CLASSICAL HC THEORY

Representation theory of finite groups of Lie type

e.g $GL_n(q) = \{ M \in Mat_{n\times n}(\mathbb{F}_q) \mid \det M \neq 0 \}$

$SL_n(q) = \{ \quad \underline{\qquad\qquad} \quad \det M = 1 \}$

$Sp_{2n}(q) = \{ M \in GL_{2n}(q) \mid MJ\,{}^tM = J \}$ with $J = \begin{bmatrix} (0) & & & 1 \\ & & \cdot^{\cdot^{\cdot}} & \\ & -1 & & \\ -1 & & & (0) \end{bmatrix}$

$\ldots\ G_2(q), \ldots, E_8(q)$

Representations over $k = \bar{k}$ with either

- char $k = 0$ (e.g. $k = \mathbb{C}$)
- char $k \nmid q$ [transverse characteristic]
- ~~char $k \mid q$ [natural characteristic]~~

} this lecture

I – Principal series representations

$\mathbb{G}$ connected reductive algebraic group

$\mathbb{B} \supseteq \mathbb{T}$ Borel with max torus

$W = N_{\mathbb{G}}(\mathbb{T})/\mathbb{T}$

Ex: $\mathbb{G} = GL_n \supseteq \mathbb{B} = \begin{bmatrix} * & & * \\ & \ddots & \\ (0) & & * \end{bmatrix} \supseteq \mathbb{T} = \begin{bmatrix} * & & (0) \\ & \ddots & \\ (0) & & * \end{bmatrix}$

$W \simeq \mathfrak{S}_n$

$\mathbb{G}$ defined over $\mathbb{F}_q$ with Frobenius $F : \mathbb{G} \to \mathbb{G}$

e.g $F : (a_{ij}) \in Gl_n \longmapsto (a_{ij}^q) \in Gl_n \rightsquigarrow \mathbb{G}^F = Gl_n(q)$

One can always choose $\mathbb{B}$ and $\mathbb{T}$ F-stable
$\rightsquigarrow$ F acts on W and preserve the set of simple reflections

<u>Motivating example</u> : $V = \mathrm{Ind}_{\mathbb{B}^F}^{\mathbb{G}^F} k = k\mathbb{G}^F/\mathbb{B}^F$

? indecomposable summands
? simple submodules / quotient] look at $\mathrm{End}_{\mathbb{G}^F}(V)$
? composition factors

For simplicity, assume that
- F acts trivially on W
- $q-1 \in k^\times$

Bruhat decomposition :

$$\mathbb{G}^F/\mathbb{B}^F = \bigsqcup_{w \in W} \mathbb{B}^F w \mathbb{B}^F/\mathbb{B}^F \quad \text{with } \# \mathbb{B}^F w \mathbb{B}^F/\mathbb{B}^F = q^{\ell(w)}$$

Let $e = \frac{1}{|\mathbb{B}^F|} \sum_{b \in \mathbb{B}^F} b$ idempotent of $k\mathbb{B}^F$

$\rightsquigarrow$ $k\mathbb{G}^F/\mathbb{B}^F = k\mathbb{G}^F e$ and $ek\mathbb{G}^F e \xrightarrow{\sim} \mathrm{End}_{\mathbb{G}^F}(k\mathbb{G}^F e)$

$exe \longmapsto (e \longmapsto exe)$

Define $T_w = q^{\ell(w)} ewe$

(1) if $\ell(ww') = \ell(w)+\ell(w')$ then $B^F w B^F w' B^F \simeq B^F ww' B^F$

hence $\forall b \in B^F$ $\quad ewbw'e = eb_1ww'b_2e = eww'e$

so that

$$T_w T_{w'} = T_{ww'}$$

(2) if $s \in W$ is a simple reflection $B^F s B^F s B^F = B^F \sqcup B^F s B^F$

hence $\forall b \in B^F$

$$esbse = \begin{cases} eb'e = e \text{ if } b \in B^F \cap sB^F s & \leftarrow \frac{|B^F|}{q} \text{ el}^{t}\text{s} \\ eb_1sb_2e = ese \text{ otw} \end{cases}$$

$\rightsquigarrow \quad T_s^2 = q^2(esese) = q^2\left(\frac{1}{q}e + \left(1-\frac{1}{q}\right)ese\right)$

$$T_s^2 = (q-1)T_s + qT_1 \qquad \left(\text{or } (T_s - q)(T_s+1) = 0\right)$$

def: The Iwahori-Hecke algebra of W with parameter q is $\mathcal{H}_q(W) = \langle t_w \, ; \, w \in W \rangle \Big/ \begin{array}{l} t_s^2 = (q-1)t_s + 1 \\ t_w t_{w'} = t_{ww'} \text{ if } \ell(ww') = \ell(w)+\ell(w') \end{array}$

$$\rightsquigarrow \quad \mathcal{H}_q(W) \xrightarrow{\sim} \mathrm{End}_{G^F}(kG^F/B^F) \quad \text{(sends basis to basis)}$$
$$t_s \longmapsto T_s$$

In part the functor $M \in kG^F\text{-mod} \mapsto \mathrm{Hom}_{G^F}(kG^F/B^F, M)$ induces a bijection

$$\{M \text{ s.t. } kG^F/B^F \twoheadrightarrow M\} = \mathrm{hd}(kG^F/B^F) \xleftrightarrow{1:1} \mathrm{Irr}_k \mathcal{H}_q(W)$$

These M are called principal series representations

Examples : (a) $\mathbb{G}^F = GL_n(q)$

* if char $k=0$ $\mathcal{H}_q(\mathfrak{S}_n) \simeq k\mathfrak{S}_n$ semisimple

$$\{\text{principal series char.}\} \longleftrightarrow \operatorname{Irr}\mathfrak{S}_n \longleftrightarrow \{\text{partitions of } n\}$$

$$\rho_\lambda \longleftarrow\!\!\!\!\mapsfrom \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$$

* if char $k>0$, let $d=$ order of q in $k^\times$ then

$$\{\text{principal series rep.}\} \longleftrightarrow \{d\text{-regular partitions of } n\}$$

multiplicity of each λ_i is $<d$

(b) $\mathbb{G}^F = Sp_{2n}(q)$

$$\{\text{principal series characters}\} \longleftrightarrow \{\text{bipartitions of } n\}$$

$$\rho_{\lambda,\mu} \longleftarrow\!\!\!\!\mapsfrom (\lambda,\mu) \text{ with } |\lambda| + |\mu| = n$$

II Parabolic induction / restriction

1- Levi decomposition

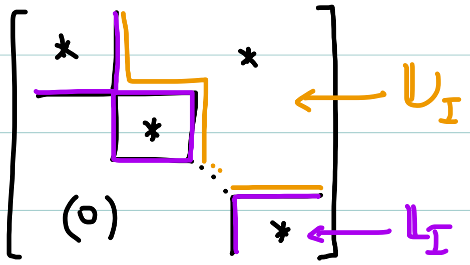
Recall S is the set of simple reflections

$I \subseteq S \rightsquigarrow$ • $W_I = \langle I \rangle \leq W$

• $\mathbb{P}_I = \mathbb{B} W_I \mathbb{B}$ parabolic subgroup

- $\mathbb{U}_I = R_u(\mathbb{P}_I)$ unipotent radical
- $\mathbb{L}_I \supseteq \mathbb{T}$ with $\mathbb{P}_I = \mathbb{L}_I \ltimes \mathbb{U}_I$ Levi complement

In addition, if I is F-stable, so are W_I, $\mathbb{P}_I$, $\mathbb{U}_I$, $\mathbb{L}_I$.

Ex: $\mathbb{G} = GL_n$ $\quad \mathbb{P}_I =$ 

$$\mathbb{L}_I \simeq GL_{n_1} \times GL_{n_2} \times \cdots \times GL_{n_r}$$

2- Induction/restriction

$$k\mathbb{L}_I^F\text{-mod} \underset{{}^*R_{\mathbb{L}_I}^{\mathbb{G}}}{\overset{R_{\mathbb{L}_I}^{\mathbb{G}}}{\rightleftarrows}} k\mathbb{G}^F\text{-mod}$$

Not the usual induction/restriction (too big!)

$$R_{\mathbb{L}_I}^{\mathbb{G}}(M) = \mathrm{Ind}_{\mathbb{P}_I^F}^{\mathbb{G}^F}\left(\mathrm{Inf}_{\mathbb{L}_I^F}^{\mathbb{P}_I^F} M\right) = k\mathbb{G}^F/\mathbb{U}_I^F \otimes_{k\mathbb{L}_I^F} M$$

$${}^*R_{\mathbb{L}_I}^{\mathbb{G}}(N) = N^{\mathbb{U}_I^F} = \mathrm{Hom}_{\mathbb{G}^F}(k\mathbb{G}^F/\mathbb{U}_I^F, N)$$

Ex: $I = \emptyset$, $\mathbb{L}_I = \mathbb{T}$ $\quad R_{\mathbb{T}}^{\mathbb{G}}(k) = k\mathbb{G}^F/\mathbb{B}^F$

Prop: (i) $R_{\mathbb{L}_I}^{\mathbb{G}}$ and ${}^*R_{\mathbb{L}_I}^{\mathbb{G}}$ are exact, biadjoint
(ii) They preserve projectivity

Transitivity: If $I \subseteq J \subseteq S$ then

$$R_{L_J}^{G} \circ R_{L_I}^{L_J} \simeq R_{L_I}^{G} \quad \text{and} \quad {}^*R_{L_I}^{L_J} \circ {}^*R_{L_J}^{G} \simeq {}^*R_{L_I}^{G}$$

3- Mackey formula

$I, J \subseteq S$ both F-stable and $x \in W$

$\mathbb{L}_I \cap {}^x\mathbb{L}_J$ is a standard Levi of both $\mathbb{L}_I$ and ${}^x\mathbb{L}_J$

Thm: $${}^*R_{L_I}^{G} \circ R_{L_J}^{G} \simeq \bigoplus_{x \in W_I^F \backslash W^F / W_J^F} R_{L_I \cap {}^x L_J}^{L_I} \circ {}^*R_{L_I \cap {}^x L_J}^{{}^x L_J} \circ \mathrm{ad}\, x$$

III Harish-Chandra theory

Goal: compute $R_{L_I}^{G}(M)$ for M "minimal"

1- Cuspidality and series

def: (i) $X \in \mathrm{Irr}\, k\mathbb{G}^F$ is cuspidal if ${}^*R_{L_I}^{G}(X) = 0$ for every F-stable proper subset $I \subsetneq S$

(ii) A cuspidal pair $(\mathbb{L}_I, X)$ is a pair with M a cuspidal $k\mathbb{L}_I^F$-module

(iii) $\mathcal{E}(\mathbb{L}_I, X) = \{ M \in \mathrm{Irr}\, k\mathbb{G}^F \text{ s.t. } R_{L_I}^{G}(X) \twoheadrightarrow M \}$

Rmk: ${}^*R^G_{L_I}(X) = 0 \iff$ every map $R^G_{L_I}(N) \to X$ is zero
$\rightsquigarrow$ X is "minimal"

Thm: $$\mathrm{Irr}\, k\mathbb{G}^F = \bigsqcup_{\text{cusp. pairs}/\sim \mathbb{G}^F} \mathcal{E}(\mathbb{L}_I, X)$$

proof: Let $M \in \mathrm{Irr}\, k\mathbb{G}^F$
Take I minimal s.t ${}^*R^G_{L_I}(M) \neq 0$ and $X \subseteq {}^*R^G_{L_I}(M)$
Transitivity + exactness $\Longrightarrow$ X is a cuspidal $k\mathbb{L}_I^F$-module
Adjunction $\Longrightarrow$ $R^G_{L_I}(X) \twoheadrightarrow M$ (non zero and M is simple)
Therefore
$$\mathrm{Irr}\, k\mathbb{G}^F = \bigcup \mathcal{E}(\mathbb{L}_I, X)$$

Let $(\mathbb{L}_I, X)$ and $(\mathbb{L}_J, Y)$ two cusp. pairs s.t

$$\begin{tikzcd}
R^G_{L_I}(X) \arrow[r, two heads] & M & R^G_{L_J}(Y) \arrow[r, two heads] & M \\
R^G_{L_I}(P_X) \arrow[u, two heads] & & & R^G_{L_I}(P_X) \arrow[u, two heads] \arrow[ul, dashed]
\end{tikzcd}$$

hence $$\mathrm{Hom}_{\mathbb{G}^F}(R^G_{L_I}(P_X), R^G_{L_J}(Y)) \neq 0$$

$$\| $$

$$\mathrm{Hom}_{\mathbb{L}_I^F}(P_X, {}^*R^G_{L_I} R^G_{L_J}(Y)) \qquad \text{[adjunction]}$$

$$\|$$

$$\bigoplus_{W_I^F \backslash W^F / W_J^F} \mathrm{Hom}_{\mathbb{L}_I^F}\Big(P_X, R^{L_I}_{L_I \cap {}^x L_J}\, \underbrace{{}^*R^{{}^xL_J}_{L_I \cap {}^x L_J}({}^x Y)}_{=0 \text{ unless } L_I \supseteq {}^x L_J}\Big)$$

by symmetry one must have $y \in W$ s.t $\mathbb{L}_J \supseteq {}^y\mathbb{L}_I$
which force $\mathbb{L}_I = {}^x\mathbb{L}_J$ and

$$\mathrm{Hom}_{\mathbb{G}^F}\left(R_{L_I}^G(P_X), R_{L_J}^G(Y)\right) \simeq \bigoplus_{{}^x\mathbb{L}_J = \mathbb{L}_I} \mathrm{Hom}_{\mathbb{L}_I^F}(P_X, {}^xY)$$

is non zero forces $X \simeq {}^xY$ for some $x \in W^F$. $\square$

2- Endomorphism algebra

Let $(\mathbb{L}_I, X)$ be a cuspidal pair

$$\begin{aligned}\mathcal{H}(\mathbb{L}_I, X) &= \mathrm{End}_{\mathbb{G}^F}\left(R_{L_I}^G(X)\right)\\ &\simeq \mathrm{Hom}_{\mathbb{L}_I^F}\left(X, {}^*R_{L_I}^G R_{L_I}^G(X)\right)\\ &\simeq \bigoplus_{{}^x\mathbb{L}_I = \mathbb{L}_I} \mathrm{Hom}_{\mathbb{L}_I^F}(X, {}^xX) \quad \dim = \left| N_{W^F}(\mathbb{L}_I, X) / W_I^F \right|\end{aligned}$$

$\mathcal{H}(\mathbb{L}, X)$ is close to be a Hecke algebra associated to $W(\mathbb{L}, X) = N_{W^F}(\mathbb{L}_I, X)/W_I^F$

Thm [Lusztig, Howlett-Lehrer, Geck-Hiss-Malle]
There is a natural basis $\{T_w\}_{w \in W(\mathbb{L}, X)}$ of $\mathcal{H}(\mathbb{L}, X)$
s.t $\quad T_w T_{w'} = \lambda(w, w') T_{ww'}$ when $\ell(ww') = \ell(w) + \ell(w')$
for some cocycle $\lambda : W \times W \longrightarrow k^\times$

<u>Particular case</u>: char $k=0$

- $W(\mathbb{L},X)$ is a Coxeter gp (with natural simple ref. $S(\mathbb{L},X)$)
- $T_s^2=(q_s-1)T_s+q_s$ for $s\in S(\mathbb{L},X)$
- λ is trivial

$\rightsquigarrow \mathcal{H}(\mathbb{L},X)\simeq \mathcal{H}_{q_s}(W(\mathbb{L},X))$

From now on we restrict to unipotent representations

<u>Examples</u> (a) $\mathbb{G}^F=GL_n(q)$

* char $k=0$ $(\mathbb{T},k)$ is the only cuspidal pair

* char $k=\ell>0$, $d=$ order of q modulo ℓ

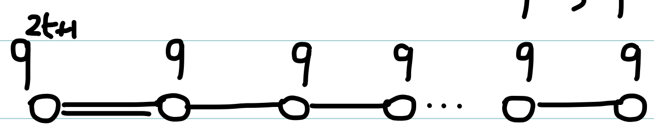
Given $\mathbb{L}$, at most one cuspidal pair $(\mathbb{L},X)$ and exactly one iff $\mathbb{L}\simeq GL_1^{n_{-1}}\times GL_d^{n_0}\times GL_{d\ell}^{n_1}\times GL_{d\ell^2}^{n_2}\times\cdots$

In that case $\mathcal{H}(\mathbb{L},X)\simeq\mathcal{H}_q(\mathfrak{S}_{n_{-1}})\otimes k\mathfrak{S}_{n_0}\otimes k\mathfrak{S}_{n_1}\otimes\cdots$

(b) $\mathbb{G}^F=Sp_{2n}(q)$

* char $k=0$ at most one cuspidal pair for a given Levi $\mathbb{L}$ and exactly one if $\mathbb{L}=Sp_{2m}\times(GL_1)^{n-m}$ with $m=t(t+1)$

In that case $\mathcal{H}(\mathbb{L},X)\simeq\mathcal{H}_{q^{2t+1},q}(W(B_{n-m}))$

i.e 

$*$ char $k = \ell > 0$??? [D-Varagnolo-Vasserot]

3- Compatibility with ind/res

Q: How to compute R_L^G inside a Harish-Chandra series
$(\mathbb{L}_I, X)$ cusp pair of $\mathbf{G}^F$ and $J \supseteq I$
$\rightsquigarrow (L_I, X)$ cuspidal pair of $\mathbb{L}_J^F$

For simplicity we assume char $k = 0$

$$\begin{tikzcd}
\mathbb{Z}\mathcal{E}^G(\mathbb{L}_I, X) \arrow[r, leftrightarrow] \arrow[d, shift left, "{}^*R_{\mathbb{L}_J}^G"] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{Z}\operatorname{Irr} W^G(\mathbb{L}_I, X) \arrow[d, shift left, "\mathrm{Res}"] \\
\mathbb{Z}\mathcal{E}^{\mathbb{L}_J}(\mathbb{L}_I, X) \arrow[r, leftrightarrow] \arrow[u, shift left, "R_{\mathbb{L}_J}^G"] & \mathbb{Z}\operatorname{Irr} W^{\mathbb{L}_J}(\mathbb{L}_I, X) \arrow[u, shift left, "\mathrm{Ind}"]
\end{tikzcd}$$

Rmk: can be generalized to positive char.

Examples: (a) $\mathbf{G}^F = GL_n(q)$ and char $k = 0$
Unipotent characters $=$ principal series char $= \{\rho_\lambda\}_{\lambda \vdash n}$

$$R_{GL_n}^{GL_{n+1}}(\rho_\lambda) = \sum \rho_{\lambda + \square}$$

e.g. $R_{GL_n}^{GL_{n+1}}\left(\ydiagram{3,2}\right) = \ydiagram{4,2} + \ydiagram{3,3} + \ydiagram{3,2,1}$

(b) $G^F = Sp_{2n}(q)$, char $k = 0$

Unipotent characters $\longleftrightarrow \{ (\lambda, \mu, t)$ with (λ,μ) bipart of $n - t(t+1) \}$

$$R_{Sp_{2n}}^{Sp_{2n+2}}\left(\rho_{(\lambda,\mu,t)}\right) = \sum \rho_{(\lambda+\square,\mu,t)} + \sum \rho_{(\lambda,\mu+\square,t)}$$