The tame inertial Langlands correspondence

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Notation

- F a non-archimedean local field;
- $\mathcal{O} \subset F$ its discrete valuation ring;
- $\mathfrak{p} \subset \mathcal{O}$ the maximal ideal;
- $k = \mathcal{O}/\mathfrak{p} \simeq \mathbb{F}_q$; $q = p^f$ the residue field;
- $\mathbf{G}$ a connected reductive group defined over F;
- $G = \mathbf{G}(F)$, with the $\mathfrak{p}$-adic topology;
- ${}^0G$ the subgroup of G generated by its compact subgroups;
- $\mathrm{Rep}(G)$ the category of smooth $\mathbb{C}$-representations of G.

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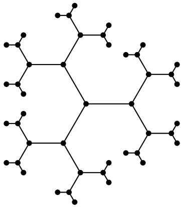
Definition

An irreducible representation π of G is *of depth zero* if there exists $x \in \mathscr{B}(G)$ such that $\pi^{G_x^+} \neq 0$.

An example

The building of $G = \mathbf{SL}_2(F)$ is a regular infinite tree of degree $q+1$:



Vertices naturally correspond to $\mathcal{O}$-lattices in F^2, and for a lattice Λ, G_Λ is the usual stabilizer of Λ. If x is not a vertex, $G_x = G_\Lambda \cap G_{\Lambda'}$ for Λ, Λ' the two neighbouring vertices.

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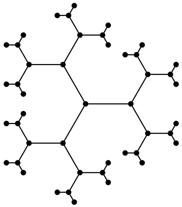

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So, up to $\mathbf{GL}_N(F)$-conjugacy, the parahoric subgroups of $\mathbf{SL}_N(F)$ are:

$$K_1 = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} \cap G; \quad K_1^+ = \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix} \cap G; \quad K_0/K_1 \simeq \mathbf{SL}(2,q); \text{ and}$$

$$K_2 = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} \end{pmatrix} \cap G; \quad K_2^+ = \begin{pmatrix} 1+\mathfrak{p} & \mathcal{O} \\ \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix} \cap G; \quad K_2/K_2^+ \simeq (\mathbb{F}_q^\times)^2.$$

Types

Theorem (Bernstein decomposition)

The category $\mathrm{Rep}(G)$ *of smooth complex representations of* G *splits as an product of indecomposable full subcategories:*

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Definition

Let $\mathfrak{S} \subset \mathfrak{B}(G)$. An $\mathfrak{S}$-*type* (J, λ) consists of an irreducible representation λ of a compact open subgroup J of G such that, for any $\pi \in \mathrm{Rep}(G)$, one has $\mathrm{Hom}_J(\lambda, \pi|_J) \neq 0$ if and only if $\pi \in \mathrm{Rep}^{\mathfrak{S}}(G) = \prod_{\mathfrak{s} \in \mathfrak{S}} \mathrm{Rep}^{\mathfrak{s}}(G)$.

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Then the irreducible objects of $\mathrm{Rep}^{[G,\pi]_G}(G)$ are $\pi \otimes \omega$, for unramified $\omega : G \to \mathbb{C}^\times$ (i.e. such that $\omega|_{{}^0G}$ is trivial).

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(iv) *For any $\mathfrak{s} \in \mathfrak{S}_\sigma$, there exists a subrepresentation τ of* $\mathrm{Ind}_{G_x}^{\tilde{G}_x} \sigma$ *such that $(\tilde{G}_x, \tau)$ is an $\mathfrak{s}$-type. In particular, there exists a $[G, \pi]_G$-type for every depth zero supercuspidal representation π of G.*

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(iii) *Any $[G, \pi]_G$-type is a refined depth zero type.*

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Let $I_F^+ \subset I_F$ be the wild inertia group, i.e. the maximal open normal pro-p subgroup, and let ${}^L\mathbf{G}$ be the Langlands dual of G; this is an algebraic group over $\mathbb{C}$ whose connected component has root datum dual to that of $\mathbf{G}$.

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Theorem (DeBacker–Reeder)

There is a surjective map with finite fibres

$$\begin{array}{ccc}
\mathcal{R}(G) & & \mathcal{L}(G) \\
\| & & \| \\
\left\{ \begin{array}{c} \textit{iso. classes of "regular" depth zero} \\ \textit{supercuspidal representations} \end{array} \right\} & \xrightarrow{\mathrm{rec}} & \left\{ \begin{array}{c} \textit{smooth "regular" } \mathrm{Frob}\textit{-semisimple} \\ \varphi : W_F/I_F^+ \to {}^L\mathbf{G}, \textit{ modulo } {}^L\mathbf{G}\textit{-conjugacy} \end{array} \right\}
\end{array}$$

The tame inertial Langlands correspondence, continued

Let $\mathcal{A}(G)$ denote the set of conjugacy classes of $[G, \pi]_G$-types (refined depth zero types), for $\pi \in \mathcal{R}(G)$, and $\mathcal{D}(G)$ the set of conjugacy classes of unrefined depth zero types contained in $\pi \in \mathcal{R}(G)$.

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Let $\mathcal{I}(G)$ denote the image under $\mathrm{Res}_{I_F}^{W_F}$ of $\mathcal{L}(G)$.

Theorem (L.)

There are unique well-defined surjective maps $\mathrm{iner} : \mathcal{A}(G) \to \mathcal{I}(G)$ *and* $\mathrm{iner}_{\mathcal{D}} : \mathcal{D}(G) \to \mathcal{I}(G)$ *such that the following diagram commutes:*

$$\begin{tikzcd}
 & \mathcal{R}(G) \arrow[r, "\mathrm{rec}"] \arrow[d] \arrow[ddl, bend right] & \mathcal{L}(G) \arrow[d, "\mathrm{Res}_{I_F}^{W_F}"] \\
 & \mathcal{A}(G) \arrow[r, "\mathrm{iner}"'] \arrow[dl] & \mathcal{I}(G) \\
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(i) $\#\mathrm{iner}_{\mathcal{D}}^{-1}(\varphi|_{I_F}) = \#\mathrm{rec}^{-1}(\varphi)$;

(ii) $\#\mathrm{iner}^{-1}(\varphi|_{I_F}) = \sum_{(G_x,\sigma)\in\mathrm{iner}_{\mathcal{D}}^{-1}(\varphi|_{I_F})} \#\mathfrak{S}_\sigma$.

Consequences for finite groups

Corollary ("Local Langlands" for finite groups)

The "regular" cuspidal representations of the parahoric subgroups of G are naturally parametrised by the set of ${}^L\mathbf{G}$-conjugacy classes of smooth Frob*-semisimple homomorphisms $I_F/I_F^+ \to {}^L\mathbf{G}$. Moreover, this parametrisation has a completely explicit description.*

This gives a natural transfer between packets of representations of finite groups of Lie type.

For example, taking $G = \mathbf{Sp}_4(F)$, there are vertices $x, y \in \mathscr{B}(G)$ with $G_x/G_x^+ = \mathbf{Sp}_4(k)$ and $G_y/G_y^+ = \mathbf{SL}_2(k) \times \mathbf{SL}_2(k)$. There exist inertial types $\varphi : I_F/I_F^+ \to {}^L\mathbf{Sp}_4(F) = \mathbf{SO}_5(\mathbb{C})$ such that $\mathrm{iner}_{\mathcal{D}}^{-1}(\varphi)$ contains cuspidal representations of both G_x/G_x^+ and G_y/G_y^+.