Products of random matrices: explicit formulas and asymptotics

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Outline

1. **Introduction to Random Matrices**
2. **Orthogonal Polynomial Ensembles**
3. **Products of Random Matrices**

1. Introduction to Random Matrices

Random Matrix Theory

- **Probability measure on some set of matrices**
- **Induced probability measure on eigenvalues**

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In many cases of interest:

- Exact formulas for eigenvalue distributions that are tractable to asymptotic analysis as size tends to infinity

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Asymptotic analysis

- Classical methods in case of contour integral representations or otherwise new methods such as Riemann-Hilbert problems

Connections and applications

Techniques and results from random matrix theory apply to other models

- **Random tiling problems**
- **Stochastic growth models**
- **Asymptotic representation theory**
- $\dots$

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Techniques and results from random matrix theory apply to **other models**

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- ...

Applications in

- Physics (nuclear physics, quantum chaos, 2D quantum gravity, ...)
- Multivariate statistics
- Number theory
- Wireless communication
- ...

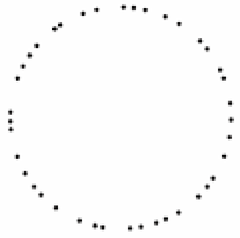
Example 1: CUE

Unitary group with Haar measure
= Circular Unitary Ensemble (CUE)

- **Joint density**

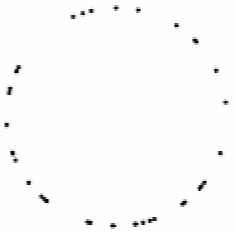
$$\frac{1}{(2\pi)^n n!} \prod_{j<k} |z_j - z_k|^2$$

with $z_1, \ldots, z_n$ on unit circle.



- **Nearest neighbour repulsion**

- **CUE eigenvalues (top) are much more regularly spaced than independent points on the circle (bottom)**



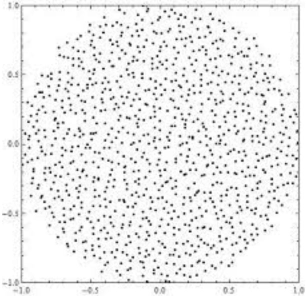
Example 2: Ginibre ensemble

Complex Ginibre matrix

- **Independent entries with standard complex Gaussian distribution**
- **Eigenvalue density**

$$\frac{1}{\pi^n \prod_{j=1}^{n} j!} \prod_{j<k} |z_j - z_k|^2 \prod_{k=1}^{n} e^{-|z_k|^2}$$

- **Ginibre eigenvalues fill out a disk**

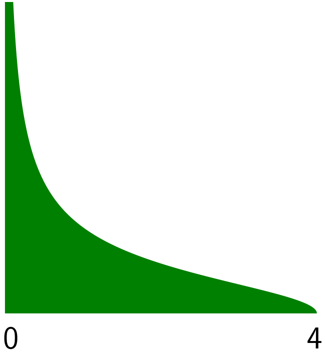


Example 3: Wishart matrix

Take $(n+\nu)\times n$ **complex Ginibre matrix** G

- **Wishart matrix** G^*G **has eigenvalue density**

$$\frac{1}{Z_n}\prod_{j<k}(x_j-x_k)^2\prod_{k=1}^{n}x_k^{\nu}e^{-x_k}, \qquad \textbf{all } x_k>0$$



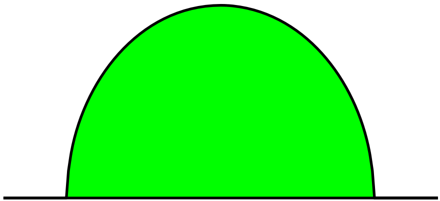
Marchenko-Pastur law **for scaled eigenvalues as size tends to infinity**

Example 4: GUE matrix

$G + G^*$ is **Gaussian Unitary Ensemble (GUE) matrix**

- **Hermitian matrix with independent complex Gaussian entries**
- **Eigenvalue density of GUE matrix**

$$\frac{1}{Z_n} \prod_{j<k} (x_j - x_k)^2 \prod_{k=1}^{n} e^{-\frac{1}{2}x_k^2}$$



Semi-circle law for scaled eigenvalues as size tends to infinity

2. Orthogonal Polynomial Ensembles

OP ensemble

$$\frac{1}{Z_n} \prod_{j<k} (x_j - x_k)^2 \prod_{k=1}^{n} w(x_k)$$

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- **Orthogonal polynomials**

$$\int_{-\infty}^{\infty} p_j(x)p_k(x)w(x)dx = \delta_{j,k}$$

are used to build the OP kernel

$$K_n(x,y) = \sqrt{w(x)w(y)}\sum_{j=0}^{n-1} p_j(x)p_j(y)$$

- **Correlation functions are expressed as determinants**

$$\det\left[K_n(x_j,x_k)\right]_{j,k=1}^{r}$$

- **Determinantal point process**

Gap probability

Probability that no eigenvalue is in $[a, b]$ is a Fredholm determinant

$$\det \left[I - K_n \mid_{[a,b]} \right] =$$

$$1 + \sum_{r=1}^{\infty} \frac{(-1)^r}{r!} \underbrace{\int_a^b \cdots \int_a^b}_{r \textbf{ integrals}} \det \left[K_n(x_j, x_k) \right]_{j,k=1}^{r} dx_1 \cdots dx_r$$

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- **Largest eigenvalue has distribution function**

$$F_n(s) = \textbf{Probability all eigenvalues } \le s$$

$$= \det\left[I - K_n\,|_{[s,\infty]}\right]$$

Tracy-Widom distribution

$$F_n(s) = \det\left[I - K_n \,|_{[s,\infty]}\right]$$

- **Limit theorem (after scaling, in many cases)**

$$\lim_{n\to\infty} F_n(a_n + tb_n) = F_{TW}(t)$$

exists. It is Tracy-Widom distribution.

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$$F_{TW}(t) = \det\left[I - K^{Airy} \,|_{[t,\infty]}\right]$$

with Airy kernel

$$K^{Airy}(x,y) = \frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\,\mathrm{Ai}(y)}{x-y}$$

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- **Also explicit formula in terms of a solution of the Painlevé II equation.**

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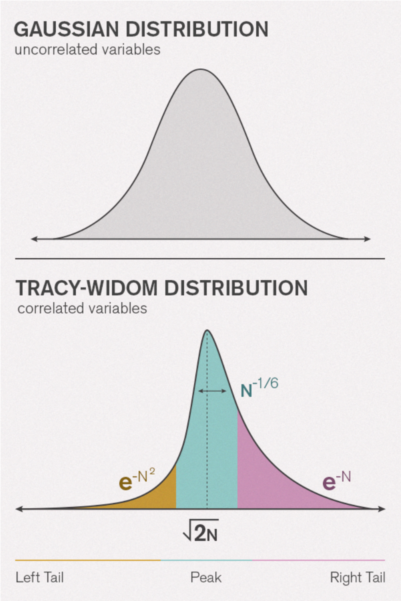

Natalie Wolchover
Mysterious statistical law may finally have an explanation
Quanta Magazine 2014

3. Products of Random Matices

Products of Ginibre matrices

Product of complex Ginibre matrices $Y = G_r \cdots G_1$

- **Eigenvalues** of Y are determinantal point process

$$\frac{1}{Z_n} \prod_{j<k} |z_k - z_j|^2 \prod_{j=1}^{n} w(|z|^2)$$

Akemann, Burda (2012)

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Akemann, Burda (2012)

- **Eigenvalues of $Y^* Y$ (squared singular values of Y) are determinantal point process on $[0, \infty)$**

$$\frac{1}{Z_n} \prod_{j<k} (y_k - y_j) \det \left[w_k(y_j) \right]_{j,k=1}^{n}$$

square matrices: Akemann, Kieburg, Wei (2013)
rectangular: Akemann, Ipsen, Kieburg (2013)

Transformation result

Theorem (K-Stivigny (2014))

Suppose squared singular values of X have joint density

$$\propto \prod_{j<k}(x_k - x_j) \det\left[f_k(x_j)\right]_{j,k=1}^{n}$$

Let G be $(n+\nu) \times n$ complex Ginibre matrix.
Then squared singular values of $Y = GX$ have density

$$\propto \prod_{j<k}(y_k - y_j) \det\left[g_k(y_j)\right]_{j,k=1}^{n}$$

with $$g_k(y) = \int_0^\infty x^\nu e^{-x} f_k\left(\frac{y}{x}\right) \frac{dx}{x}$$

g_k **is the Mellin convolution of** f_k **with** $x^\nu e^{-x}$

Transformation result

Complex Ginibre G_1 of size $(n \times \nu_1) \times n$

- **Joint density**

$$
\begin{aligned}
&\frac{1}{Z_n} \prod_{j<k} (x_k - x_j)^2 \prod_{j=1}^{n} x_j^{\nu_1} e^{-x_j} \\
=&\frac{1}{Z_n} \prod_{j<k} (x_k - x_j) \det \left[x_j^{k-1} \right]_{j,k=1}^{n} \prod_{j=1}^{n} x_j^{\nu_1} e^{-x_j} \\
=&\frac{1}{Z_n} \prod_{j<k} (x_k - x_j) \det \left[x_j^{k+\nu_1-1} e^{-x_j} \right]_{j,k=1}^{n}
\end{aligned}
$$

- **Apply transformation result $r-1$ times to find density for $G_r \cdots G_1$**

Product of Ginibre matrices

$Y = G_r \cdots G_1$ **with complex Ginibre matrices** G_j **of sizes**

$$(n+\nu_j) \times (n+\nu_{j-1}) \qquad \textbf{all } \nu_j \geq 0 \textbf{ and } \nu_0 = 0$$

Theorem (Akemann-Ipsen-Kieburg (2013))

Squared singular values of Y **have joint density**

$$\frac{1}{Z_n} \prod_{j<k} (y_k - y_j) \det \left[w_k(y_j) \right]_{j,k=1}^{n}$$

where functions w_k **are** $(r-1)$**-fold Mellin convolutions of gamma densities.**

They have Mellin transforms **Meijer G-functions**

$$\int_0^\infty x^{s-1} w_k(x) dx = s^{k-1} \prod_{j=1}^{r} \Gamma(s+\nu_j), \qquad \mathrm{Re}\, s > 0.$$

Transformation of kernel

Suppose $Y = GX$ **as before**

Transformation result for correlation kernel of the determinantal point process

- **Correlation kernel** $K_n^X \mapsto K_n^Y$

$$K_n^Y(x,y) = \frac{1}{2\pi i} \oint_\Sigma \frac{ds}{s} \int_0^\infty \frac{dt}{t} \left(\frac{t}{s}\right)^\nu e^{s-t} K_n^X\left(\frac{x}{s}, \frac{y}{t}\right)$$

Claeys-K-Wang (2015)

Correlation kernel

Suppose $Y = G_r \cdots G_1$ as before and apply the transformation result for the kernel repeatedly.

Correlation kernel K_n for the squared singular values of Y is a double contour integral

$$K_n(x,y) =$$

$$\frac{1}{(2\pi i)^2} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} ds \oint_{\Sigma} dt \prod_{j=0}^{r} \frac{\Gamma(s+\nu_j+1)}{\Gamma(t+\nu_j+1)} \frac{\Gamma(t-n+1)}{\Gamma(s-n+1)} \frac{x^t y^{-s-1}}{s-t}$$

- **Σ is a contour encircling the positive real axis, and not intersecting the vertical line $\mathrm{Re}\, s = -1/2$.**
- **Double integral is convenient for limit $n \to \infty$.**

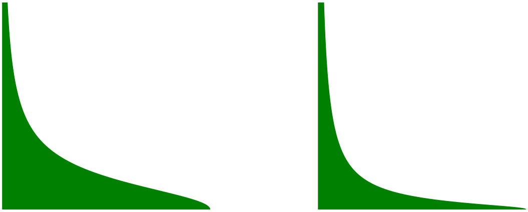
Macroscopic limit

- **Fix** $\nu_1, \ldots, \nu_r$ **and let** $n \to \infty$**.**
- **Largest eigenvalue of** Y^*Y **grows like** n^r**.**

Macroscopic limit

- **Fix $\nu_1, \ldots, \nu_r$ and let $n \to \infty$.**
- **Largest eigenvalue of Y^*Y grows like n^r.**

- **Limiting density $\rho_r(x)$ of rescaled eigenvalues exists**



$r = 1$ $\qquad$ $r = 3$

Density blows up $\quad \rho_r(x) \sim x^{-r/(r+1)} \quad$ **as** $x \to 0+$

- **Exponent depends on** r

Microscopic limit: universality

- **Scaling limits of**

$$K_n(x,y) = \frac{1}{(2\pi i)^2} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} ds \oint_{\Sigma} dt$$

$$\left(\prod_{j=0}^{r} \frac{\Gamma(s+\nu_j+1)}{\Gamma(t+\nu_j+1)} \right) \frac{\Gamma(t-n+1)}{\Gamma(s-n+1)} \frac{x^t y^{-s-1}}{s-t}$$

as $n \to \infty$ **with** $\nu_1, \ldots, \nu_r$ **fixed?**

We find

- **sine kernel in the bulk and Airy kernel at the right edge** **Liu-Wang-Zhang (to appear 2016)**
- **something new at the hard edge** 0**.**

Hard edge scaling limit

Theorem (K-Zhang (2014))

$$\lim_{n\to\infty} \frac{1}{n} K_n\left(\frac{x}{n}, \frac{y}{n}\right) = K_{\nu_1,\ldots,\nu_r}(x,y), \qquad x, y > 0$$

exists with limiting kernel

$$K_{\nu_1,\ldots,\nu_r}(x,y) =$$

$$\frac{1}{(2\pi i)^2} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} ds \int_{\Sigma} dt \left(\prod_{j=0}^{r} \frac{\Gamma(s+\nu_j+1)}{\Gamma(t+\nu_j+1)} \right) \frac{\sin \pi s}{\sin \pi t} \frac{x^t y^{-s-1}}{s-t}$$

It implies new limiting law for smallest squared singular value

$$\det \left[I - \left(K_{\nu_1,\ldots,\nu_r}\right) |_{[0,s]} \right]$$

- **Problem: What is the connection with Painlevé-type equations?**

Thank you for your attention