Maximal Edit Distance to a Synchronizing Coloring of a Graph

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Introduction

Goal: What number of coloring changes is sufficient to make it synchronizing?

- Examples of graphs, for which that number is big.
- How big that number can be for a graph on n vertices - upper bound.

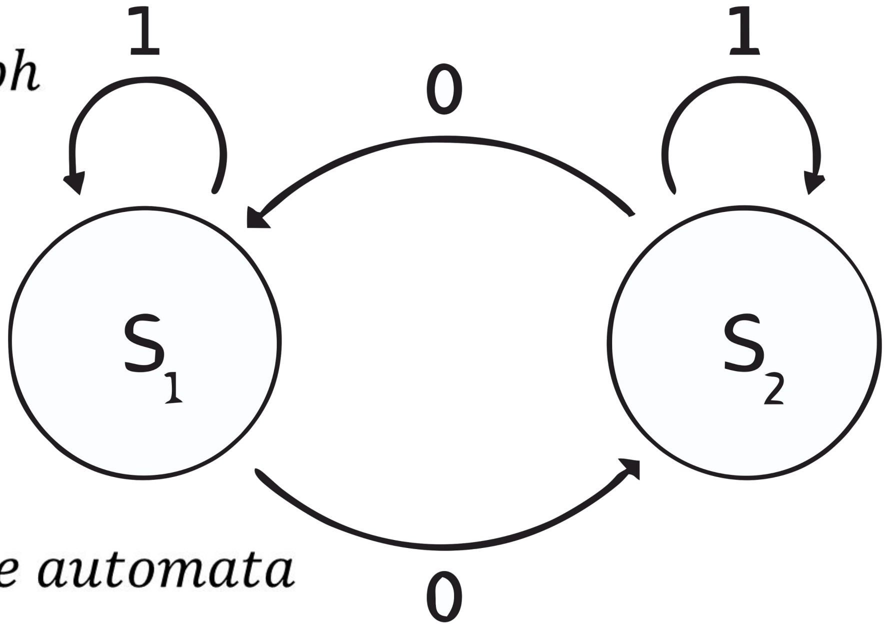
Graphs and Automata

$G = \langle V, E \rangle - directed\ multigraph$

- $|V| = n\ - number\ of\ vertices$
- $\forall v \in V, out\text{-}degree(v) = k$

$A = \langle Q, \Sigma \rangle - deterministic\ finite\ automata$

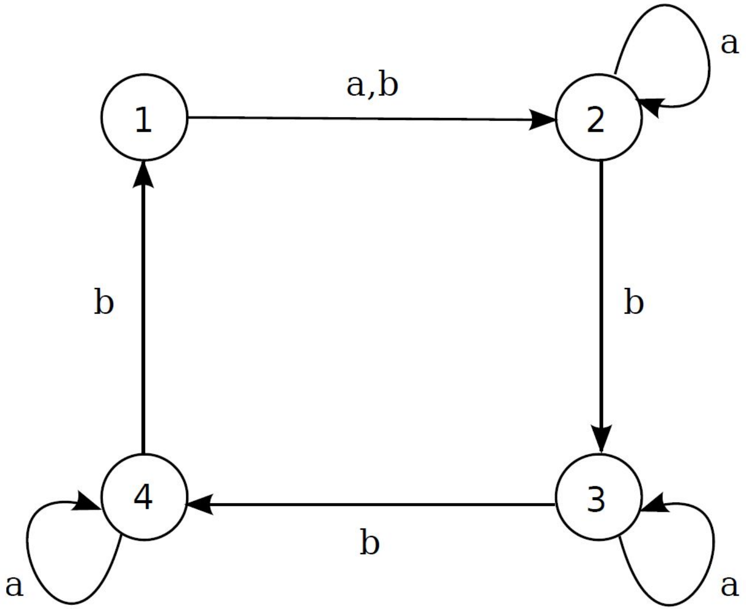
- $|Q| = n\ - number\ of\ states$
- $|\Sigma| = k\ - number\ of\ symbols\ (colors)$



Synchronizing automata

Word $w \in \Sigma^*$ $- \mathit{synchronizing\ (reset)\ word}$

$$q_1 \cdot w = q_2 \cdot w, for\ all\ q_1, q_2 \in Q$$



Example: *Černý automata*

Synchronizing word: $(ab^3)^2a$

Motivation

When graph has a synchronizing coloring?

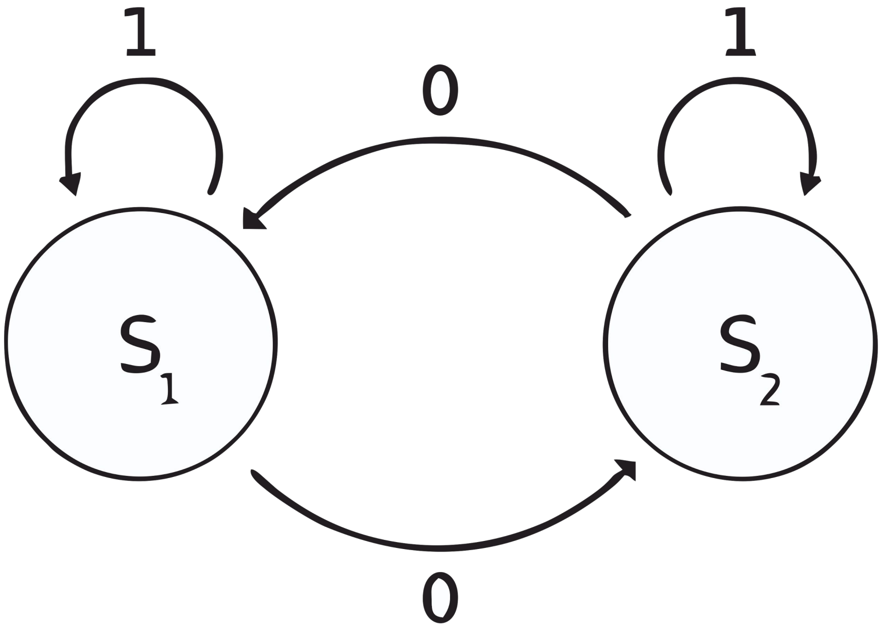
1. Graph has a unique reachable sink component.

2. Graph is aperiodic (primitive) - the gcd of lengths of all its cycles is equal to 1.

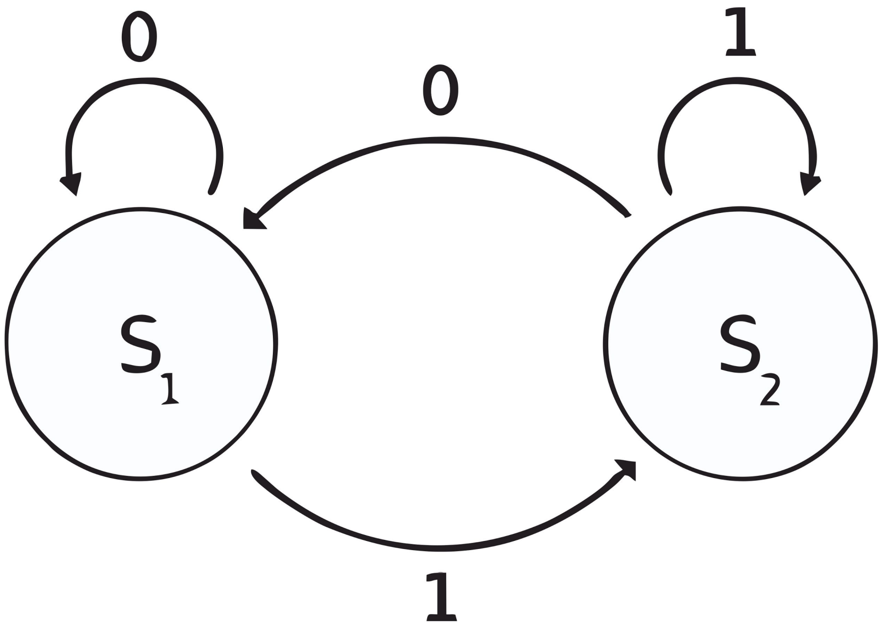
Road coloring theorem:
A graph has a synchronizing coloring if and only if it satisfies those two condition.

Problem:
What is the number of *changes,* that is sufficient to transform any random coloring to a synchronizing one?

Coloring change



Not synchronizing



Synchronizing

Problem formulation

Sync-range *of an automata* A*, denoted* $\boldsymbol{\varrho(A)}$*,*
is the minimal distance to a synchronizing coloring

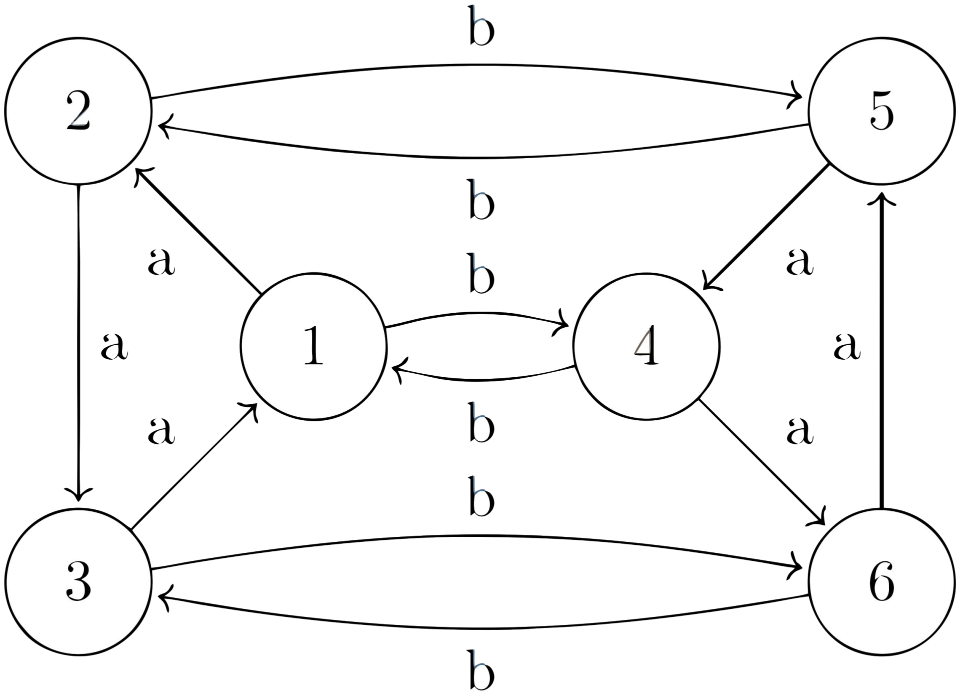
Sync-range *of a graph* G*, denoted* $\boldsymbol{\varrho(G)}$*,*
is the maximal sync-range of its colorings.

How large can $\varrho(G)$ *be, for a graph on* n *vertices?*

Upper bound: $n(k-1)$

Experiments

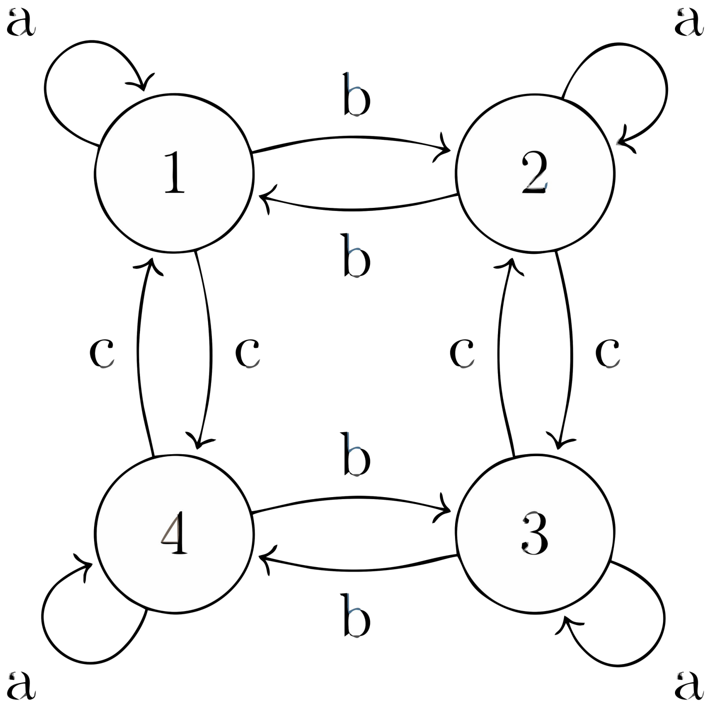
Simple enumeration of graphs for k = 2 up to 6 vertices gives us:



$$\varrho(G) = 2$$

Experiments

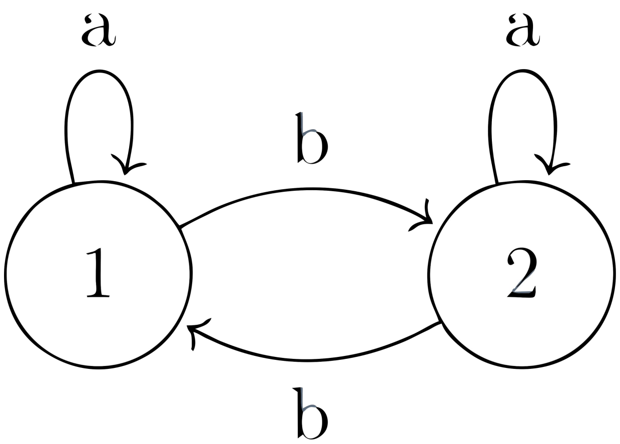
Simple enumeration of graphs for k = 3 up to 4 vertices gives us:



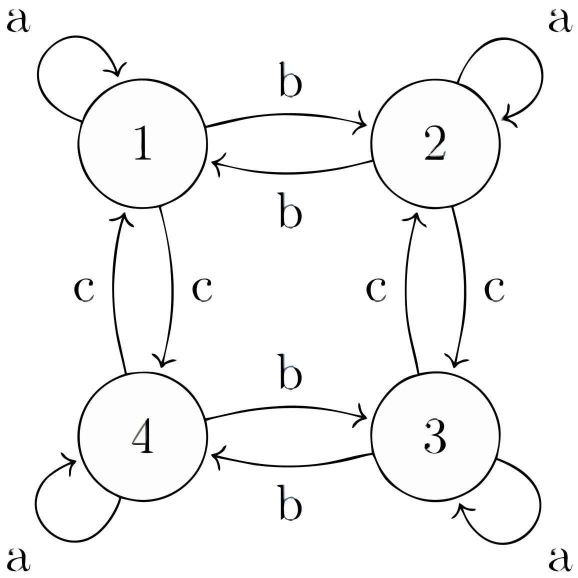
$$\varrho(G) = 2$$

Experiments

Enumeration of graphs up to n=10 for k=2 and to n=7 for k=3 gives: nothing new...



$\varrho(G) = 1$



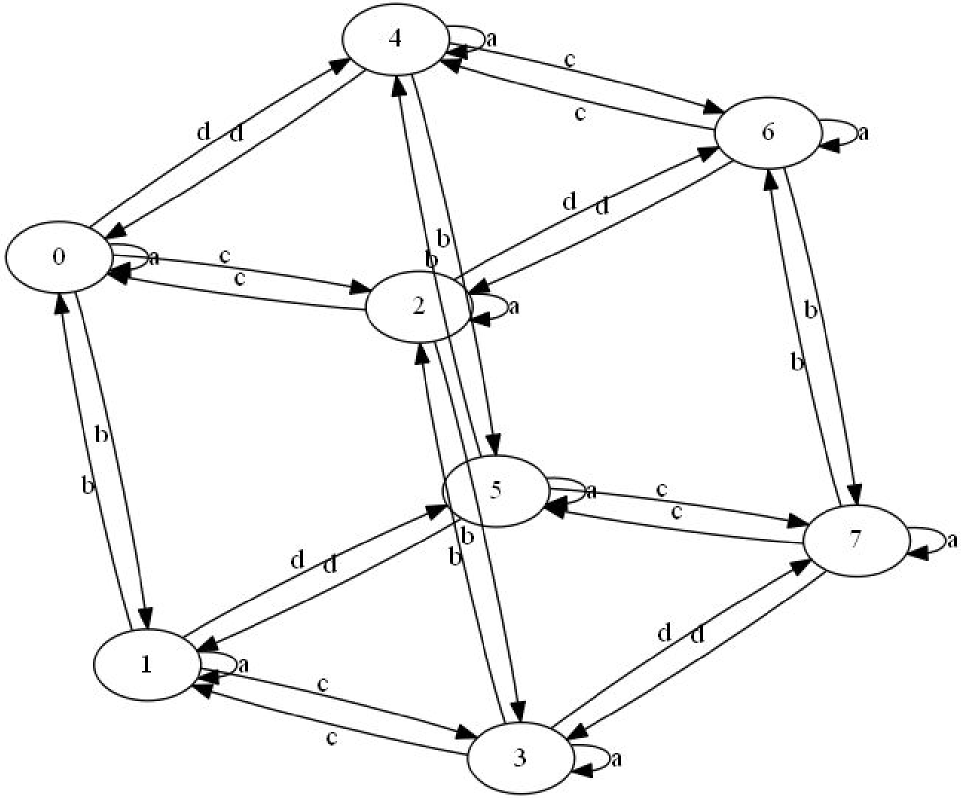
$\varrho(G) = 2$

Experiments

It is easy to guess the next one.
This is the graph of **d**-dimensional hypercube with loops.

$$Hypercube_d$$

- $n = 2^d$
- $k = d + 1$
- $\varrho(Hypercube_d) = d$



$$\varrho(G) = 3$$

Cayley graphs

$Let\ \mathfrak{G} - be\ a\ group\ and\ S \subseteq \mathfrak{G}\ - generating\ set$

$The\ Cayley\ graph\ \Gamma\ (\mathfrak{G}, S) = \ \langle V, E\rangle, is\ automata\ where$:

- $V = \mathfrak{G}$
- $\Sigma = S$
- $For\ each\ element\ g \in \mathfrak{G}, and\ generator\ s \in S$ $there\ is\ an\ edge\ (g, gs)\ in\ E, having\ color\ s$.

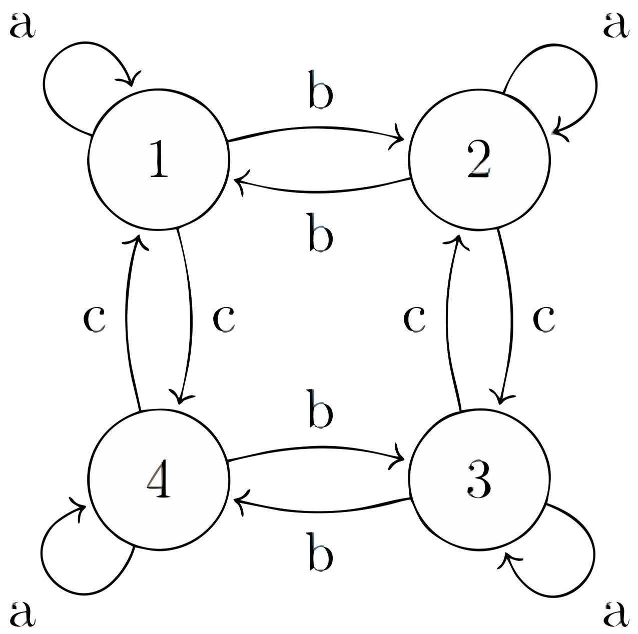
$Note\ that\ \mathfrak{G}\ is\ uniquely\ defined\ by\ S, and\ can\ be\ omited$.

Hypercube graphs

It is easy to see, that every $Hypercube_d$ *graph is in fact Cayley graph of a permutation group with generating set composed of*:

- *Identity permutation*
- *Cycle* $(1, 2)$
- *Cycle* $(3, 4)$
- $\dots$
- *Cycle* $(2d - 1, 2d)$

For example: $Hypercube_2$ *is a graph of a group of permutations on* 4 *elements, with generating set* $\{id, (1,2), (3,4)\}$



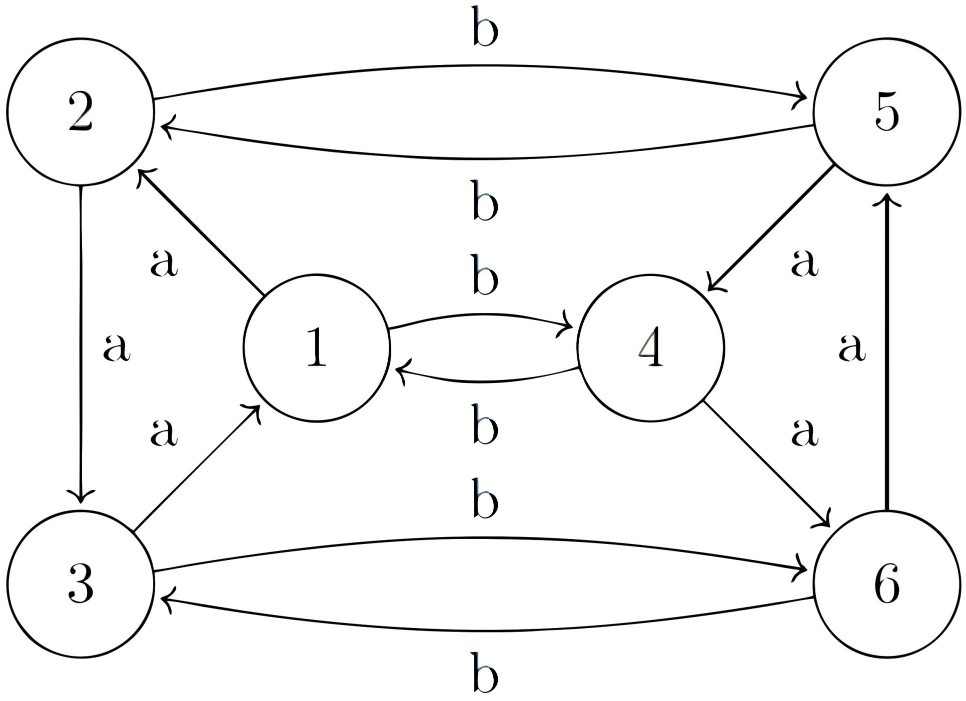
Hyperprism graphs

$Hyperprism_d$

- $n = 2 \cdot 3^{d-1}$
- $k = d$
- $\varrho(Hyperprism_d) = d$

It is Cayley graph of:

- $(1,2), (1,2,3)$
- $(4,5), (4,5,6)$
- …
- $(3d-2, 3d-1)$
- $(3d-2, 3d-1, 3d)$



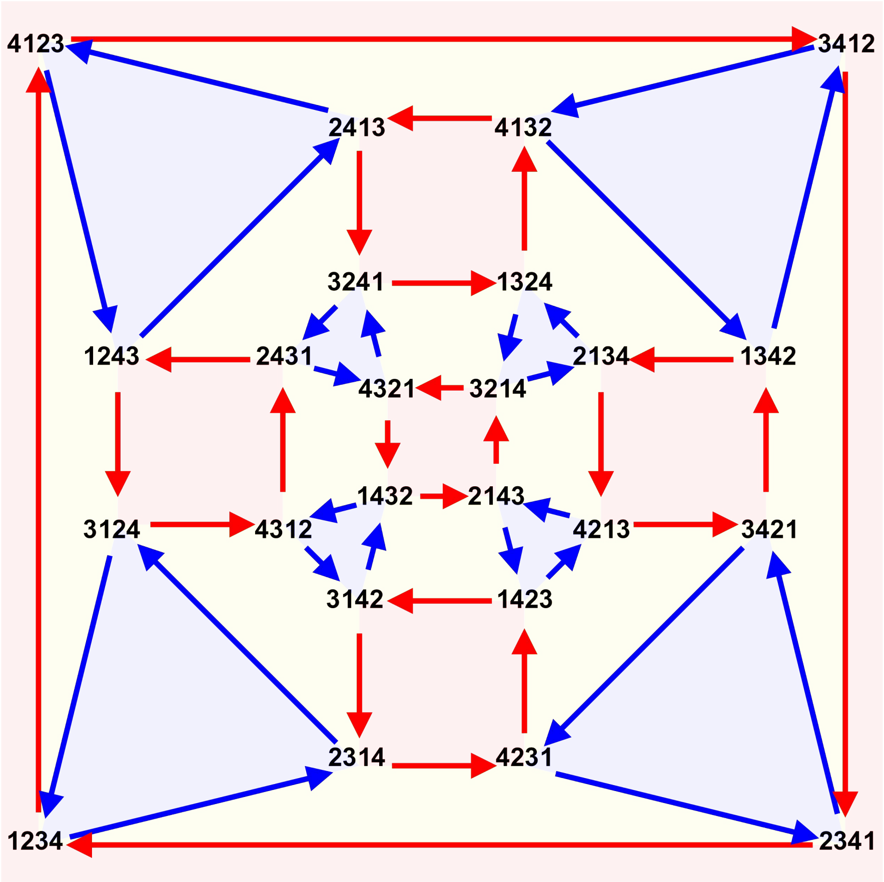
Symmetric group graph

Cayley graph of generated by:

- $(1, 2, \ldots, d-1)$
- $(1, 2, \ldots, d)$

Is a graph of the symmetric group S_d, *and has*:

- $n = d!$
- $k = 2$
- $\varrho(G) = d$



Determining synchronization

$\boldsymbol{Lemma}$:

$Automata\ A\ is\ synchronizing\ iff$

$\forall q_1, q_2 \ \in Q, \exists w \in \Sigma^* : q_1 \cdot w = q_2 \cdot w$

$\boldsymbol{Algorithm}$:

$Consider\ graph\ \mathcal{P}^{[2]}(A) = \langle Q', \Sigma \rangle, where$

$Q' = Q \cup \{(q_1, q_2) | q_1, q_2 \in Q\}.$

$Check\ that\ for\ every\ (q_1, q_2)\ there\ is$

$some\ reachable\ singleton\ q.$

Upper bound

$Denote\ SCC(A) - number\ of\ sink\ components\ in\ \mathcal{P}^{[2]}(A).$

$A\ is\ synchronizing\ \Leftrightarrow\ SCC(A) = 1.$

$\boldsymbol{Observation}$:
$If\ for\ every\ non\text{-}synchronizing\ coloring\ A, there\ is$
$a\ change\ to\ a\ coloring\ B\ having\ SCC(B) \leq SCC(A)/2$

$Then, \varrho(A) \leq \log_2 n.$

Summary

$$We\ have\ upper\ bound\colon\ \varrho(G) \leq \log_2 n$$
$$It\ is\ reached\ by\ Hypercube_d,$$
$$for\ k = d = \log_2 n\ .$$

$$For\ k = 2, upper\ bound\ must\ be\ smaller\colon$$
$$For\ a\ graph\ having\ n = \mathrm{d}!, \varrho(G) = d.$$
$$But\ the\ upper\ bound\ is\ \ln d! \approx d \cdot \ln d.$$