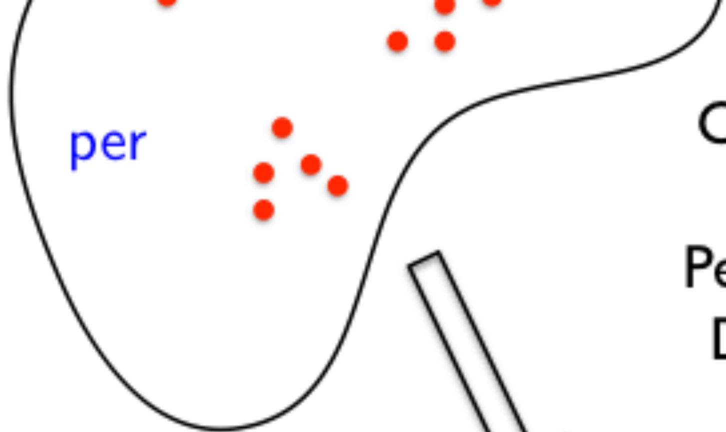
Virginia Tech
M. Paul
M. Xu

General Homology Software
chomp.rutgers.edu

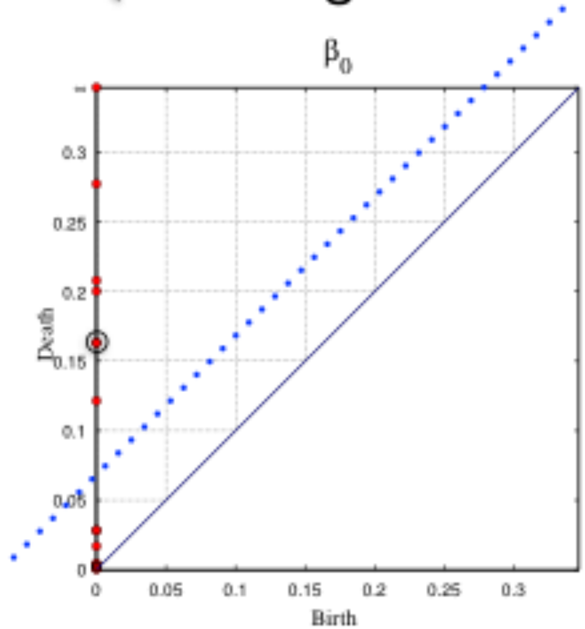
Persistent Homology Software
www.math.rutgers.edu/~vidit/perseus.html

Clustering Fixed Points





Compute β_0 Persistence Diagram of Persistence Diagrams



Seven Distinct Equilibria