Using Generalized B-splines in Isogeometric Analysis

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joint work with
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Isogeometric Analysis (IgA)

[Hughes, Cottrel, Bazilevs; CMAME 2005]

- Isogeometric Analysis (IgA) is a unifying framework for
  - Computed aided design (CAD)
  - Finite element analysis (FEA)
- same functions (CAD primitives)
  - to describe the geometry of the domain
  - to define the approximation space
- Includes standard FEA, but offers other possibilities:
  - Precise and efficient geometric modeling
  - Simplified mesh refinement
  - Smooth basis functions with compact support
  - Superior approximation properties per dofs
  - ...
- Applications
  - structural analysis
  - electromagnetism
  - fluids
  - shape optimization
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Outline

B-splines

polynomial spline spaces

$$a = x_0 < x_1 < \ldots < x_{n+1} = b$$

$$\{s \in C^r[a,b] : \ s_{|[x_i,x_{i+1})} \in \mathbb{P}_p, \ \ i = 0, \ldots, n\}$$

$$\mathbb{P}_p := < 1, x, \ldots, x^{p-2}, x^{p-1}, x^p >,$$

- B-splines are a special form to represent any spline function/curve/surface
- Why are B-splines so popular?
  - they are the best way to represent splines both from the geometric and computational point of view
  - there exist efficient and stable algorithms for their evaluation/manipulation

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- Given a set of knots $\Xi := \{\xi_1 \leq \xi_2 \leq \cdots \leq \xi_{n+p+1}\}$
  usually $\xi_1 = \cdots = \xi_{p+1} < \cdots < \xi_{n+1} = \cdots = \xi_{n+p+1}$
- B-spline basis functions are defined recursively

  $B_{i,\Xi}^{(p)}$: i-th B-spline, of degree p, with knots Ξ

  - they have minimum support
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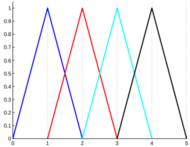
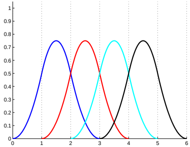
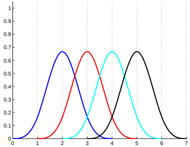


$p = 1$



$p = 2$



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Beyond polynomials: constrained curves/surfaces

- in CAGD curves/surfaces are often subjected to constraints
  - reproduction constraints
    exact reproduction of main curves/surfaces (conic sections, ...)
  - shape constraints
    curvature orientation, torsion signs,...
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Conic sections $\Rightarrow$ NURBS: Non Uniform Rational B-Splines

Given $\{B_{i,\Xi}^{(p)}(t),\ i=1,\cdots\},\ W:=\{w_i\geq 0,\ i=1,\cdots\},$ weights

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Alternatives: reproducing conic sections, cycloids ....

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alternatives to the rational model retaining properties of B-splines?

Alternatives: Polynomials vs Tchebycheff

- $\mathbb{P}_p :=< 1, t, \dots, t^{p-2}, t^{p-1}, t^p >$
- $\mathbb{P}_p \to \mathbb{T}_p$ extended Tchebycheff (EC) space on $[a, b]$
  - any non trivial element has at most p zeros in $[a, b]$ (counting multiplicity)
  - kernel of differential operators with real (constant) coefficients
- $\mathbb{T}_p \subset C^p$ containing constants
- $\Rightarrow$ EC spaces are good for design

  [Goodman, Mazure, JAT, 2001]
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- $\Rightarrow$ EC spaces are good for design

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Alternatives: Polynomials vs Tchebycheff

- $$\mathbb{P}_p :=< 1, t, \ldots, t^{p-2}, t^{p-1}, t^p >$$
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  - exponential functions $< 1, t, \ldots, t^{p-2}, e^{\alpha t}, e^{-\alpha t} >$
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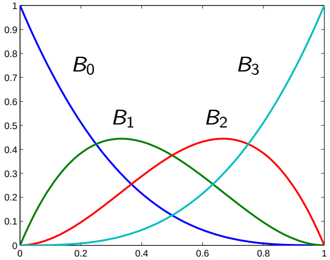
$\mathbb{P}_p^{u,v}$ possesses a Bernstein-like basis

$$\begin{aligned}
\sum_{j=0}^{p} B_j(t) &= 1 \\
\frac{\mathrm{d}^j}{\mathrm{d}t^j} B_k(0) &= 0, j = 0, \ldots, k-1, \\
\frac{\mathrm{d}^j}{\mathrm{d}t^j} B_k(1) &= 0, \; j = 0, \ldots, p-k-1.
\end{aligned}$$



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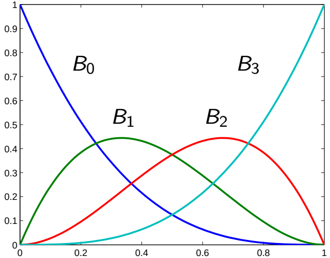

[Goodman, Mazure, JAT, 2001]
[Mainar, Peña, Sánchez-Reyes, J, CAGD 2001]
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Alternatives to the rational model

- rational model: $\mathbb{P}_p \to$ B-splines $\to$ NURBS
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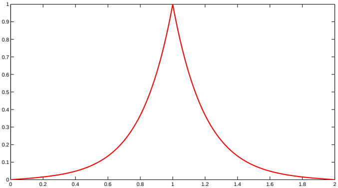

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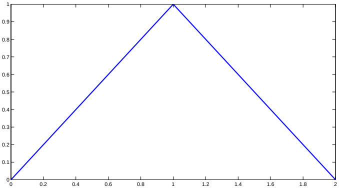

$B^{(1)}_{i,\Xi}$

- All Tchebycheffian spline spaces good for design can be built by means of integral recurrence relations, [Mazure M.L., NM 2011]

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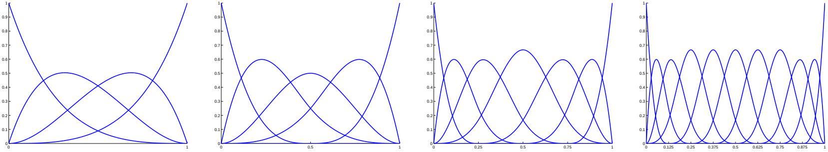
$\mathrm{EXP}_3 = \mathbb{P}_3^{u,v} :=< 1, t, e^{\alpha t}, e^{-\alpha t} >$ $\quad \alpha \to 0$: B-splines



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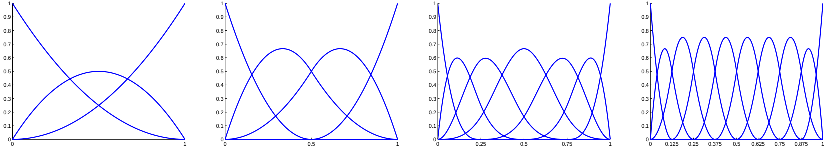
$\mathrm{TRIG}_2 = \mathbb{P}_2^{u,v} :=< 1, \cos\alpha t, \sin\alpha t >$ $\quad \alpha \to 0$: B-splines



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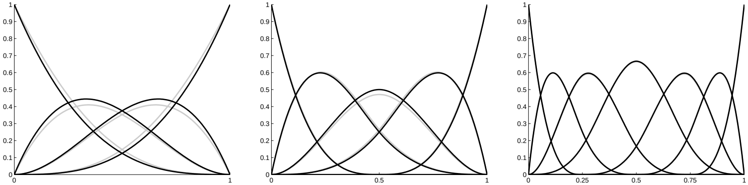

$< 1, t, \cos \alpha t, \sin \alpha t >, \alpha = \frac{2}{3}\pi$ (light) and algebraic C^2 cubics (dark)

Tensor-product structures:

- multivariate setting: Tensor-product
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Tensor-product structures:DRAWBACKS

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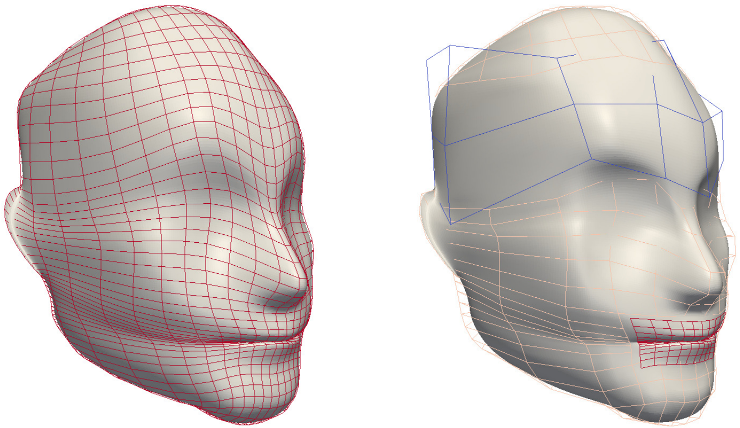

Local Tensor-product structures

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  Alternatives local tensor-product structure (polynomial splines):
  - T-splines/Analysis-Suitable T-splines
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  - LR splines
    [Dokken, Lyche, Pettersen, CAGD 2013],
  - Hierarchical splines
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    [Forsey, Bartels, CG 1988] [Kraft, 1997] [Giannelli, Juttler, Speleers, CAGD 2012], ...
  - Splines over T-meshes
    [Deng, Chen, Feng, JCAM 2006], [Schumaker, Wang, NM 2011], [Schumaker, Wang, CAGD 2012]

Local Tensor-product structures

- ☹tensor-product structure NO efficient local refinements
  Alternatives local tensor-product structure (polynomial splines):
  - T-splines/Analysis-Suitable T-splines
    [Sederberg et al., ACMToG, 2003], [Bazilevs, et al. CMAME 2010], [Beirão da Veiga, et al. CMAME, 2012]...
  - LR splines
    [Dokken, Lyche, Pettersen, CAGD 2013],
  - Hierarchical splines
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Local Tensor-product structures: Generalized splines

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  Alternatives:
  - Generalized T-splines
    [Bracco et al. CMAME 2014] [Bracco and Cho, CMAME 2014]
  - LR Generalized B-splines
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  - Hierarchical Generalized B-splines
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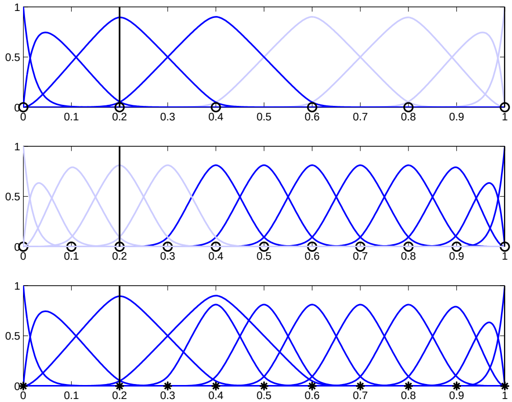
LR/Hierarchical B-splines/GB-splines

- perspective: consider a set of functions and study their properties
  - positivity, partition of unity
  - linear independence
  - spanned space

Hierarchical Generalized B-spline model

Hierarchical B-splines/Generalized B-splines: same construction

- 1D Example: $< 1, t, \exp^{\alpha_i t}, \exp^{-\alpha_i t} >, \alpha_i = 50$



[Manni, Pelosi, Speleers, LNCS 2014]

Hierarchical Generalized B-spline model

Hierarchical B-splines/Generalized B-splines: same construction & same properties

- ☺linearly independence
- ☺nested spaces:
- ☺positivity
- ☺partition of unity by using truncated bases
  [Giannelli, Jüttler, Speleers, CAGD 2012]
  [Giannelli, Jüttler, Speleers, AiCM 2013]
    - ☺good quasi-interpolants for free from good quasi-interpolants in building block spaces
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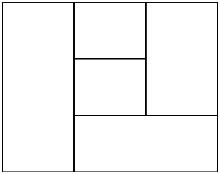
Generalized Spline spaces over T-meshes

- T-mesh $\mathcal{T}$ partition of a domain by axis-aligned rectangles



- suitable spaces : exponential, trigonometric
- smoothness cond.: Bernstein like representation
- construction and properties: as for polynomial splines

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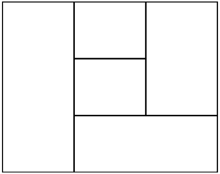

$$\widehat{\mathbb{S}}^{\mathbf{r}}_{\mathbf{p}}(\mathcal{T}) := \left\{ s(x,y) \in C^{\mathbf{r}},\ s(x,y)_{|\tau_i} \in \mathbb{P}^{u_1,v_1}_{p_1} \times \mathbb{P}^{u_2,v_2}_{p_2},\quad \tau_i \in \mathcal{T} \right\},$$

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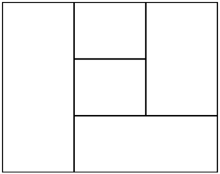

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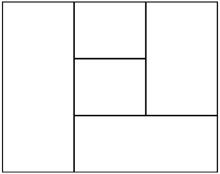

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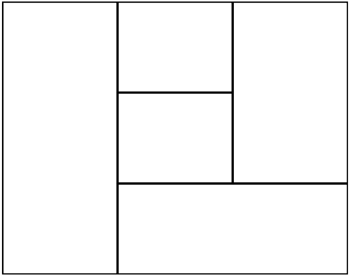

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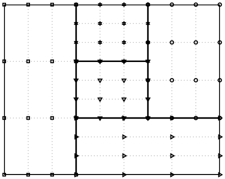
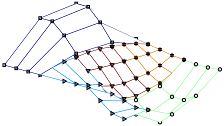
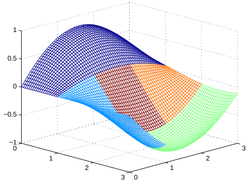
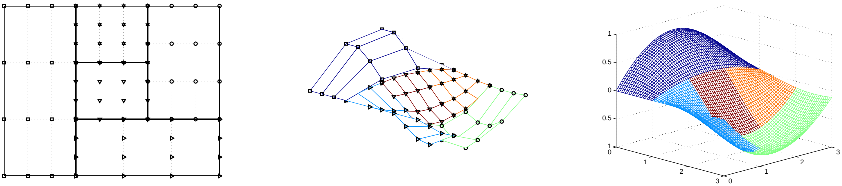






C^1 bi-cubics

Generalized Spline spaces over T-meshes



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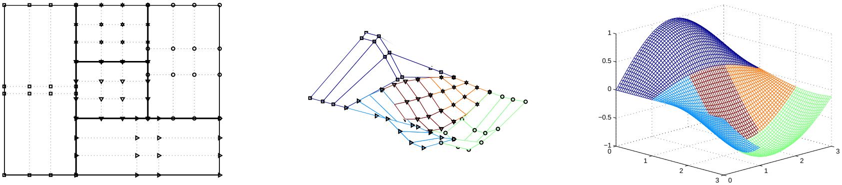
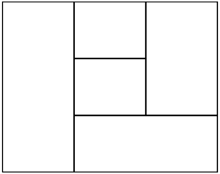


$< 1, t, \cos \alpha t, \sin \alpha t >$: C^1 trigonometric (bi-cubics), $\alpha = \frac{2}{5}\pi$

Generalized Spline spaces over T-meshes

- T-mesh $\mathcal{T}$ partition of a domain by axis-aligned rectangles



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- space structure
- dimension: homological techniques full Tchebycheffian splines [Bracco, Lyche, Manni, Roman, Speleers, CAGD 2016]

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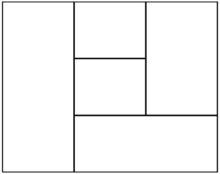

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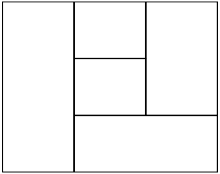

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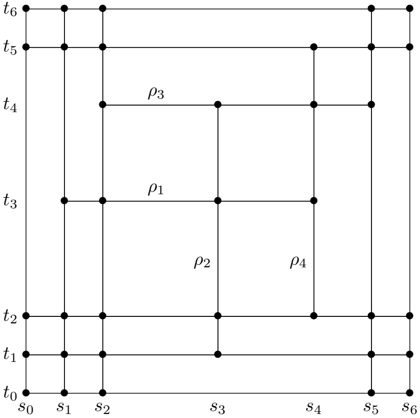
Dimension of the spline space $\widehat{\mathbb{S}}_{\boldsymbol{p}}^{\boldsymbol{r}}(\mathcal{T})$: instability

☺ stable dimension: only depending on degree, smoothness, topology

Dimension of the spline space $\widehat{\mathbb{S}}^{\boldsymbol{r}}_{\boldsymbol{p}}(\mathcal{T})$: instability

☹ NO stable dimension

$$\boldsymbol{p} = (2,2), \ \boldsymbol{r} = (1,1)$$



$$36 \leq \dim\left(\widehat{\mathbb{S}}^{\boldsymbol{r}}_{\boldsymbol{p}}(\mathcal{T})\right) \leq 37, \quad \forall \langle 1, u, v\rangle \ ECT$$

[Bracco, Lyche, Manni,Speleers, 2016]

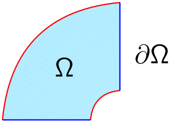
GB-splines in simulation:

GB-splines IgA Galerkin and Collocation methods

The problem

- Second order (elliptic) partial differential equation (PDE),

$$\mathcal{L}u = \begin{cases} Lu = & \mathrm{f}, \quad in\ \Omega \\ \Gamma u = & \mathrm{g} \quad on\ \partial\Omega \end{cases}$$



- weak formulation:

$$\text{Find} \quad u \in \mathcal{V}, \quad \text{such that} \quad a(u,v) = F(v), \quad \forall\ v \in \mathcal{V}$$

$a : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ bilinear form depending on L
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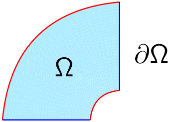

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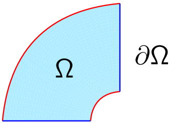

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$$\text{Find} \quad u \in \mathcal{V}, \ \text{such that} \quad a(u,v) = F(v), \ \forall\ v \in \mathcal{V}$$

Example: $\begin{cases} -\Delta u = & \mathrm{f}, \quad in\ \Omega \\ u = & 0 \quad on\ \partial\Omega \end{cases}$

$\rightsquigarrow$ find $\quad u \in \mathcal{V} := H_0^1(\Omega)$, such that

$$a(u,v) := \int_\Omega \nabla u \nabla v \ \mathrm{dxdy} = \int_\Omega \mathrm{f}v \ \mathrm{dxdy} =: F(v), \ \forall\ v \in \mathcal{V}$$

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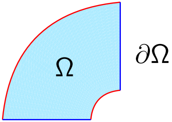

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$$\text{Find} \quad u \in \mathcal{V}, \quad \text{such that} \quad a(u, v) = F(v), \quad \forall\ v \in \mathcal{V}$$

- Galerkin approach: $\mathcal{V}_h : < \phi_1, \phi_2, \ldots, \phi_{n_h} > \subset \mathcal{V}$

$$\text{Find} \quad u_h \in \mathcal{V}_h, \ \text{such that} \quad a(u_h, v_h) = F(v_h), \quad \forall v_h \in \mathcal{V}_h$$

$$u_h = \sum_{i=1}^{n_h} q_i \phi_i \quad \rightarrow \quad \text{linear system } A\mathbf{q} = \mathbf{f}$$

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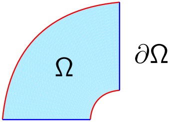

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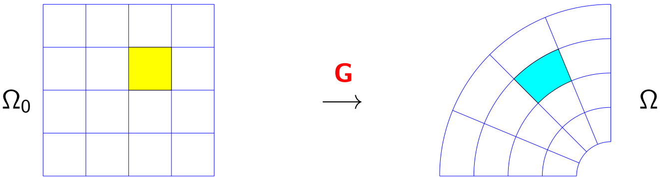
- different choices of $\mathcal{V}_h$ $\Rightarrow$ different methods (FEM,...)

Generalized B-splines based IgA

- $\Omega_0 := [0,1]^2$: parametric domain, Ω : physical domain
- global geometry function $\mathbf{G} : \Omega_0 \to \Omega$:

$$\mathbf{G}(\xi) = \sum_{i=1}^{n_h} B_i(\xi)\mathbf{c}_i, \qquad \{B_1, \cdots, B_{n_h}\} : \text{ basis}$$



$\xi \in \Omega_0$ $\qquad$ $x(\xi) \in \Omega$

$$u_h(x) = \sum_{i=1}^{n_h} \underbrace{B_i \circ \mathbf{G}^{-1}(x)}_{\text{basis functions } \phi_i} q_i, \quad x \in \Omega,$$

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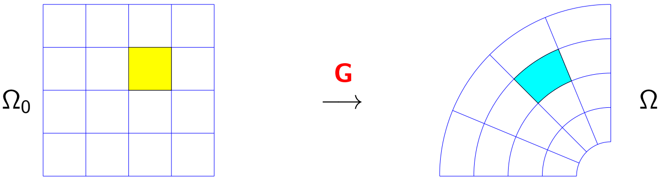

$\xi \in \Omega_0$ $x(\xi) \in \Omega$

IgA Galerkin based on NURBS

$\boxed{\text{basis functions in } \Omega_0\text{: tensor-product NURBS}}$

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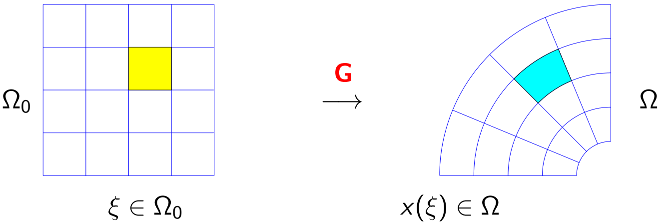

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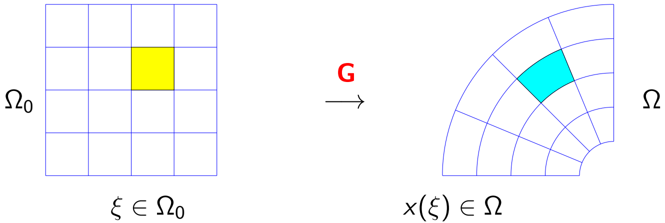

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IgA Galerkin based on GB-splines

$\boxed{\text{basis functions in } \Omega_0\text{: tensor-product GB-splines}}$

$\boxed{\text{NURBS and Generalized B-splines are plug-to-plug in IgA}}$

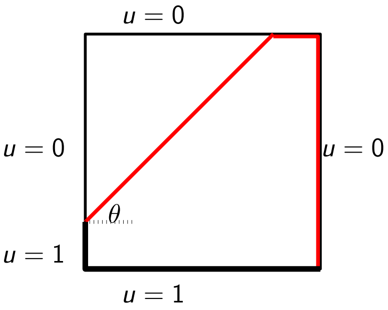
Generalized B-splines based IgA: Galerkin

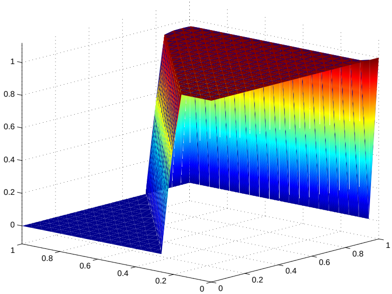
$\boxed{\text{Section spaces to be selected with a problem-dependent strategy}}$

Generalized B-splines based IgA: Galerkin

Section spaces to be selected with a problem-dependent strategy

strong gradients/thin layers $\Rightarrow$ Exp.or Variable degree B-splines





$$\varepsilon \triangle u + \mathbf{b} \cdot \nabla u = 0, \qquad \mathbf{b} = (\cos(\theta), \sin(\theta)), \ \varepsilon = 10^{-6}$$

[Manni, Pelosi, Sampoli, JCAM 2011]

Generalized B-splines based IgA: Galerkin

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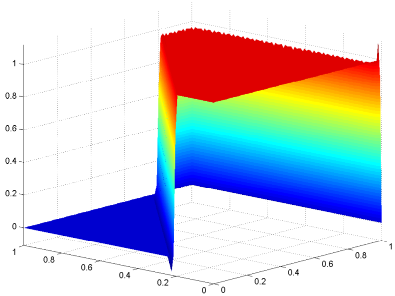
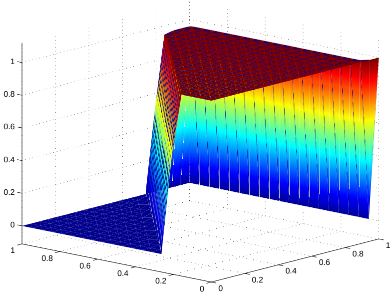
strong gradients/thin layers $\Rightarrow$ Exp.or Variable degree B-splines





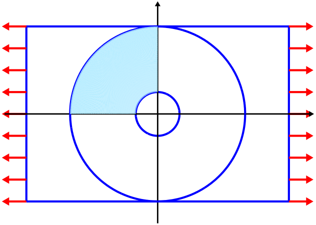
C^0 FEM $\mathbb{P}_2$ three-directional mesh 40×40 C^3 quintic VD mesh 40×40

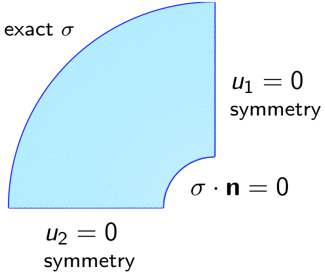
Generalized B-splines based IgA: Hierachical bases

[Manni, Pelosi, Sampoli, CMAME 2011], [Manni, Pelosi, Speleers, LNCS 2014]

- Infinite plate with circular hole, uniform tension in x-direction

$$\operatorname{div} \sigma(\mathbf{u}) = 0 \quad \text{in} \quad \Omega$$





- exact geometry requires trigonometric GB-splines $< 1, \cos((\frac{\pi}{2})t), \sin((\frac{\pi}{2})t) >$

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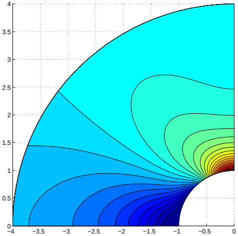

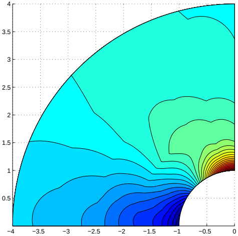

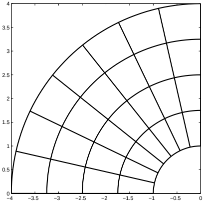

EXACT

TRIG$_2$

dof $= 96$

approx σ_{11}

Generalized B-splines based IgA: Hierachical bases

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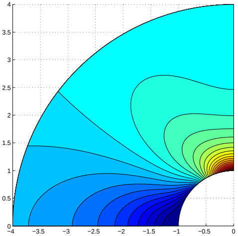

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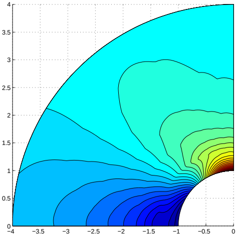
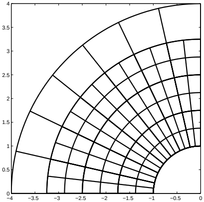


TRIG_2



dof $= 230$

approx σ_{11}

Generalized B-splines based IgA: Hierachical bases

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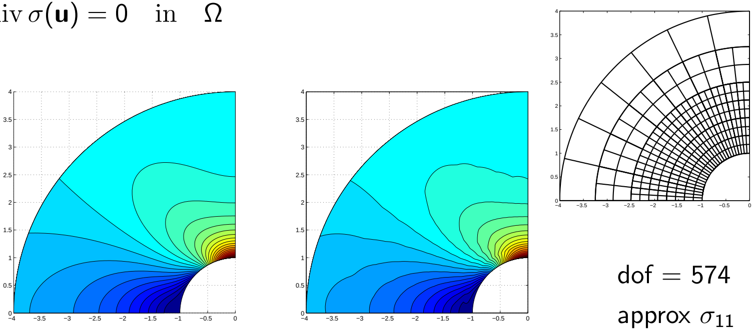

EXACT TRIG$_2$

dof $=$ 574

approx σ_{11}

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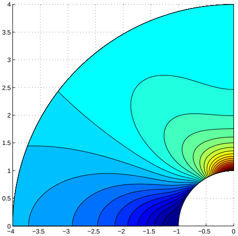

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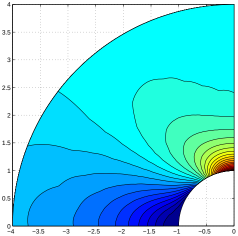

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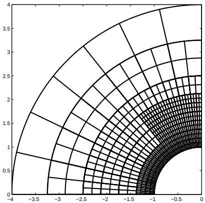

dof $= 1122$

approx σ_{11}

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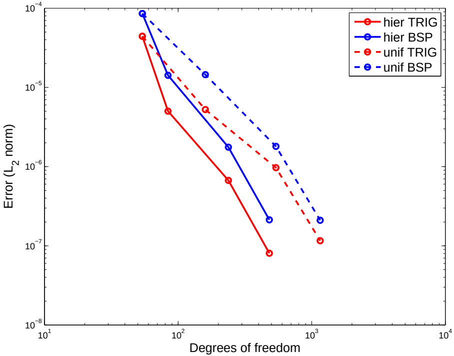

IgA: Galerkin vs Collocation

- the efficiency of the Galerkin method deeply depends on the numerical quadrature rules for the construction of the linear systems
- in FEA elementwise Gauss quadrature is known to be optimal; not the same for IgA
- ☹GB-splines IgA Galerkin methods also suffer from the quadrature issue
- a minimum number of point evaluations per degree of freedom is even more attractive in the context of GB-splines than for classical polynomial B-splines/NURBS.
- ☺high regularity of the basis functions $\Rightarrow$ discretization of the strong form of (high order) PDEs
- ☺GB-splines present the same smoothness properties and can be adjusted to any order (degree) as classical B-splines.

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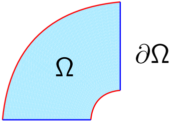
$\boxed{\text{IgA NURBS/GB-splines Collocation}}$

IgA Collocation

- Second order (elliptic) partial differential equation (PDE),

$$\mathcal{L}u = \begin{cases} Lu = & \mathrm{f}, \quad in\ \Omega \\ \Gamma u = & \mathrm{g} \quad on\ \partial\Omega \end{cases}$$



- Collocation:
  - Collocation space: $\mathcal{V}_h : <\phi_1, \phi_2, \dots, \phi_{n_h}> \subset \mathcal{V}$
    Collocation points: $\tau_1, \tau_2, \dots, \tau_{n_h} \in \overline{\Omega}$
  - find $u_h \in \mathcal{V}_h$, such that

$$\begin{cases} Lu_h(\tau_i) = & \mathrm{f}(\tau_i), \quad \tau_i \in \Omega \\ \Gamma u_h(\tau_j) = & \mathrm{g}(\tau_j) \quad \tau_j \in \partial\Omega \end{cases}$$

$$u_h = \sum_{i=1}^{n_h} q_i \phi_i \quad \longrightarrow \quad \text{linear system } A\mathbf{q} = \mathbf{f}$$

  - different choices of $\mathcal{V}_h \Rightarrow$ different collocation methods

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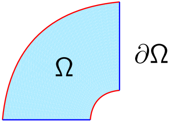

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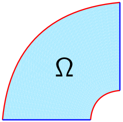

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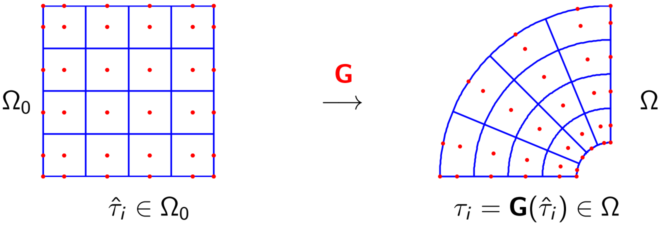
IgA Collocation

- $\Omega_0 := [0,1]^2$: parametric domain, Ω : physical domain
- global geometry function $\mathbf{G} : \Omega_0 \to \Omega$:
  $\mathbf{G}(\xi) = \sum_{i=1}^{n_h} B_i(\xi)\mathbf{c}_i,\qquad \{B_1, \cdots, B_{n_h}\}$: basis



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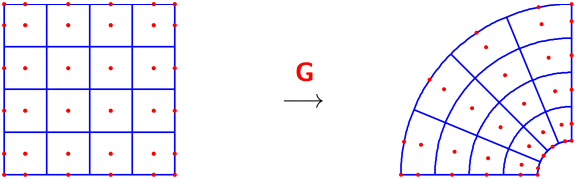
Ω_0  Ω

$$\hat{\tau}_i \in \Omega_0 \qquad\qquad \tau_i = \mathbf{G}(\hat{\tau}_i) \in \Omega$$

$$u_h(x) = \sum_{i=1}^{n_h} \underbrace{B_i \circ \mathbf{G}^{-1}(x)}_{} q_i, \quad x \in \Omega,$$

basis functions ϕ_i

collocation points $\tau_i = \mathbf{G}(\hat{\tau}_i)$

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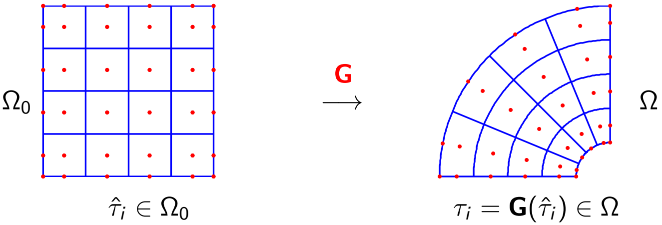

IgA collocation based on NURBS

$\boxed{\text{basis functions in } \Omega_0\text{: tensor-product NURBS}}$

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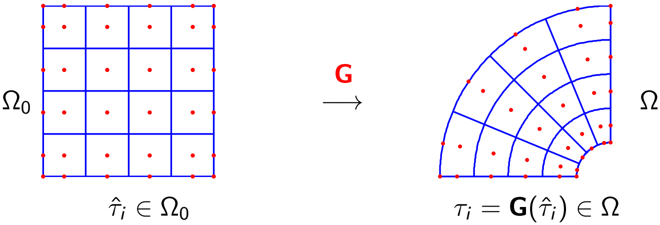

IgA collocation based on NURBS

basis functions in Ω_0: tensor-product NURBS

IgA collocation based on GB-splines

basis functions in Ω_0: tensor-product GB-splines

IgA Collocation: Pros/Cons

- ☺extremely cheap: one degree of freedom for evaluation
- ☺extremely easily to implement
- almost optimal approximation order

$$\begin{cases} h^p & p: even, \\ h^{p-1} & p: odd \end{cases} \qquad L_2$$

- ☹almost no theoretical understanding (partial results $d = 1$)
- Collocation points? Usually Greville abscissae

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Example: Advection-Diffusion Prob.

[Manni, Reali, Speleers, CMA 2015]

$$\begin{cases} -\kappa \Delta u + \boldsymbol{\beta} \cdot \nabla u = \mathrm{f}, & \text{in } \Omega := (0,1) \times (0,1) \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

$$\kappa = 10^{-3}, \quad \beta = [1,0]^T, \quad \mathrm{f} = 1$$

- hyperbolic GB-splines $< 1, t, \dots, t^{p-2}, \cosh(\alpha t), \sinh(\alpha t) >$
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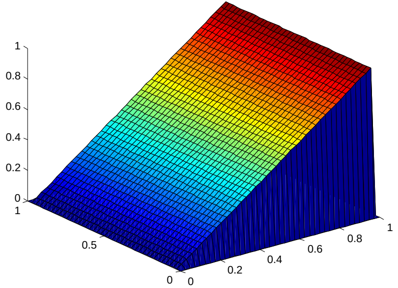

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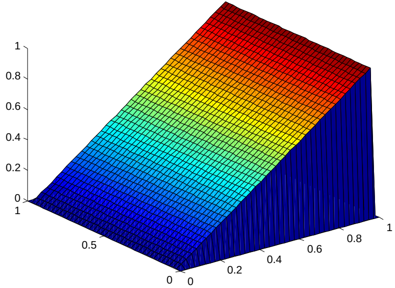

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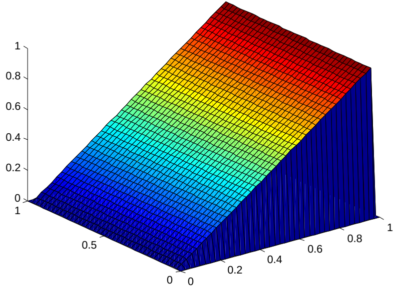

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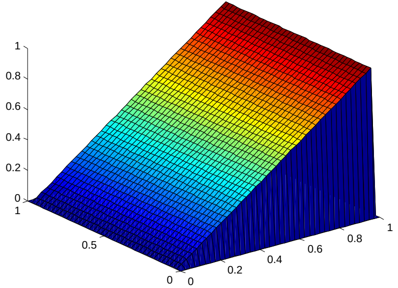

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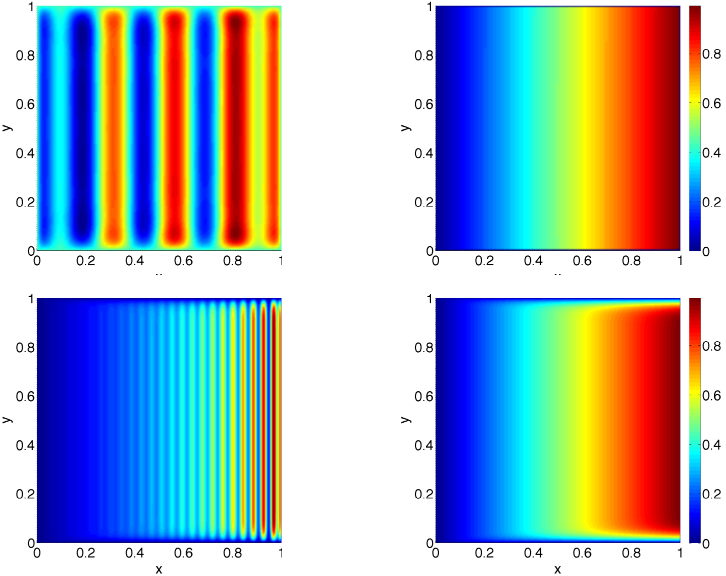

10×10 | 10×10

50×50 | 50×50

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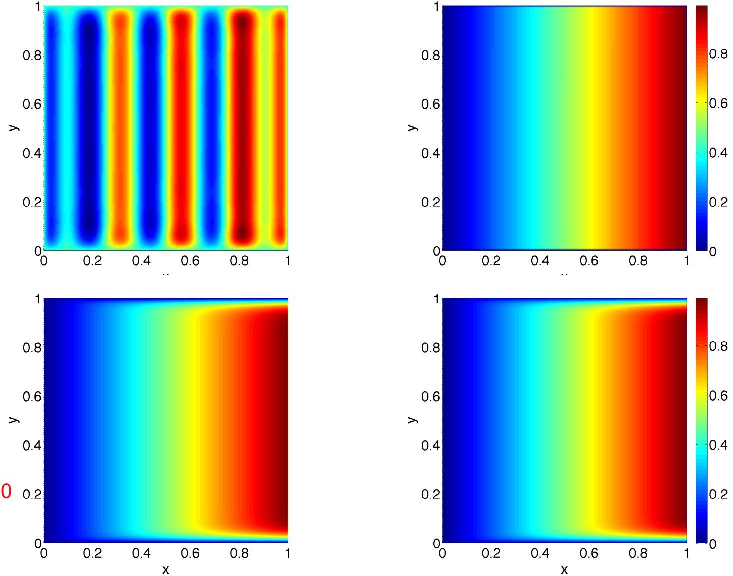

10×10 | 10×10

300×300 | 50×50

Spectral Analysis of matrices in IgA

$$\boxed{\text{Linear PDE } \mathcal{L}u = g}$$

$$\Downarrow$$

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Spectral Analysis of matrices in IgA

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- Informal Meaning: the eigenvalues of A_n are approximately a uniform sampling of f

$$\boxed{\{A_n\}_n \sim_\lambda f}$$

Symbol: WHY

- analysis of the spectral properties of A_n for large n
- design of fast (iterative) solvers
  [Donatelli, Garoni, Manni, Serra-Capizzano, Speleers, CMAMEa 2015, CMAMEb 2015], [Donatelli, Garoni, Manni, Serra-Capizzano, Speleers, SINUM 2016]
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Spectral analysis for B-spline IgA (Galerkin/Collocation)

- Elliptic model problem

$$\begin{cases} -\nabla \cdot K\nabla u + \boldsymbol{\beta} \cdot \nabla u + \gamma u = \mathrm{f}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^d$, $K : \Omega \to \mathbb{R}^{d\times d}$ is SPD, $\boldsymbol{\beta} : \Omega \to \mathbb{R}^d$ and $\gamma \geq 0$

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canonical: similar to FD/FE discretizations
- ill-conditioning at the high frequencies for large p ($\theta = \pi$)
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  $\Rightarrow$ classical multigrid methods present problems for large p

[Garoni, Manni, Pelosi, Serra-Capizzano, Speleers, NM 2014]

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Constructing symbol: building block

$$\begin{cases} -u'' = \mathrm{f}, & \text{in } \Omega = (0,1), \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

Galerkin uniform grid

$$f_p(\theta) := -\ddot{\phi}_{[2p+1]}(p+1) - 2\sum_{k=1}^{p} \ddot{\phi}_{[2p+1]}(p+1-k)\cos(k\theta) = (2-2\cos\theta)h_{p-1}(\theta)$$

$$h_{p-1}(\theta) := \phi_{[2p-1]}(p) + 2\sum_{k=1}^{p-1} \phi_{[2p-1]}(p-k)\cos(k\theta)$$

$\phi_{[p]}$ cardinal B-spline of degree p

$$\boxed{\left\{\frac{1}{n}A_n\right\}_n \sim_\lambda f_p}$$

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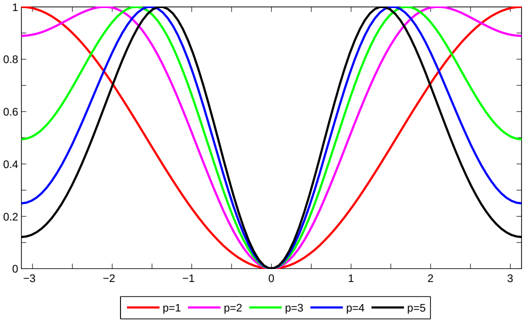

Symbol: general case

$$\begin{cases} -\nabla \cdot K\nabla u + \boldsymbol{\beta} \cdot \nabla u + \gamma u = \mathrm{f}, & \text{in } \Omega \subset \mathbb{R}^d, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

$$\mathbf{G} : [0,1]^d \to \overline{\Omega}$$

$$\begin{cases} -\mathbf{1}(K_{\mathbf{G}} \circ H\hat{u})\mathbf{1}^T + \boldsymbol{\beta}_{\mathbf{G}} \cdot \nabla \hat{u} + \gamma(\mathbf{G})\hat{u} = \mathrm{f}(\mathbf{G}), & \text{in } (0,1)^d, \\ \hat{u} = 0, & \text{on } \partial((0,1)^d), \end{cases}$$

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HOW: GLT (Generalized Locally Toeplitz) sequences

[Serra-Capizzano, LAA 2003], [Serra-Capizzano, LAA 2006],

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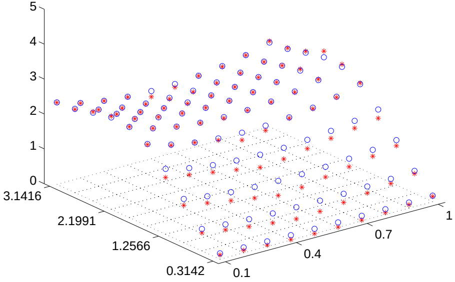
Example: symbol 1D

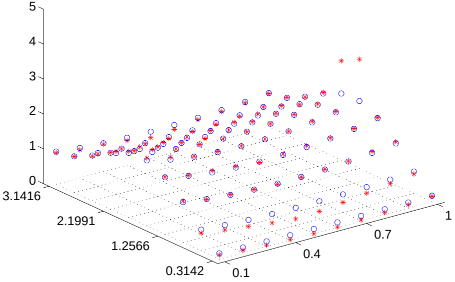
$$\begin{cases} -k(x)u'' + \beta u' + \gamma u = \mathrm{f}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

$$G : [0,1] \to \overline{\Omega},\ x = G(\hat{x}),\ \hat{x} \in [0,1]$$

$$\left\{ \frac{1}{n} A_n \right\}_n \sim_\lambda \frac{k(G(\hat{x}))}{|G'(\hat{x})|} f_p(\theta)$$

$$k(x) = e^x, \qquad G(\hat{x}) = \frac{1}{2}\hat{x}(\hat{x}+1), \qquad \beta = \gamma = 0$$





$p = 2, n = m^2 - p + 2, \ m = 10$ $\qquad$ $p = 3, n = m^2 - p + 2, \ m = 10$

o symbol samples $\qquad$ $*$ eigenvalues

Spectral analysis for GB-spline IgA (Galerkin/Collocation)

[Roman, Manni, Speleers, NM 2016]

- nested trigonometric/hyperbolic GB splines
  $< 1, t, \ldots, t^{p-2}, \cos \alpha t, \sin \alpha t >, \quad < 1, t, \ldots, t^{p-2}, e^{\alpha t}, e^{-\alpha t} >$
  same symbol as polynomial B-splines of the same degree
- not nested trigonometric/hyperbolic GB splines
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  same structure and properties of the symbol as polynomial B-splines with building blocks

$$f_p^{\mathbb{T}_\alpha}, f_p^{\mathbb{H}_\alpha}$$

$$\lim_{\alpha \to 0} f_p^{\mathbb{T}_\alpha} = \lim_{\alpha \to 0} f_p^{\mathbb{H}_\alpha} = f_p$$

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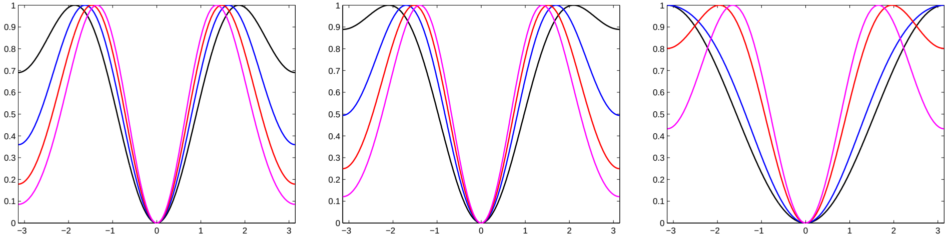

$p = 2, 3, 4, 5$

Concluding Message

- Bernstein/B-splines representations
    - crucial for efficiency of NURBS based IgA
    - not confined to (piecewise) polynomial spaces
- GB-splines
    - enjoy the same properties of B-splines
    - support local refinement based on local tensor-product structure
- Generalized (trigonometric/exponential...) B-splines behave similarly to NURBS in IgA, with problem-dependent improvements
    - B-splines/GB-splines plug-to-plug compatible in IgA
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Announcement

CIME Summer School

"Splines and PDEs: Recent Advances from Approximation Theory to Structured Numerical Linear Algebra"

Organizers:

Tom Lyche, Carla Manni, Hendrik Speleers

Date:

July 2–8, 2017

Place:

Hotel S. Michele, Cetraro, Italy