Approximating Riemannian Voronoi Diagrams and Delaunay triangulations.

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Outline

- Introduction and previous work
- Power Protection and stability
- Discrete approximations of the Voronoi diagram
- Combinatorial correctness of the discrete Voronoi Diagram: Protection and Sperner's lemma
- Distortion and straight simplices; Delaunay triangulations
- Experimental results

Introduction and previous work

How to prescribe anisotropy

Metric field g

Continuous map $g : p \in \Omega \mapsto g_p$, positive symmetric definite matrix

Normally:

$$\mathrm{length}(\gamma) = \int \langle \dot\gamma, \dot\gamma \rangle_g^{1/2} \mathrm{d}t = \int \sqrt{\dot\gamma^t(t) g_{\gamma(t)} \dot\gamma(t)} \mathrm{d}t$$

$$d_{\mathcal{M}}(p, q) = \inf_\gamma \mathrm{length}(\gamma)$$

g_p also defines an anisotropic distance if domain $\subset \mathbb{R}^d$:

$$d_{g_p}(a, b) = d_p(a, b) = \sqrt{(a-b)^t g_p (a-b)}$$

Local approximations

Locally $d_{g_p}(a, b)$ approximates $d_{\mathcal{M}}(p, q)$.

Moreover g_p defines and inner product, which after a linear transformation is the Euclidean.

Metric distortion ψ:
x, y in Ω, we have

$$\frac{1}{\psi} d_{G'}(x, y) \leq d_G(x, y) \leq \psi d_{G'}(x, y).$$

For $G' = g_p$ and $G = g$ the Metric distortion decreases if Ω smaller, $\|g_p - g_x\|$ small, for $x, p \in \Omega$.

Blanket assumption: point sets (ϵ, μ)-nets!

Riemannian Voronoi diagrams and approximations

Riemannian Voronoi diagrams

The Riemannian Voronoi cell of the site p is given by

$$V_{\mathcal{M}}(p_i) = \{x \in \Omega \mid \ d_{\mathcal{M}}(p_i, x) \leq d_{\mathcal{M}}(p_j, x), \forall p_j \in \mathcal{P} \setminus p_i\}.$$

Anisotropic Voronoi diagrams

The anisotropic Voronoi cell of the site p is given by:
Labelle and Shewchuk

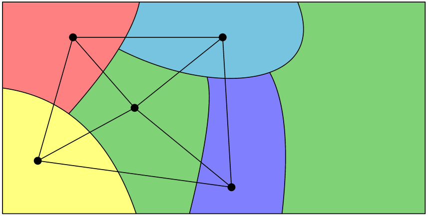
$$V_{LS}(p) = \{x \in \mathbb{R}^d : d_{g_p}(p, x) \leq d_{g_q}(q, x), \forall q \in \mathcal{P}, q \neq p\}$$

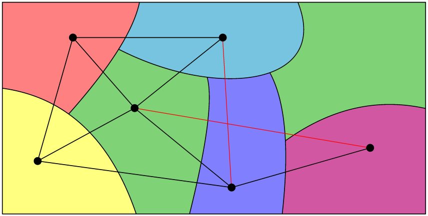
Du and Wong

$$V_{DW}(p) = \{x \in \mathbb{R}^d : d_{g_x}(p, x) \leq d_{g_x}(q, x), \forall q \in \mathcal{P}, q \neq p\}$$

Anisotropic Voronoi diagrams: Labelle and Shewchuk and Du and Wong

- Each site is within its cell
- Possibility of orphans (connected cells)
- The dual may not be a triangulation





Termination of up sampling and quality bounds can be proven in 2D [Labelle, Shewchuk 03]

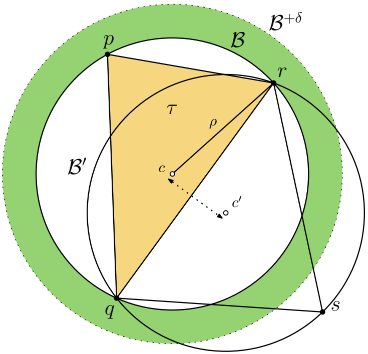
Riemannian Voronoi Diagrams

- Each site is within its cell
- No orphans (connected cells)
- Theoretical guarantees for dual to be a triangulation (Boissonnat, Dyer, Ghosh)

Protection [Boissonnat et al. 13]

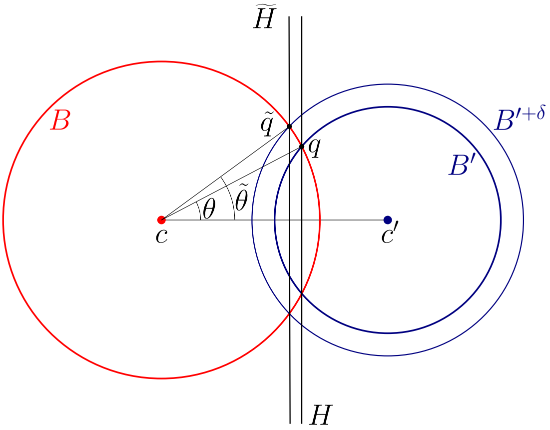
If $B(c, \rho)$ circum ball, then no alien vertices in $B(c, \rho + \delta)$.
Gives:

- lower bounds on height
- stability of the circumcentre



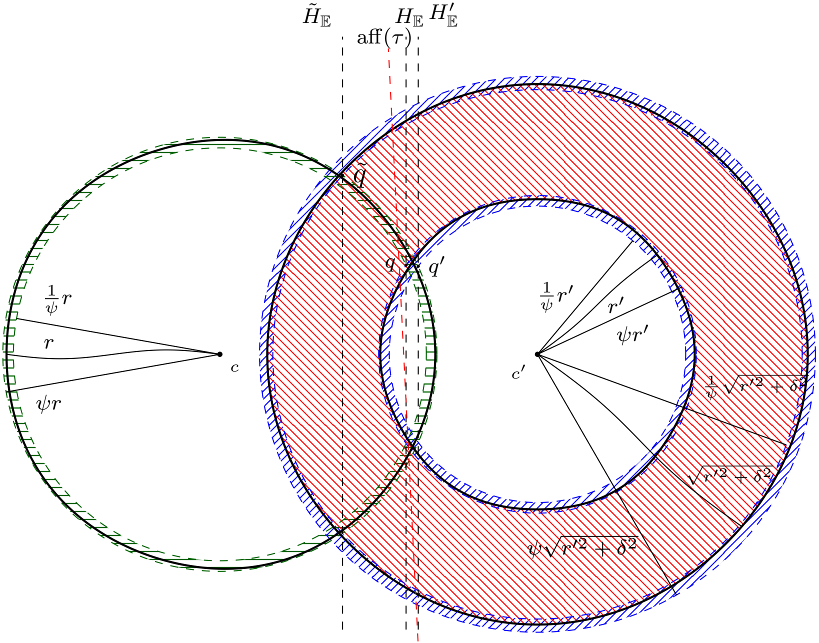
Power Protection and stability

Power protection and height



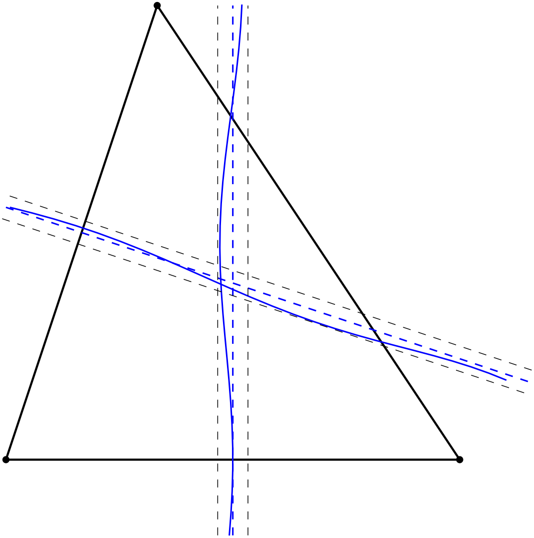
B and B' circumspheres of simplices, that share a face. Common face lies in H. Distance H, $\tilde{H}$ lower bounds height.

Lower bound height, gives bounds on angles, because (ϵ, μ)-net.

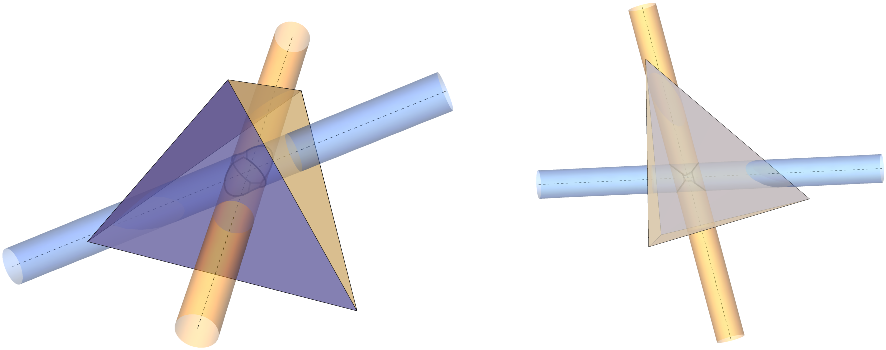

$\tilde{H}_{\mathbb{E}}$
$H_{\mathbb{E}}$
$H'_{\mathbb{E}}$
$\mathrm{aff}(\tau)$
$\tilde{q}$
q
q'
$\frac{1}{\psi}r$
r
ψr
c
c'
$\frac{1}{\psi}r'$
r'
$\psi r'$
$\frac{1}{\psi}\sqrt{r'^2+\delta^2}$
$\sqrt{r'^2+\delta^2}$
$\psi\sqrt{r'^2+\delta^2}$

Stability circumcentre

The circumcentre is stable with respect to perturbations and even with metric distortion



Stability in higher dimensions



With metric distortion intersection of the bisectors of 3 vertices of a face is constraint to a cylinder (in fact a parallel piped) orthogonal to the face, the circumcentre lies in the intersection of such cylinders. Induction on dimension is now possible.

Discrete approximations of the Voronoi diagram

Discrete Riemannian Voronoi diagrams

Geodesic distances cannot be computed exactly: must discretize

Canvas

The underlying structure used to compute geodesic distances (typically, an isotropic triangulation)

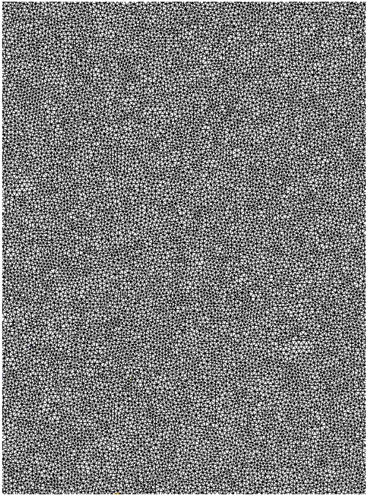
Many methods exist to compute geodesic distances and paths:

- Fast marching methods [Konukoglu et al. 07]
- Heat-kernel based methods [Crane et al. 13]
- Short-term vector Dijkstra [Campen et al. 13]

Computing geodesics from multiple sites is a natural extension from the propagation of geodesic distances from one site.

Construction of the discrete diagram

- Canvas generated fine enough so that the discrete diagram has the "correct" dual
- Color each vertex of the canvas with the closest site
- New sites of the discrete diagram are inserted through a farthest point refinement algorithm driven by a sizing field
- Connectivity is extracted



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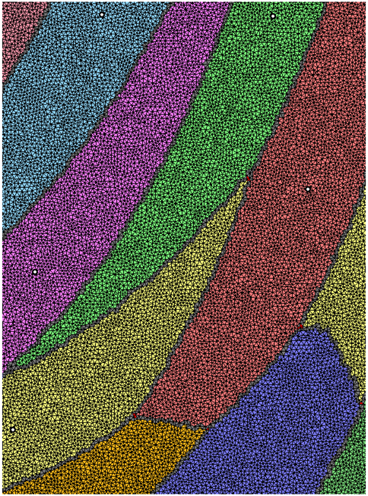

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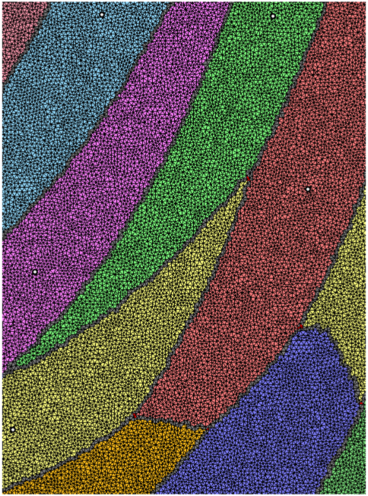

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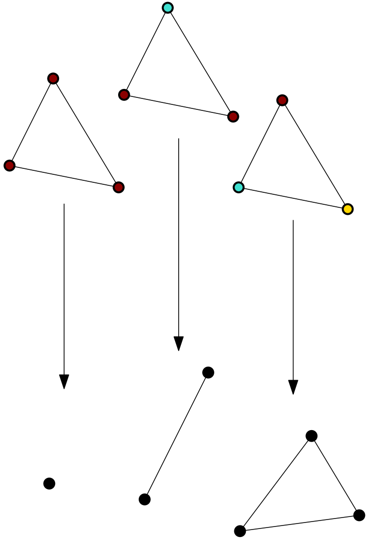
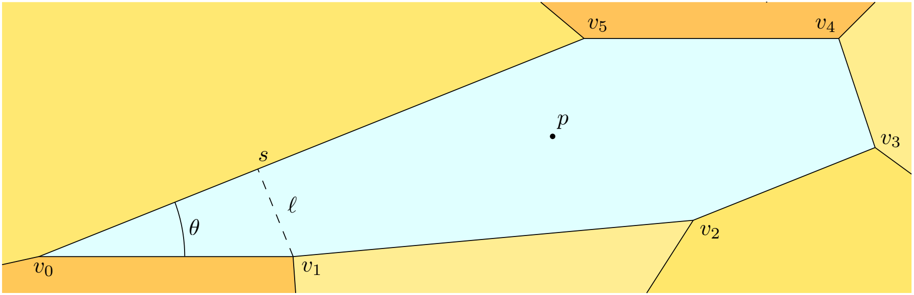


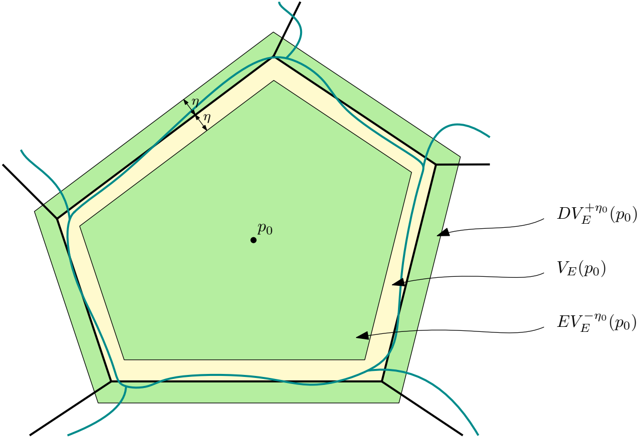
Combinatorial correctness of the discrete Voronoi diagram: Protection and Sperner's lemma

Combinatorial correctness: 2D



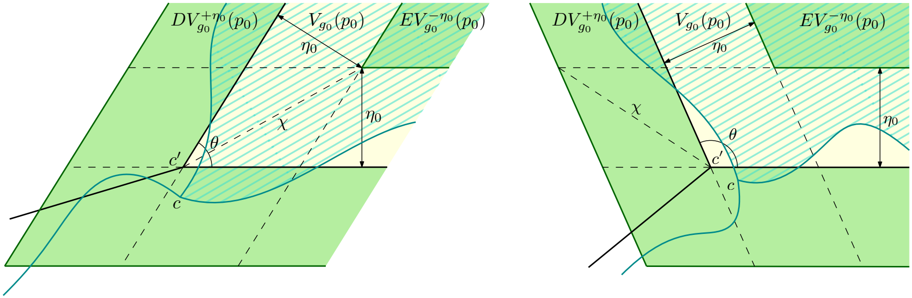
Thanks to protected (ϵ, μ)-net well shaped Voronoi cells; distance between foreign objects is large. It is impossible for triangle to be coloured in a way that does not correspond to a Voronoi vertex.

Combinatorial correctness: 2D with metric distortion



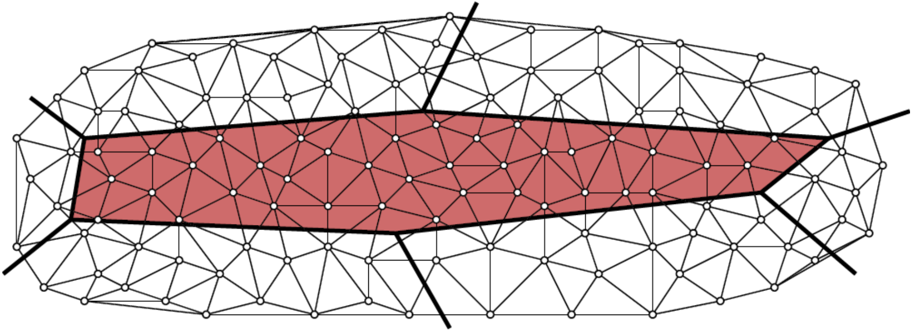
With metric distortion one needs a margin.

Combinatorial correctness: 2D with metric distortion



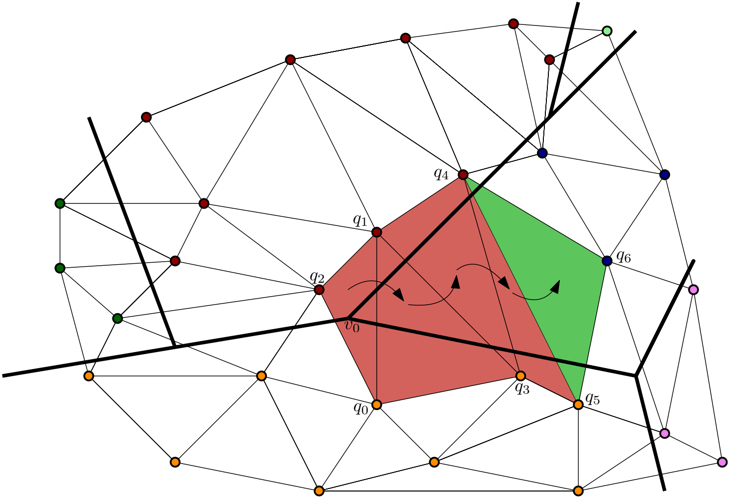
The Riemannian Voronoi vertices are close to the Euclidean vertices if the distortion is small.

Combinatorial correctness: 2D with metric distortion



The canvas has to be dense

Combinatorial correctness: 2D

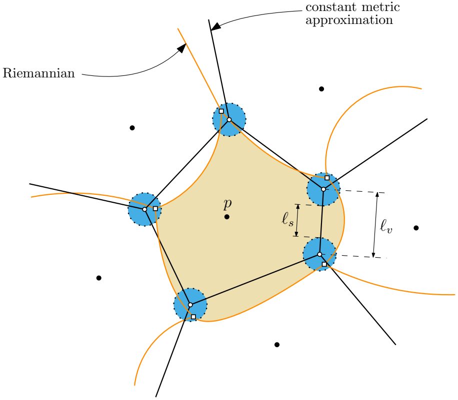


Thanks to induction one finds a three coloured simplex for every Voronoi vertex.

Higher dimensions

Combinatorial correctness:

- You have everything: Sperner's lemma
- You don't get more than this (power) protection (same as in 2D)



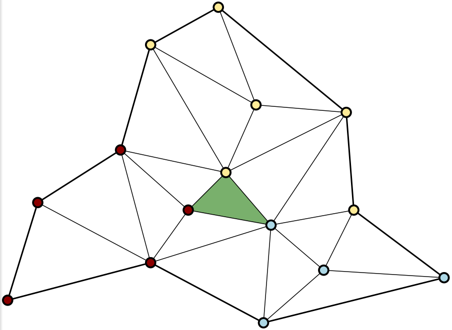
Sperner's Lemma

Theorem (Sperner's lemma)

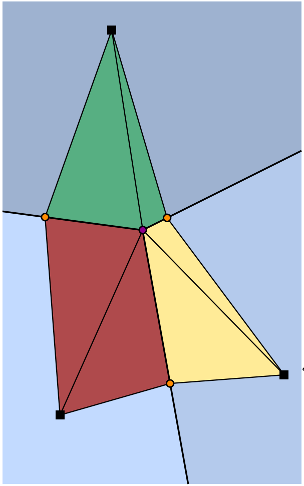
Let $\sigma = (e_0, \ldots, e_n)$ be an n-simplex and T_σ a triangulation of the simplex. Let $e' \in T_\sigma$ be colored such that:

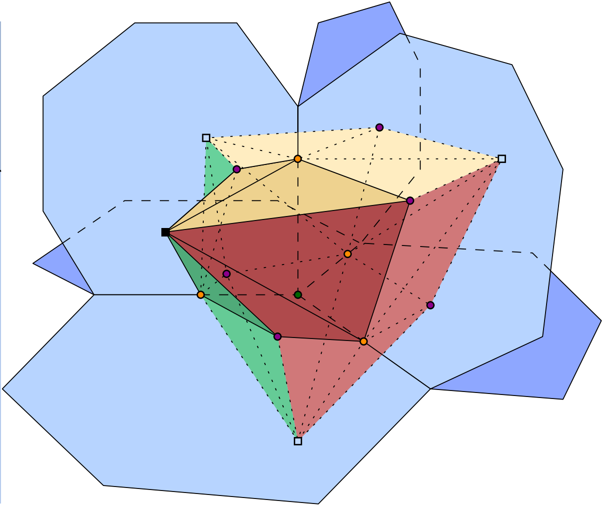
- *The vertices e_i of σ all have different colors.*
- *If e' lies on a k-face $(e_{i_0}, \ldots e_{i_k})$ of σ, then e' has the same color as one of the vertices of the face, that is e_{i_j}.*

Then, there exists at least one simplex in T_σ whose vertices are colored with all $n+1$ colors.



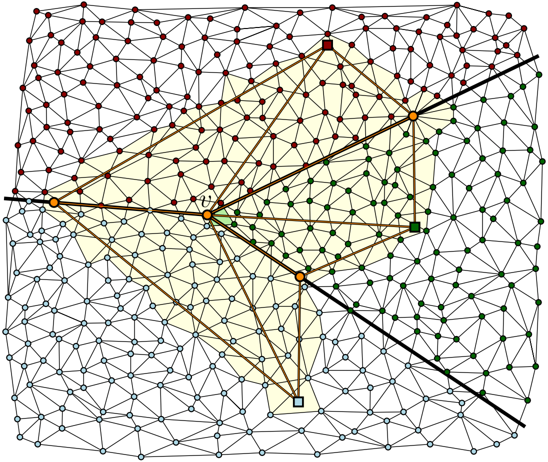
Construction of Sperner simplex





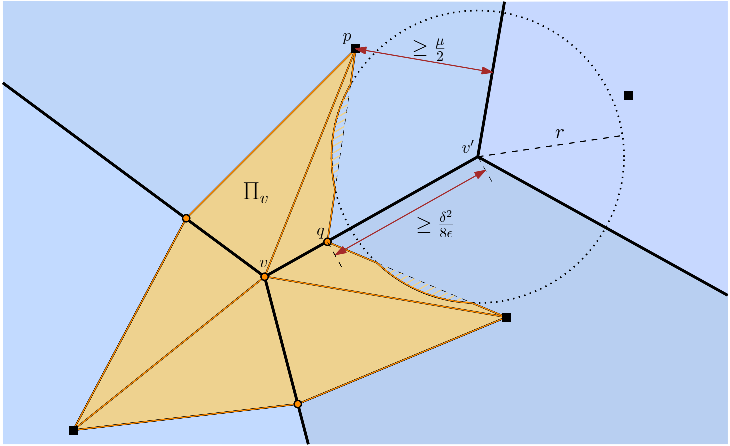
Find safe points (thanks to protection) on Voronoi objects.

Construction of Sperner simplex



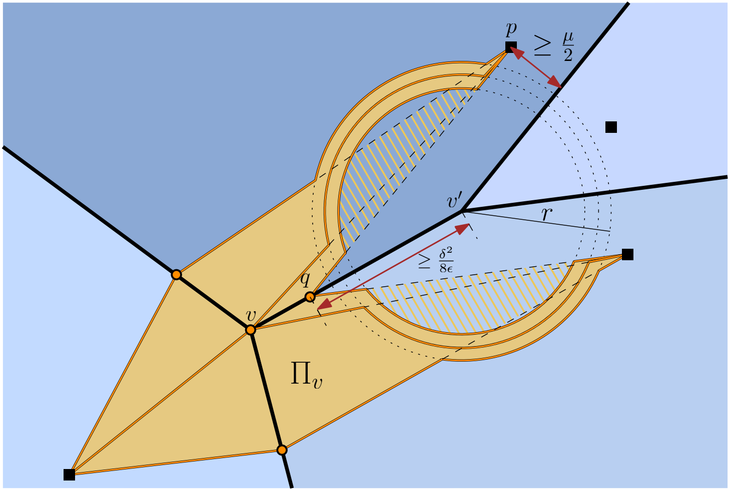
The simplex with the canvas.

Construction of Sperner simplex



Technical details: shift away from the voronoi vertices.

Construction of Sperner simplex

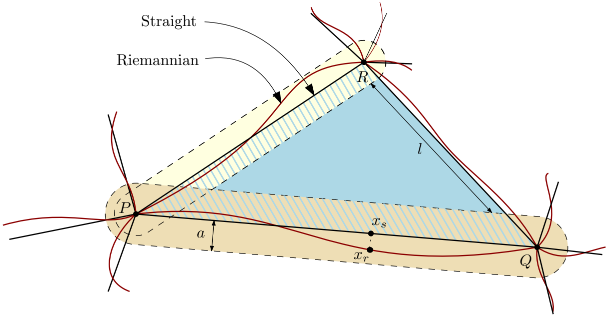


Technical details: shift away from the voronoi vertices.

Distortion and straight simplices; Delaunay triangulations

Embeddability of the straight dual: 2D

Low distortion: geodesics are close to straight edges

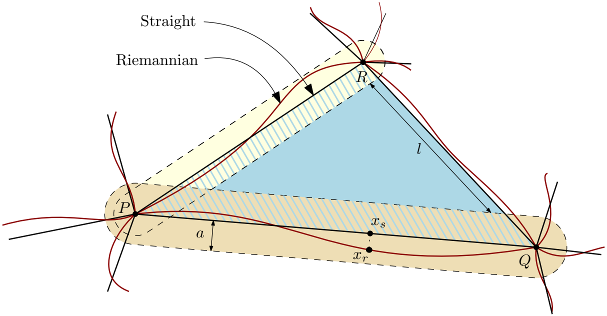


Embeddability

(ϵ, μ)-net, geodesics can be "straightened" without creating inversions.
Thus: embeddability of the Riemannian dual $\Longrightarrow$ embeddability of the straight dual

Embeddability of the straight dual: 2D

Low distortion: geodesics are close to straight edges



Embeddability

(ϵ, μ)-net, geodesics can be "straightened" without creating inversions.
Thus: embeddability of the Riemannian dual $\Longrightarrow$ embeddability of the straight dual

Centres of mass in Euclidean space

Weighted average of points

$$\sum \mu_i p_i,$$

with $\sum \mu_i = 1$.
Generalizes to

$$\int p \, \mathrm{d}\mu(p),$$

which is where the minimum of

$$P_{\mathbb{R}^n}(x) = \frac{1}{2} \int \|x - p\|^2 \mathrm{d}\mu(p)$$

is attained.

Theorem (Karcher)

Let $\mathcal{M}$ be a manifold whose sectional curvature K is bounded, that is $\Lambda_l \leq K \leq \Lambda_u$. Let $P_{\mathcal{M}}$ the function on B_ρ defined by

$$P_{\mathcal{M}}(x) = \frac{1}{2} \int d_{\mathcal{M}}(x,p)^2 \mathrm{d}\mu(p),$$

where $\mathrm{d}\mu$ is a positive measure and the support of $\mathrm{d}\mu$ is contained in B_ρ. We now give two conditions on ρ:

- *ρ is less than half the injectivity radius,*
- *if $\Lambda_u > 0$ then*

$$\rho < \frac{\pi}{2\sqrt{\Lambda_u}}.$$

If these conditions are met then P_M has a unique critical point in B_ρ, which is a minimum.

Riemannian simplices using Riemannian centres of mass

Definition (Riemannian simplex)

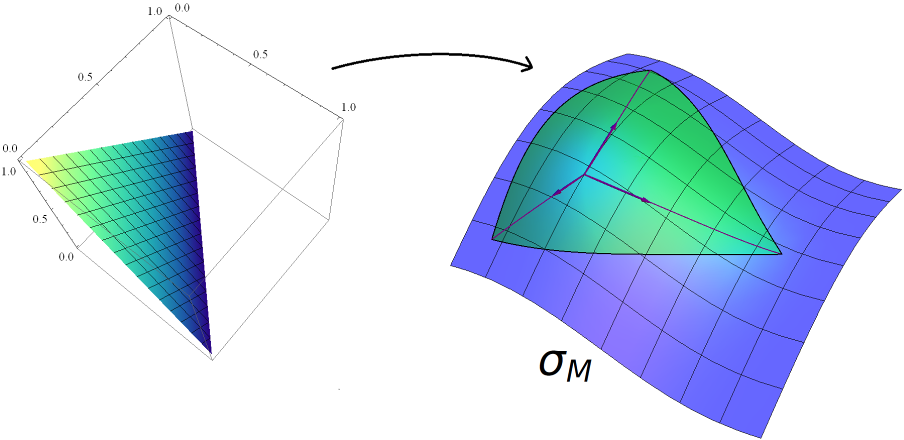
$$\mathcal{E}_\lambda(x) = \frac{1}{2}\sum_i \lambda_i d_{\mathcal{M}}(x, p_i)^2$$

barycentric coordinates: $\lambda_i \geq 0$, $\sum \lambda_i = 1$

$$\begin{aligned} \mathcal{B}_{\sigma^j} : & \triangle^j \to \mathcal{M} \\ & \lambda \mapsto \operatorname*{argmin}_{x \in \bar{B}_\rho} \mathcal{E}_\lambda(x) \end{aligned}$$

$\triangle^j$ the standard Euclidean j-simplex, $\sigma_{\mathcal{M}}$ image

Smooth map



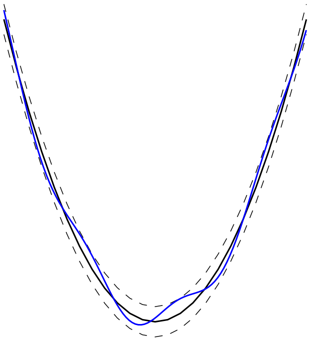
Distortion

We assume that $d_g(x,y)^2 = |x-y|^2 + \delta d_g^2(x,y)$,

$$\sum_i \lambda_i |x - p_i|^2 = \left| x - \sum_i \lambda_i p_i \right|^2 + \text{const.}$$

$\operatorname{argmin} \sum_i \lambda_i d_g(x,p_i)^2$ close to $\operatorname{argmin} \sum_i \lambda_i |x-p_i|^2$

Works also for continuous distributions and negative weights!

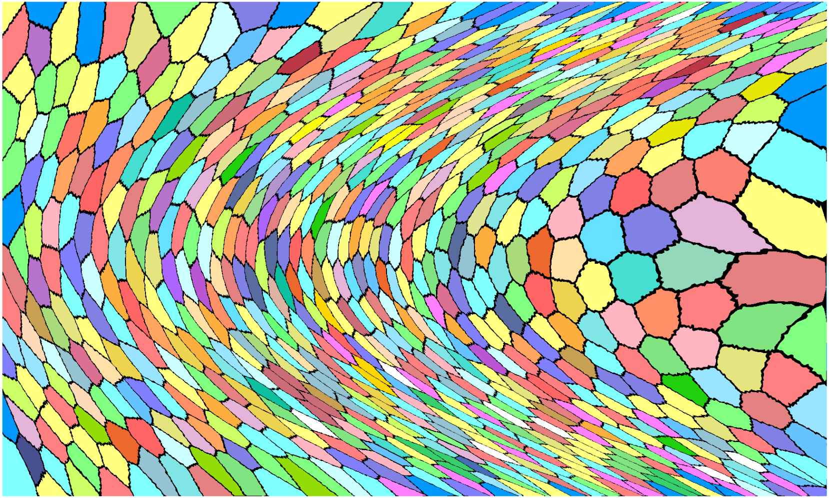


Riemannian simplex close to Euclidean simplex: straight simplex triangulation.

Experimental results

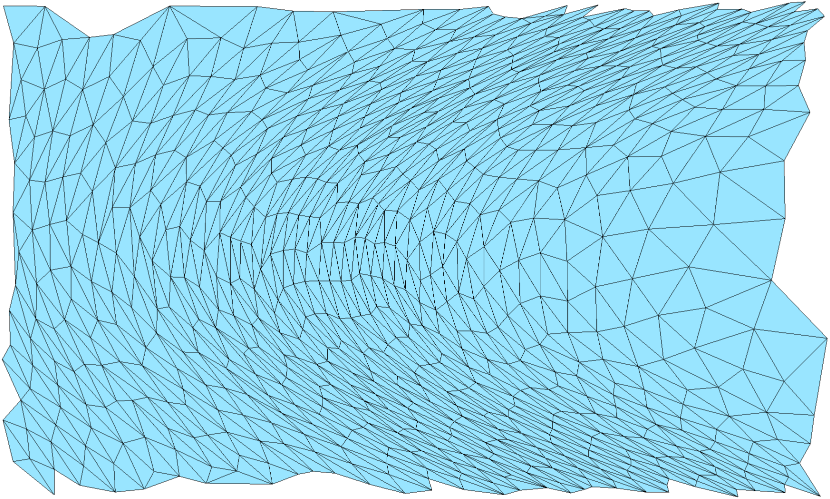
Discrete Riemannian Voronoi diagram: Result

Voronoi diagram of 750 sites



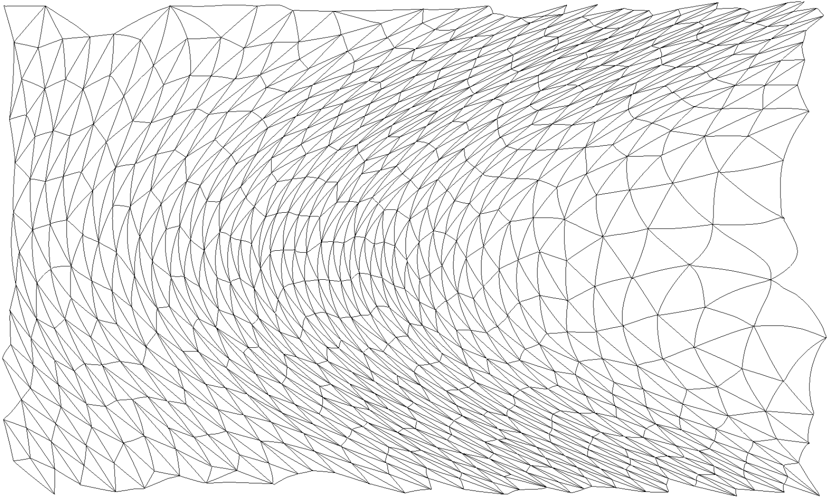
Straight Delaunay triangulation

(Straight) dual of the previous diagram



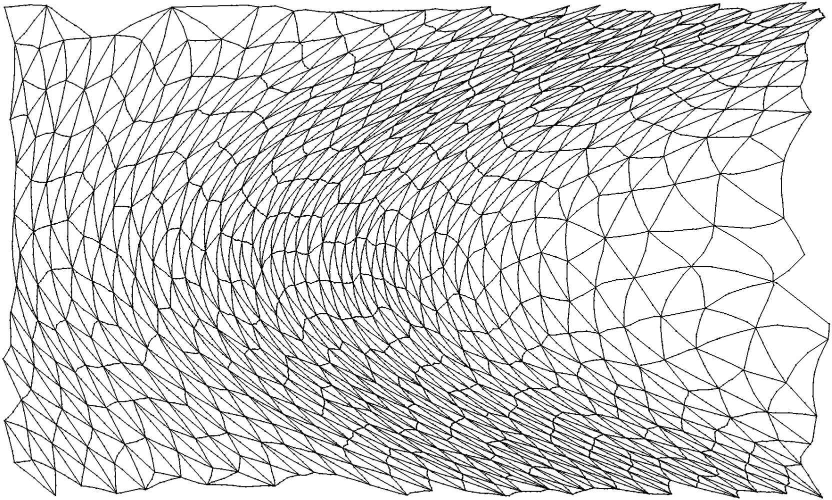
Riemannian Delaunay triangulation

Anisotropic and curved elements



Approximating Riemannian simplices

Piecewise flat approximation of Riemannian simplices



Acknowledgments

Thanks to Ramsay Dyer for suggesting Sperner's lemma, instead of dimension theory.

Questions?