CATEGORICAL ACTION OF LIE ALGEBRAS

<u>Recall</u> : $\mathrm{Irr}_k G = \bigsqcup \mathcal{E}(\mathbb{L}, X)$

$\uparrow$ simple quotients of $R_{\mathbb{L}}^{G}(X)$

with $\mathcal{E}(\mathbb{L}, X) \overset{1:1}{\longleftrightarrow} \mathrm{Irr}_k \mathcal{H}(\mathbb{L}, X)$

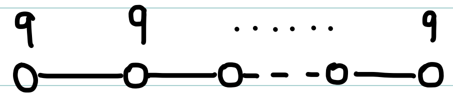
$\uparrow$ deformation of the group algebra of a (almost) Coxeter gp

+ compatibility between R_M^G, ${}^*R_M^G$ with ind/res in gp algebras

<u>Questions</u>
- cuspidal representations?
- parameters of the deformations?

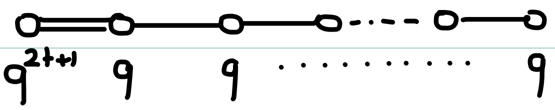
<u>Ex</u> : for char $k = 0$ and unipotent representations

(a) $G = GL_n(q)$
- $(\mathbb{L}, X) = (\mathbb{T}, k)$
- $\mathcal{H}(\mathbb{L}, X) \simeq \mathcal{H}_q(\mathfrak{S}_n)$



- $\mathrm{Irr}_k \mathcal{H} \leftrightarrow \mathrm{Irr}_k \mathfrak{S}_n \longleftrightarrow \{\text{partitions of } n\}$

(b) $G = Sp_{2n}(q)$
- $(\mathbb{L}, X) = (Sp_{2m} \times GL_1^{n-m}, \text{cusp})$ with $m = t(t+1)$ $t \geq 0$
- $\mathcal{H}(\mathbb{L}, X) \simeq \mathcal{H}_{q^{2t+1}, q}(W(B_{n-m}))$



- $\mathrm{Irr}_k \mathcal{H} \longleftrightarrow$ bipartition of $n-m$

and parabolic ind/res corresponds to adding/removing boxes in the pair of Young diagrams

$$\mathrm{Unip}(Sp_{2n}(q)) \longleftrightarrow \left\{ (\lambda,\mu;t) \;\middle|\; \begin{matrix} t \geqslant 0 \\ (\lambda,\mu) \vdash n - t(t+1) \end{matrix} \right\}$$

$$\begin{matrix} \cup\!\!\mid & & \cup\!\!\mid \\ \mathcal{E}(T,k) = \text{principal series} & \longleftrightarrow & \{ (\lambda,\mu;0) \} \end{matrix}$$

I Lie algebra action

Idea: construct a Lie algebra action on the Grothendieck group of the category of rep. coming from HC ind/res

(Recall that $[R, {}^*R]$ is needed for the Mackey formula)

Lie bracket

Assume $G = Gl_n(q)$

$$R_{Gl_n}^{Gl_{n+1}}(\rho_\lambda) = \sum \rho_{\lambda+\square}$$

$${}^*R_{Gl_n}^{Gl_{n+1}}(\rho_\lambda) = \sum \rho_{\lambda-\square}$$

The q-content of a partition is defined on its Young diagram as follows:

$$\begin{array}{|c|c|c|c|c}
\hline
1 & q & q^2 & q^3 & \cdots \\
\hline
q^{-1} & 1 & q & \cdots & \\
\hline
q^{-2} & \cdots & \cdots & & \\
\hline
\vdots & & & & \\
\end{array}$$

<u>Define</u> $f_i(e_\lambda) = \sum e_{\lambda + \boxed{q^i}}$

e_i ——— $\lambda - \boxed{q^i}$

<u>Prop</u>: e_i, f_i $i \in \mathbb{Z}$ define on a action of a Lie algebra $\mathfrak{g}$ on $V = \bigoplus \mathbb{C} e_\lambda$ with $\mathfrak{g} = \begin{cases} \mathfrak{sl}_\infty & \text{if } q \text{ has } \infty \text{ order} \\ \widehat{\mathfrak{sl}}_d & \text{——— order } d \end{cases}$

Serre cat. generated by unipotent reps.

<u>Now</u>: let $\mathcal{V} = \bigoplus_{n \geq 0} kGL_n(q)\text{-unip}$

- $F = \bigoplus_{n \geq 0} R_{GL_n}^{GL_{n+1}}$ and $E = \bigoplus_{n \geq 0} {}^*R_{GL_n}^{GL_{n+1}}$ are exact endofunctors of $\mathcal{V}$

- $K_0(\mathcal{V})$ is a $\mathbb{Z}$-free module with basis $\{e_\lambda\}_{\lambda \text{ partition}}$ and $[F]$ acts as $\sum f_i$
  $[E]$ ——— $\sum e_i$

Same for $Sp_{2n}(q)$ with some modifications

$\rightsquigarrow$ $V = K_0(\mathcal{V}) \otimes_{\mathbb{Z}} \mathbb{C}$ is a $\mathfrak{g}$-module

and
- simple modules are weight vectors
- cuspidal modules are heighest weight vectors

II - Categorical action of $\mathfrak{g}$ (action on $\mathfrak{g}$ on $\mathcal{V}$?)

E and F are exact functors yielding $\sum e_i$ and $\sum f_i$

(P1) Construct E_i and F_i yielding e_i and f_i

Harish-Chandra theory $\rightsquigarrow$ study of $\mathrm{End}(F^m X)$ for X cusp
↳ Hecke algebra

(P2) Understand $\mathrm{End}(F^m)$ before $\mathrm{End}(F^m X)$

For example, for $G = Sp_{2n}(q)$ we have

$$\begin{tikzcd}
?? \arrow[r] \arrow[d] & \mathrm{End}\, F^m \arrow[d, two heads] \\
\mathcal{H}_{q^{2t+1}, q}(W(B_m)) \arrow[r, "\sim"] & \mathrm{End}_G(F^m \mathrm{cusp})
\end{tikzcd}$$

The solution to (P1) and (P2) involves affine Hecke algebras and is given by the following thm

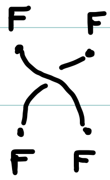
Thm [Chuang-Rouquier, D-Varagnolo-Vasserot]

Assume $G = GL_n(q)$ or $Sp_{2n}(q)$

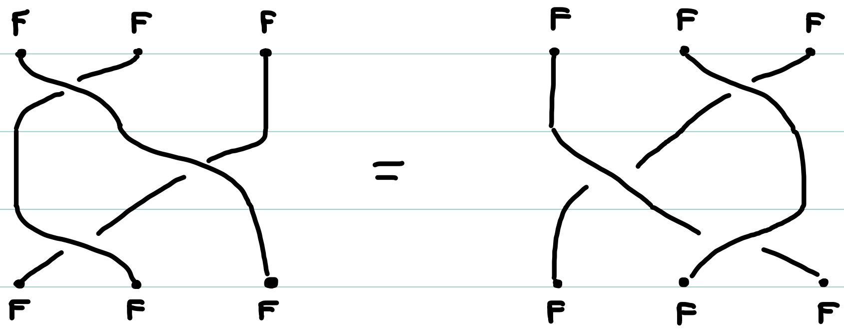
$\exists\, X \in \mathrm{End}(F)$ and $T \in \mathrm{End}(F^2)$ st

- $T1_F \circ 1_F T \circ T1_F = 1_F T \circ T1_F \circ 1_F T$ in $\mathrm{End}(F^3)$
- $(T - q1_F)\circ(T + 1_F) = 0$ in $\mathrm{End}(F^2)$
- $T \circ (1_F X) \circ T = q X 1_F$ in $\mathrm{End}(F^2)$

T is a braid operator can be pictured as



the relation in $\mathrm{End}\,F^3$



X should be thought of as a Jucys-Murphy elt

Let $\mathcal{H}_q(\tilde{A}_{m-1})$ be the affine Hecke algebra with

- generators $X_1^{\pm 1}, \dots, X_m^{\pm 1}$ and $T_1, \dots, T_{m-1}$
- relations: $T_1, \dots, T_{m-1}$ satisfy relations of $\mathcal{H}_q(\mathfrak{S}_m)$

  $X_i^{\pm 1}$ commute and $X_i X_i^{-1} = X_i^{-1} X_i = 1$

  $T_i X_i T_i = q X_{i+1}$

Then
$$\begin{aligned} \mathcal{H}_q(\tilde{A}_{m-1}) &\longrightarrow \mathrm{End}(F^m) \\ X_i &\longmapsto 1_{F^{i-1}} X 1_{F^{m-i}} \\ T_i &\longmapsto 1_{F^{i-1}} T 1_{F^{m-i-1}} \end{aligned}$$
is an algebra homomorphism

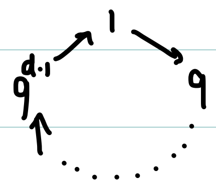
Given $i \in k$, let E_i = generalized i-eigenspace of X on E
F_i = ―――――――――――――― F

Thm [CR, DVV] $[E_i], [F_i]$ $i \in k$ induce an action of a Lie algebra $\mathfrak{g}$ on $V = K_0(\mathcal{V}) \otimes_{\mathbb{Z}} \mathbb{C}$

<u>What is $\mathfrak{g}$</u>: it is the Kac-Moody algebra associated with the quiver $(\operatorname{Spec} X) \circlearrowleft q$

<u>Ex</u>: (a) $G = GL_n(q) \rightsquigarrow$ quiver $q^{\mathbb{Z}} \circlearrowleft q$

$\longrightarrow \mathfrak{g} = \begin{cases} \mathfrak{sl}_\infty & \cdots \longrightarrow 1 \longrightarrow q \longrightarrow \cdots \\ \widehat{\mathfrak{sl}}_d & \end{cases}$



(b) $G = Sp_{2n}(q) \rightsquigarrow$ quiver $(q^{\mathbb{Z}} \cup -q^{\mathbb{Z}}) \circlearrowleft q$

$\longrightarrow \mathfrak{g} = \mathfrak{sl}_\infty^{\oplus 2}, \widehat{\mathfrak{sl}}_d^{\oplus 2}$ or $\widehat{\mathfrak{sl}}_d$

depending on whether $-1 \in q^{\mathbb{Z}}$ (ie d even)

In the end we get a dictionary between the properties of the $\mathfrak{g}$-module V and the Harish-Chandra theory in $\mathcal{V}$

$V = K_0(\mathcal{V})\otimes_{\mathbb{Z}} \mathbb{C}$	$\mathcal{V}$ abelian category
$V = \bigoplus V_\omega$ weight space decomposition	$\mathcal{V} = \bigoplus \mathcal{V}_\omega$ block decomp. with $K_0(\mathcal{V}_\omega)\otimes_{\mathbb{Z}}\mathbb{C} = V_\omega$
$V_\omega \underset{f_i}{\overset{e_i}{\rightleftarrows}} V_{\omega+\alpha_i}$	$\mathcal{V}_\omega \underset{F_i}{\overset{E_i}{\rightleftarrows}} \mathcal{V}_{\omega+\alpha_i}$ exact functors biadjoint
$V_\omega \overset{s_i}{\xrightarrow{\sim}} V_{s_i(\omega)}$ with $s_i = \exp f_i \exp(-e_i) \exp f_i$	$D^b(\mathcal{V}_\omega) \overset{\Theta_i}{\xrightarrow{\sim}} D^b(\mathcal{V}_{s_i(\omega)})$ with $[\Theta_i] = s_i$ complex of functors [Rickard]
highest weight vector x of weight $\omega \in X^+$	cuspidal representation X
$V = \bigoplus L(\omega)$ simple rep. of hw $\omega \in X^+$	$L(\omega) \longleftrightarrow$ Harish-Chandra series whose Hecke algebra can be computed from ω

<u>Ex</u>: in $Sp_{2m}(q)$ with $m = t(t+1)$ and char $k = 0$

⇝ X cuspidal rep. with weight $\Lambda_{q^t} + \Lambda_{-q^{-1-t}}$

$$\mathcal{H}(\mathbb{L},X) \simeq \operatorname{End} F^m X \simeq \mathcal{H}_q(\tilde{A}_{m-1}) \big/ (X_1 - q^t)(X_1 + q^{-1-t}) \simeq \mathcal{H}_{q^{2t+1},q}(B_m)$$