Representations of p-adic groups and Hecke algebras

§1 Groups

F locally compact complete valued field with the associated notation from Vincent's talk

$\mathcal{O} \supset \mathcal{P}$ $\quad k = \mathcal{O}/\mathcal{P}$ residue field $\quad q = \# k \quad p = \operatorname{char} k$

t a uniformizer : $\mathcal{P} = t\mathcal{O}$

$v : F \longrightarrow \mathbb{Z} \cup \{\infty\}$ valuation , $v(t) = 1$

$|x| = q^{-v(x)}$ if $x \neq 0$

G a "reductive p-adic group", i.e., the group of F-rational points of a connected reductive group defined over F.

Examples: ① $GL_n(F)$

② Symplectic group $Sp_{2n}(F)$

G is a topological group with the topology coming from F. It is locally compact, totally disconnected.

G has many compact open subgroups (cosg) - they form a fundamental system of neighbourhoods of the identity.

Examples:

⓪ $F^\times \supset \mathcal{O}^\times$ is compact open

$\supset 1 + \mathfrak{P}^r \ , \ r > 0$

$\{1 + \mathfrak{P}^r \mid r > 0\}$ is a filtration of $\mathcal{O}^\times$.

① $GL_n(F)$. The maximal cosg's are conjugate to $GL_n(\mathcal{O})$

$K = K_0 = GL_n(\mathcal{O})$ has a filtration $(K_r)_{r>0}$, where

$K_r = 1 + \mathrm{Mat}_{n\times n}(\mathfrak{P}^{\lceil r \rceil}) \ , \ r \in \mathbb{R}_{>0}$

Put $K_{r+} = \bigcup_{s>r} K_s = 1 + \mathrm{Mat}_{n\times n}(\mathfrak{P}^{\lfloor r \rfloor + 1})$ for $r \geq 0$

Properties:

① $\{K_r \mid r \in \mathbb{R}_{>0}\}$ is a fund. sys. of nbhds of 1

$\bigcap_{r>0} K_r = \{1\}$

② The commutator group $[K_r, K_s] \subset K_{r+s}$ for $r,s > 0$

③ K_r / K_{r+} abelian for $r > 0$

$\Big[$ In fact, $K_r / K_{r+} = \begin{cases} 1 & , \text{ if } r \notin \mathbb{Z} \\ \mathrm{Mat}_{n\times n}(k) & , \text{ if } r \in \mathbb{Z}_{>0} \end{cases}$ $\Big]$

④ $\{r \in \mathbb{R}_{>0} \mid K_r \neq K_{r+}\}$ is discrete in $\mathbb{R}_{>0}$

We get an exact sequence:

$$\begin{tikzcd} 1 \arrow[r] & K_{0+} \arrow[r] & K_0 = GL_n(\mathcal{O}) \arrow[r, "\text{reduction mod } p"'] & GL_n(k) \arrow[r] & 1 \end{tikzcd}$$

The inverse image in K_0 of a parabolic subgroup is called a <u>parahoric subgroup</u>; the inverse image of a Borel subgroup is called an <u>Iwahori subgroup</u>.

So an Iwahori subgroup is conjugate to $\begin{pmatrix} \mathcal{O}^\times & & \mathcal{O} \\ & \ddots & \\ P & & \mathcal{O}^\times \end{pmatrix}$

and a parahoric ——— " ——— $K = \begin{pmatrix} \boxed{GL_{n_1}(\mathcal{O})} & & \mathcal{O} \\ & \ddots & \\ P & & \boxed{GL_{n_e}(\mathcal{O})} \end{pmatrix}$ $\sum_{i=1}^{e} n_i = n$

Parahoric subgroups have filtrations $(K_r)_{r\geq 0}$ with nice properties (① - ④ above)

$$K_{0^+} = 1 + \begin{pmatrix} Mat_{n\times n}(P) & & \mathcal{O} \\ & \boxed{} & \\ & & \ddots \\ P & & \boxed{P} \end{pmatrix}$$

We get an exact sequence:

$$1 \longrightarrow \underbrace{K_{0^+}}_{\substack{\text{pro-unipotent} \\ \text{radical}}} \longrightarrow \underset{\substack{\| \\ K}}{K_0} \longrightarrow \underbrace{\prod_{i=1}^{e} GL_{n_i}(k)}_{\text{reductive quotient of } K_0} \longrightarrow 1$$

<u>In general</u>, we have:

- Iwahori subgroups, which are minimal
- Parahoric subgroups $K = K_0$, which have filtrations $(K_r)_{r\geq 0}$ with K_0 / K_{0^+} a finite connected reductive group.

Example ② In $Sp_{2n}(F)$ there are $(n+1)$ conjugacy classes of maximal parabolic subgroups, represented by

$$K_0^{(i)} = \left(\begin{array}{c|c|c} \mathcal{O} & \mathcal{O} & \mathfrak{p}^{-1} \\ \hline \mathfrak{p} & \mathcal{O} & \mathcal{O} \\ \hline \mathfrak{p} & \mathfrak{p} & \mathcal{O} \\ \underbrace{\qquad}_{i} & \underbrace{\qquad}_{2(n-i)} & \underbrace{\qquad}_{i} \end{array}\right) \cap Sp_{2n}(F) \quad \text{for } 0 \leq i \leq n$$

$$1 \longrightarrow K_{0+}^{(i)} \longrightarrow K_0^{(i)} \longrightarrow Sp_{2i}(k) \times Sp_{2(n-i)}(k) \longrightarrow 1$$

$$K_{0+}^{(i)} = \left(1 + \left(\begin{array}{c|c|c} \mathfrak{p} & \mathcal{O} & \mathcal{O} \\ \hline \mathfrak{p} & \mathfrak{p} & \mathcal{O} \\ \hline \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \underbrace{\qquad}_{i} & \underbrace{\qquad}_{2(n-i)} & \underbrace{\qquad}_{i} \end{array}\right)\right) \cap Sp_{2n}(F)$$

§2 Representations

G a closed subgroup of a reductive p-adic group
C a fixed alg. closed field of char ℓ.

Def^n A smooth representation of G is a pair (π, V) with V a C-vector space, $\pi : G \longrightarrow \mathrm{Aut}_C(V)$ is a homomorphism s.t. $\forall\, v \in V$ Stab(v) is open.

Equivalently, writing

$V^H = \{ v \in V \mid \pi(h)v = v \ \forall h \in H \}$

V is smooth if

$V = \bigcup_{K \text{ cosg}} V^K$

Write

$\mathcal{R}(G)$ – the category of smooth representations of G

$\mathrm{Irr}(G)$ – the set of classes of irred. smooth rep^s of G

Prop. If G is compact and $l = 0$, then $\mathcal{R}(G)$ is semisimple.

Examples:

(1) A quasi-character $\chi : F^\times \longrightarrow \mathbb{C}^\times$ is smooth iff $\mathrm{Ker}(\chi)$ contains $1 + \mathfrak{P}^r$ for $r >> 0$.

(2) $G = GL_n(F)$. Suppose $\pi \in \mathrm{Irr}(G)$ is finite-dimensional.
If $\mathcal{B}$ is a basis for V_π, then $\ker(\pi) \supseteq \bigcap_{v \in \mathcal{B}} \mathrm{Stab}_G(v)$ is open, so contains $K_r = 1 + \mathrm{Mat}_{n \times n}(\mathfrak{P}^r)$ for some $r > 1$
But every unipotent elt of $GL_n(F)$ is conjugate to an element of K_r, so is contained in $\mathrm{Ker}(\pi)$.
Since $SL_n(F)$ is generated by unipotent elts, $\ker(\pi) \supset SL_n(F)$ and π factors through $\det : GL_n(F) \longrightarrow F^\times$. Since $F^\times$ is abelian and π is irred., $\pi = \chi \circ \det$ for some χ as in (1), $\boxed{\text{1-dim}}$

Schur's Lemma: Suppose $\pi \in \mathrm{Irr}(G)$. If either $\mathbb{C}$ is uncountable or π is admissible [V^K is fin.dim^l $\forall$ cosg K], then $\mathrm{End}_G(\pi) = \mathbb{C}$.

Question: What plays the role of the group algebra?

Put $\mathcal{H}(G) = C_c(G) = \{$locally constant $f : G \to \mathbb{C}$ with compact support $\}$

$\hookrightarrow$ $\mathrm{supp}(f) = \overline{\{g \in G \mid f(g) \neq 0\}}$

E.g. If Y is open-closed and compact (e.g. a coset of a cosg) then $\mathbb{1}_Y$ = char. function of Y is in $\mathcal{H}(G)$

G acts by left (or right) translation on $\mathcal{H}(G)$.

For K cosg $\mathcal{H}(G/\!/K) = \{K$-bi-invariant functions in $\mathcal{H}(G)\}$ has basis $\{\mathbb{1}_{KgK} \mid KgK \in K\backslash G/K\}$ and $\mathcal{H}(G) = \bigcup_{K \text{ cosg}} \mathcal{H}(G/\!/K)$

To define a convolution on $\mathcal{H}(G)$ we need a measure

A left Haar measure in G is a non-zero linear form $\mu : \mathcal{H}(G) \longrightarrow \mathbb{C}$ which is invariant under left translation.

Write $\mu(f) = \int_G f(g)\,dg$ and for $Y \subset G$, $\mu(Y) = \mu(\mathbb{1}_Y)$

Rmk There is a Haar measure because $\ell \neq p$, but you can get cosgs of measure 0 e.g. in $GL_2(F)$ if $\ell \mid q^2 - 1$, then $\mu(GL_2(\mathcal{O})) = 0$.

Now, if $f, f' \in \mathcal{H}(G)$, define $f * f' \in \mathcal{H}(G)$ by

$$f * f'(h) = \int_G f(g) f'(g^{-1}h)\, dg$$

$\mathcal{H}(G)$ is the *(global) Hecke algebra*.

$\mathcal{H}(G)$ has no unit, but lots of idempotents e.g. if K is compact open and $\mu(K) \neq 0$, then $e_K := \frac{1}{\mu(K)} \mathbb{1}_K$ is idempotent and $e_K \mathcal{H}(G) e_K = \mathcal{H}(G /\!/ K)$

Def^n An $\mathcal{H}(G)$-module is *smooth* if $M = \bigcup_{K \text{ cosg}} e_K M$

$\mathcal{H}(G)$-Mod = the category of smooth $\mathcal{H}(G)$-modules.

Thm There is an eq. of categories

$$\begin{array}{ccccl} R(G) & \longrightarrow & \mathcal{H}(G)\text{-Mod} & & \\ (\pi, V) & \longmapsto & V & & \text{with } \mathbb{1}_{gK} \cdot v = \mu(K)\pi(g)v \text{ for } v \in V^K \\ (\pi, M) & \longleftarrow & M & & \text{with } \pi(g)m = \mu(K)^{-1}\mathbb{1}_{gK} \cdot m \text{ for } m \in e_K M \end{array}$$

§3 Adjunctions & H-C theory

H is a closed subgp of G

The restriction functor Res_H^G has a right adjoint Ind_H^G

i.e., $\mathrm{Hom}(\mathrm{Res}_H^G \pi, \rho) = \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \rho)$

$\mathrm{Ind}_H^G \rho$ is given by the right regular action of G on

$$\left\{ f : G \longrightarrow V_\rho \;\middle|\; \begin{array}{l} f(hg) = \rho(h) f(g) \quad \text{for } h \in H, g \in G \\ \exists \text{ cosg } K \text{ s.t. } f(gk) = f(g) \text{ for } g \in G, k \in K \end{array} \right\}$$

Ind_H^G is exact and transitive.

When H is also open in G, Res_H^G has a left adjoint ind_H^G (compact induction)

$\mathrm{ind}_H^G \rho$ is given by the right regular action on

$\{ f \in \mathrm{Ind}_H^G \rho \mid \mathrm{supp}(f) \text{ is compact modulo } H \}$

e.g. $\mathrm{ind}_{\mathcal{O}^\times}^{F^\times} \mathbb{1} = \{ f : \underbrace{F^\times / \mathcal{O}^\times}_{\mathbb{Z} = \langle t \rangle} \longrightarrow \mathbb{C} \text{ with finite support} \} \simeq \mathbb{C}[X, X^{-1}]$

has infinite length, with quotients every q-char $\chi : F^\times \longrightarrow \mathbb{C}^\times$ which is trivial on $\mathcal{O}^\times$ (unramified q-chars) but no irred. subrepn^s

Rmk If H is a parabolic sbgp of reductive p-adic G, then $H \backslash G$ is compact, so Ind_H^G and ind_H^G coincide.

Inflation

If H is closed and normal in G

$$\left(\begin{array}{lcc} \text{e.g. unipotent radical in a parabolic} & & \\ \qquad\qquad K_{0+} & \text{in} & K_0 \\ \quad \text{pro-unipotent radical} & & \text{parahoric} \end{array} \right)$$

then $\mathrm{Infl}_{G/H}^{G}$ has a left adjoint given by taking co-invariants.

For (π, V) a smooth repⁿ of G,

$V_H := V / V(H)$ where $V(H) = \langle \pi(h)v - v \mid h \in H, v \in V \rangle$

largest quotient on which H acts trivially.

If H is the union of its compact subgroups

– e.g. the unipotent radical of a parabolic $\mathcal{U} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} = \bigcup_{r \in \mathbb{Z}_{>0}} \begin{pmatrix} 1 & p^r \\ 0 & 1 \end{pmatrix}$

then $V \longmapsto V_H$ is exact.

Parabolic induction & restriction

$$\begin{array}{ccl} P = & L\,\mathcal{U} & \text{parabolic subgroup} \\ & \nearrow\ \nwarrow & \\ \text{Levi} & & \text{unipotent radical} \end{array}$$

We define functors
$$\begin{tikzcd} R(L) \arrow[r, bend left, "\ell_P^G"] & R(G) \arrow[l, bend left, "r_P^G"] \end{tikzcd}$$

normalized parabolic induction/ restriction

$\ell_P^G(\rho) = \mathrm{Ind}_P^G \, \mathrm{Infl}_L^P (\rho \otimes \delta_P^{1/2})$

$r_P^G(\pi) = (\mathrm{Res}_P^G \pi)_{\mathcal{U}} \otimes \delta_P^{-1/2}$

where δ_P is the

modulus character

For G a closed sbgp of a p-adic group

$\delta_G : G \longrightarrow \mathbb{C}^\times$ defined by

$$\delta_G(g) = \frac{(gKg^{-1} : gKg^{-1} \cap K)}{(K : gKg^{-1} \cap K)}$$ for any cosg K.

E.g. $\delta_G = \mathbb{1}$ if
- G is compact
- G is reductive p-adic

e.g. $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subset GL_2(F)$

$$\delta_B \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = |ad^{-1}|$$

Parabolic induction and restriction are transitive, exact, preserve finite length

$\rightsquigarrow$ Get notions of cuspidal and of cuspidal pair

↳ $r_P^G(\pi) = 0$ ↳ (L, ρ), ρ a cuspidal of L

Thm (Harish-Chandra) For any $\pi \in \mathrm{Irr}(G)$, there are a Levi sbgp L and a cuspidal rep^n ρ of L such that $\pi \hookrightarrow I_P^G(\rho)$ for some parabolic $P = LU$.

Moreover, (L, ρ) is unique up to G-conjugacy.

(L, ρ) (or its conjugacy class) is called the cuspidal support of π.

§4 Bernstein decomposition

Def^n Two cuspidal pairs (L_1,ρ_1), (L_2,ρ_2) are <u>inertially equivalent</u> if $\exists$ $g \in G$ and $\chi : L_2 \longrightarrow \mathbb{C}^\times$ trivial on all compact subgroups, s.t. $L_2 = {}^gL_1$ and $\rho_2 = {}^g\rho_1 \otimes \chi$. Such χ is called <u>unramified</u>.

e.g. $G = GL_n(F)$, the unramified q-chars are $\psi \circ \det(\ )$ for some unramified quasi-character $\psi : F^\times \longrightarrow \mathbb{C}^\times$.

Write $[L,\rho]_G$ for the equiv. class of (L,ρ)

$\mathfrak{B}(G)$ = the set of inertial equiv. classes

Def^n For $\mathfrak{s} \in \mathfrak{B}(G)$, define $\mathcal{R}^{\mathfrak{s}}(G)$ to be the full subcategory of $\mathcal{R}(G)$ with objects those reps all of whose irred. s/quotients have cusp. support in $\mathfrak{s}$.

Thm (Bernstein) $\ell = 0$

$\mathcal{R}(G) = \prod_{\mathfrak{s}\in\mathfrak{B}(G)} \mathcal{R}^{\mathfrak{s}}(G)$ is a block decomposition

i.e.,

any V decomposes $V = \bigoplus_{\mathfrak{s}} V^{\mathfrak{s}}$ with $V^{\mathfrak{s}}$ in $\mathcal{R}^{\mathfrak{s}}(G)$

$\mathrm{Hom}_G(V,W) = \prod_{\mathfrak{s}} \mathrm{Hom}_G(V^{\mathfrak{s}}, W^{\mathfrak{s}})$

and each $\mathcal{R}^{\mathfrak{s}}(G)$ is indecomposable.

Problem For each S, find an algebra $\mathcal{H}_S$ s.t. $\mathcal{R}^S(G)$ is equiv. to $\mathcal{H}_S$-Mod.

§5 Theory of types & covers

From now on, $l = 0$.
$S \in \mathcal{B}(G)$.

Def^n A pair (K, τ) with K a cosg of G and $\tau \in \mathrm{Irr}(K)$ is called an S-type if, for $\pi \in \mathrm{Irr}(G)$,
$\mathrm{Hom}_K(\tau, \pi) \neq 0 \iff$ cusp. support of π is in S.
Given (K,τ) such a pair, define the spherical Hecke algebra
$\mathcal{H}(G,\tau) = \mathrm{End}_G(\mathrm{ind}_K^G \tau)$

$$\mathcal{H}(G,\tau) \xrightarrow{\sim} \left\{ f : G \to \mathrm{End}_{\mathbb{C}}(V_\tau) \,\middle|\, \begin{array}{l} f(kgk') = \tau(k) f(g) \tau(k') \ \ k,k' \in K, g \in G \\ \mathrm{supp}(f) \text{ compact} \end{array} \right\}$$

$$\phi \longmapsto f_\phi : g \mapsto (v \mapsto \phi(T_v)(g)) \quad \text{where}$$

$T_v \in \mathrm{ind}_K^G \tau$ is given by $T_v(h) = \begin{cases} \tau(h)v & \text{if } h \in K \\ 0 & \text{otherwise} \end{cases}$

Then $\mathrm{supp}(\phi) = \mathrm{supp}(f_\phi)$ is a finite union of (K,K)-double cosets and $\mathrm{supp}\, \mathcal{H}(G,\tau) = \{ g \in G \mid \mathrm{Hom}_{{}^gK \cap K}({}^g\tau, \tau) \neq 0 \}$.

e.g. If $\tau = \mathbb{1}_K$, then $\mathcal{H}(G, \mathbb{1}_K) = \mathcal{H}(G/\!\!/K)$

If K is an Iwahori subgroup, then this is the affine I-H algebra

Thm (Bushnell - Kutzko) If (K,τ) is an s-type, then we get an equivalence of categories

$$\begin{array}{ccc} \mathcal{R}^s(G) & \longrightarrow & \mathcal{H}(G,\tau)\text{-Mod} \\ \pi & \longmapsto & \operatorname{Hom}_K(\tau,\pi) \\ & & \rotatebox{90}{$=$} \\ & & \operatorname{Hom}_G(\operatorname{ind}_K^G \tau, \pi) \end{array}$$

New strategy: For each $\mathfrak{s}$, find an $\mathfrak{s}$-type (K,τ) and compute $\mathcal{H}(G,\tau)$

How does this fit with H-C theory?

Covers

Given a cuspidal pair (L,ρ) we get

$\mathfrak{s} = [L,\rho]_G$ an inertial class in $\mathcal{B}(G)$

$\mathfrak{s}_L = [L,\rho]_L$ ——— " ——— in $\mathcal{B}(L)$

Suppose we have an $\mathfrak{s}_L$-type (K_L, τ_L)

Example: $G = GL_n(F)$, $L = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \subseteq P = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$, $\rho = \mathbb{1}_L$

Then $K_L = \begin{pmatrix} \mathcal{O}^\times & & \\ & \ddots & \\ & & \mathcal{O}^\times \end{pmatrix}$ and $\tau_L = \mathbb{1}_{K_L}$ gives an $[L,\rho]_L$-type

$K = \begin{pmatrix} \mathcal{O}^\times & & \mathcal{O} \\ & \ddots & \\ \mathfrak{p} & & \mathcal{O}^\times \end{pmatrix} = \begin{pmatrix} 1 & & \\ & \ddots & \\ \mathfrak{p} & & 1 \end{pmatrix} \begin{pmatrix} \mathcal{O}^\times & & \\ & \ddots & \\ & & \mathcal{O}^\times \end{pmatrix} \begin{pmatrix} 1 & & \mathcal{O} \\ & \ddots & \\ & & 1 \end{pmatrix}$

$\tau = \mathbb{1}_K$

Def^n A pair (K,τ) is called a cover of (K_L,τ_L) if

① For $P = LU$ parabolic with opposite $\overline{P} = L\overline{U}$

$K = (K\cap \overline{U})(K\cap L)(K \cap U) \longleftarrow$ Iwahori decomposition

with $K \cap L = K_L$.

② $\tau|_{K_L} = \tau_L$ and τ is trivial on $K\cap U$, $K \cap \overline{U}$

③ There is an invertible element $\phi \in \mathcal{H}(G,\tau)$ s.t.

$\mathrm{supp}(\phi) = KJK$ with $J \in Z(L)$ s.t. $\bigcap_{r\geq 0} J^r(K\cap U)J^{-r} = \{1\}$

strongly positive $\bigcap_{r\geq 0} J^{-r}(K\cap \overline{U})J^{r} = \{1\}$

In the previous example, $J = \begin{pmatrix} t^{n-1} & & & \\ & t^{n-3} & & \\ & & \ddots & \\ & & & t^{1-n} \end{pmatrix}$ will do

(K,τ) is a cover of (K_L,τ_L)

Thm (Bushrell-Kutzko)

Suppose (K,τ) is a cover of (K_L,τ_L), an S_L-type.

Then (K,τ) is an S-type and there is an algebra embedding

$t_P : \mathcal{H}(L,\tau_L) \longrightarrow \mathcal{H}(G,\tau)$ s.t.

$$\begin{tikzcd}
R^s(G) \arrow[r, "M_\tau"] \arrow[d, bend left, "\mathrm{proj}_{s_L}\cdot r_P^c"] & \mathcal{H}(G,\tau)\text{-Mod} \arrow[d, bend left, "(t_P)^*"] \\
R^{s_L}(L) \arrow[r, "M_{\tau_L}"] \arrow[u, bend left, "i_P^c"] & \mathcal{H}(K_L,\tau_L)\text{-mod} \arrow[u, bend left, "(t_P)_*"]
\end{tikzcd}$$

From now on, $C = \mathbb{C}$.

<u>Example</u>: $G = SL_2(F)$, $p \neq 2$

$\overset{\rotatebox{90}{$\subseteq$}}{L} = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \middle| \, x \in F^\times \right\} \simeq F^\times \supset \mathcal{O}^\times = K_L$, $\tau_L = \mathbb{1}_L$

(K_L, τ_L) is an $[L, \mathbb{1}_L]_L$-type

$$\begin{tikzcd}
\mathbb{1}_I & I = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} \end{pmatrix} \cap G \arrow[r, hook] \arrow[d] & SL_2(\mathcal{O}) = K^{(1)} \arrow[d, "\text{reduction}"', "\text{(mod p)}"] & = I \sqcup I s_1 I, \\
\mathbb{1}_{B(k)} \arrow[u, "\text{Infl}"'] & B(k) \arrow[r, hook] & SL_2(k) & s_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\end{tikzcd}$$

$\mathcal{H}(G, \mathbb{1}_I) \hookleftarrow \{ f \mid supp(f) \subseteq K^{(1)} \} = \mathcal{H}(K^{(1)}, \mathbb{1}_I) \simeq \mathcal{H}(SL_2(k), \mathbb{1}_{B(k)})$

$\rotatebox{90}{$\in$}$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$

$T_1 \qquad\qquad \{ f : SL_2(k) \to \mathbb{C} \mid f(bhb') = f(h) \;\; b, b' \in B(k), h \in SL_2(k) \}$

$\shortparallel$

with $\qquad\qquad\qquad\qquad \mathrm{End}_{SL_2(k)}\left(\mathrm{ind}_{B(k)}^{SL_2(k)} \mathbb{1}\right)$

$\rotatebox{90}{$\in$}$

$supp(T_1) = I s_1 I \longleftarrow \bar{T}_1$ with $supp(\bar{T}_1) = B(k) \bar{s}_1 B(k)$

$(T_1 + 1)(T_1 - q) = 0 \qquad\qquad (\bar{T}_1 + 1)(\bar{T}_1 - q) = 0$

T_1 is invertible as $q \neq 0$

Also $I \hookrightarrow \begin{pmatrix} \mathcal{O} & \mathfrak{p}^{-1} \\ \mathfrak{p} & \mathcal{O} \end{pmatrix} \cap G = K^{(2)} = \begin{pmatrix} t^{-1} & \\ & 1 \end{pmatrix} K^{(1)} \begin{pmatrix} t & \\ & 1 \end{pmatrix} = I \sqcup I s_2 I$

By the same arguments. $\qquad s_2 = \begin{pmatrix} & -t^{-1} \\ t & \end{pmatrix}$

$\mathcal{H}(G, \mathbb{1}_I) \supset \mathcal{H}(K^{(2)}, \mathbb{1}_I) \simeq \mathcal{H}(SL_2(k), \mathbb{1}_{B(k)})$

$\rotatebox{90}{$\in$} \qquad\qquad\qquad\qquad \rotatebox{90}{$\in$}$

$T_2 \longleftarrow \bar{T}_2$

with $supp(T_2) = I s_2 I$

$(T_2 + 1)(T_2 - q) = 0$

Now put $\phi = T_1 T_2$. Then

$$\mathrm{supp}(\phi) \subseteq I s_1 I s_2 I$$

$$= I \underbrace{s_1 (I \cap \bar{U}) s_1^{-1}}_{\subseteq I \cap U} \underbrace{s_1 (I \cap L) s_1^{-1}}_{= I \cap L} s_1 s_2 \underbrace{s_2^{-1} (I \cap U) s_2}_{\subseteq I \cap \bar{U}} I = I s_1 s_2 I$$

and $\zeta = s_1 s_2 = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ which is strongly positive.

Hence ϕ is as required and $(I, \mathbb{1}_I)$ is a cover of $(K_L, \mathbb{1}_{K_L})$.

Rmk We can make the same computations with $\mathbb{1}_L$ replaced by the tame quadratic character

$\omega : F^\times \twoheadrightarrow F^\times / \langle t \rangle \simeq \mathcal{O}^\times \twoheadrightarrow k^\times \longrightarrow \{\pm 1\}$ non-trivial

$K_L = \mathcal{O}^\times$, $\omega_L = \omega|_{K_L}$

I, $\tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \omega(a)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in I$

The only thing that change is the quadratic relations, which become $(T_i + 1)(T_i - 1) = 0$.

In both cases we get a Hecke algebra of the form

$\langle T_1, T_2 \mid (T_i - 1)(T_i + q^{a_i}) = 0 \rangle$ with $a_i \in \mathbb{Z}_{\geq 0}$

Strategy: ① Find cuspidal types, for pairs $[L, \rho]_L$.

② Find covers of these types.

③ Compute the Hecke algebras and the embedding t_P.

④ Interpret to find irred. subquotients of parabolic induction.

Example: $G = SL_2(F)$, $s_L = [L, 1\!\!1_L]_L$, $L \simeq F^\times$

$\mathrm{Irr}^{s_L}(L) = \{$ unramified q-char of $F^\times \}$

↑ of the form $x \longmapsto |x|^s$ for some $s \in \mathbb{C}/(\frac{2\pi i}{\log q})\mathbb{Z}$

Quadratic unramified character $\omega_0(x) = (-1)^{val(x)} = |x|^{\frac{\pi i}{\log q}}$

$$\begin{tikzcd}[row sep=small]
\mathcal{H}(L, 1\!\!1_{O^\times}) \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & = \{ f : L \longrightarrow \mathbb{C} \mid f(kxk') = f(x) \text{ for } x \in F^\times, k,k' \in O^\times, \text{supp} f \text{ cpt} \} \\
Z \arrow[d, mapsto] & = \{ f : F^\times/O^\times \longrightarrow \mathbb{C} \text{ with finite support} \} \\
q^{-1}\Phi \arrow[d, phantom, "\rotatebox{-90}{$\in$}", "\omega"'] & = \mathbb{C}[Z, Z^{-1}] \\
\mathcal{H}(G, 1\!\!1_I) & = \langle T_1, T_2 \mid (T_i + 1)(T_i - q) = 0 \rangle \quad \Phi = T_1 T_2
\end{tikzcd}$$

$\mathcal{H}(G, 1\!\!1_I)$ has four 1-dim^l reps given by $T_i \longmapsto -1, q$ and these must be the irred. subquotients of the reducible inductions coming from $Z \longmapsto q^{-1}, -1, -1, q$.

i.e. For χ unramified

$$\mathrm{Ind}_B^G \chi \text{ is reducible} \iff \chi = \underbrace{\overset{\omega}{|\cdot|^{\pm 1}}}_{\substack{\text{composition factors}\\ 1\!\!1_G,\ St_G}} \text{ or } \underbrace{\overset{\omega\omega_0}{\omega_0}}_{\text{semisimple length 2}}$$

More generally?

Let $\pi \in \mathrm{Irr}(G)$ and recall that we have parahoric subgroups $K = K_0$ with filtrations $(K_r)_{r \in \mathbb{R}_{\geq 0}}$, and $K_{r+} = \bigcup_{s > r} K_s$.

Def^n The depth of π is $\min\{ r \geq 0 \mid \pi^{K_{r+}} \neq 0 \text{ for some } (K_r)_{r \geq 0} \}$ which is well-defined.

Suppose π is depth zero & cuspidal, K parahoric s.t.

$\pi^{K_{0+}} \neq 0$

$\circlearrowleft$

K_0/K_{0+} is a canonical finite reductive group and

- τ is cuspidal
- K is a maximal parahoric

Example ①: $G = GL_m(F) \supset K = GL_m(\mathcal{O})$

τ the inflation to K of a cuspidal rep^n of $GL_m(k)$

Extend to a rep^n τ of $F^\times K$

Then
- $\pi = \mathrm{ind}_{F^\times K}^{G} \tau$ is irreducible and cuspidal
- (K, τ) is a $[G, \pi]_G$-type.

Rmk All known cuspidals (for all G) are of the form $\mathrm{ind}_{\tilde{K}}^{G} \tilde{\tau}$ for $\tilde{\tau}$ an irred. rep^n of a compact-mod-centre subgroup $\tilde{K}$. Other cuspidals for $GL_n(F)$ come from a type $\lambda = K \otimes \tau$

$\nearrow$ arithmetic magic $\qquad \nwarrow$ cuspidal rep^n of a finite reductive group.

Example ② : $G = Sp_{2n}(F) \supset K = \underbrace{\begin{pmatrix} & \mathcal{O} & \mathfrak{p} \\ \mathfrak{p} & & \end{pmatrix}}_{n_1 \quad 2n_2 \quad n_1} \cap G \twoheadrightarrow Sp_{2n_1}(k) \times Sp_{2n_2}(k)$

τ is the inflation of a cuspidal representation $\tau_1 \otimes \tau_2$

Then $\pi = \mathrm{ind}_K^G \tau$ is irred. & cuspidal.

Thm $(GL_m, Sp_{2n}, \dots)$

For each S there is an S-type which is a cover.

GL_m the Hecke algebra is a tensor product of affine Hecke algebras of type A with known parameters powers of q.

Sp_{2n} If L is a maximal, then the Hecke algebra is either commutative or $\langle T_1, T_2 \mid (T_i+1)(T_i - q^{a_i}) = 0 \rangle$ and there is a recipe to commute a_1, a_2.

Example: $G = Sp_{2(n+1)}(F) \supset GL_1(F) \times Sp_{2n}(F) = L$

$$\rho = \mathbb{1} \otimes \underset{\substack{\shortparallel \\ \mathrm{ind}_K^G \tau}}{\pi}$$

τ inflated from $\tau_1 \otimes \tau_2$

<u>unipotent cuspidals</u>

so $n_i = m_i(m_i+1)$

The parameters are q^{2m_i+1} and

$\mathrm{Ind}_P^G \chi \otimes \pi$ is reducible $\iff \chi = |\ |^{\pm r_1}$ or $\omega_0 |.|^{\pm r_2}$

where $\{r_1, r_2\} = \{m_1 + m_2 + 1, m_1 - m_2\}$