Automorphisms of low complexity subshifts

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Basic topological notions: Topological dynamical system

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(Y, S) is a (topological) factor of (X, T), or (X, T) is an extension of (Y, S), if there exists a continuous surjective $\phi\colon X \to Y$ such that

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Basic topological notions: Automorphism

Definition

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- Q: How does $\mathrm{Aut}(X, T)$ acts on X? On T-invariant measures?

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A word u is an element of the free monoid A^* generated by A.

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The open sets are unions of cylinders:

$$[u.v] := \{(x_n)_n \in A^{\mathbb{Z}} : x_{-|u|} \dots x_{|v|-1} = uv\}; \qquad u, v \in A^*$$

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The shift map

$$\begin{aligned} \sigma\colon A^{\mathbb{Z}} &\to A^{\mathbb{Z}} \\ (x_n)_{n\in\mathbb{Z}} &\mapsto (x_{n+1})_{n\in\mathbb{Z}} \end{aligned}$$

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For a closed set $X \subset A^{\mathbb{Z}}$, shift invariant ($\sigma(X) = X$), a subshift is the dynamical system $(X, \sigma|_X)$.

Similarly

$$X = \{(x_n)_n \in A^{\mathbb{Z}}; x_i \cdots x_{i+m} \notin \mathcal{F} \ \forall m, i\}, \text{ where } \mathcal{F} \subset A^*.$$

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The language

$$\mathcal{L}(X) := \{u \in A^* : u = x_0 \cdots x_{|u|-1} \text{ for some } (x_n)_n \in X\}.$$

$$\mathcal{L}_n(X) := \mathcal{L}(X) \cap A^n.$$

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The system (X, σ) is expansive: $\exists \epsilon > 0$, $x \neq y \in X$,

$$\sup_{n \in \mathbb{Z}} d(\sigma^n(x), \sigma^n(y)) > \epsilon.$$

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Exercice $stab_G(g \cdot x) = g[stab_G(x)]g^{-1}$.

Algebraic motivations

The (topological) full group of a subshift (X, σ) is

$$[[\sigma]] := \{\psi \in \mathrm{Homeo}(X); \exists n \colon X \to \mathbb{Z} \text{ cont. } \psi(x) = \sigma^{n(x)}(x) \, \forall x \in X\}.$$

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Matui (06), Juschenko-Monod (12):
If (X, σ) is a minimal subshift (without proper subshift)
$[[\sigma]]'$ is finitely generated, simple and amenable.
(first example known !).

See L. Bartholdi's lecture

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Giordano-Putnam-Skau (1999): If (X, σ) is minimal (without proper subshift)

$$\mathrm{Out}([[\sigma]]) \simeq \{\phi \in \mathrm{Homeo}(X) : \phi \circ \sigma = \sigma^{\pm} \circ \phi\}/\langle\sigma\rangle.$$

$$\{\phi \in \mathrm{Homeo}(X) : \phi \circ \sigma = \sigma^{\pm} \circ \phi\}_{/\mathrm{Aut}(X,\sigma)} \subset \mathbb{Z}/2\mathbb{Z}.$$

Basic topological notions

Theorem (Curtis-Hedlund-Lyndon)

An automorphism ϕ *of* (X, σ) *is a sliding block code*,
i.e. *there exists a* block map $\hat{\phi}\colon \mathcal{L}_{2r+1}(X) \to A$ *s.t.*

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$$x = \ldots 01001 \quad 1.10 \quad 10101000111\ldots$$

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$$\phi(x) = \ldots 0100111.0101010000111 \ldots$$

Basic topological notions

Theorem (Curtis-Hedlund-Lyndon)

An automorphism ϕ *of* (X, σ) *is a sliding block code*, i.e. *there exists a* block map $\hat{\phi}\colon \mathcal{L}_{2r+1}(X) \to A$ *s.t.*

$$\phi(x)_n = \hat{\phi}(x_{n-r} \cdots x_{n+r}) \textit{ for any } n \in \mathbb{Z}.$$

e.g. $A = \{0,1\}$, $\hat{\phi}$:

000	001	010	011	100	101	110	111
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
0	1	0	1	0	1	0	1

$$x = \quad \ldots 010011.10101010000111 \ldots$$

$$\phi(x) = \quad \ldots 0100111.0101010000111 \ldots = \sigma(x)$$

Basic topological notions

Theorem (Curtis-Hedlund-Lyndon)

Let ϕ be an automorphism of (X, σ)
There exists a local map $\hat{\phi}$: $\mathcal{L}_{2r+1}(X) \to A$ *s.t.*

$$\phi(x)_n = \hat{\phi}(x_{n-r} \cdots x_{n+r}) \textit{ for any } n \in \mathbb{Z}.$$

Corollary

$Aut(X, \sigma)$ *is countable.*
$Aut(X, \sigma)$ *is a discrete subgroup of Homeo(X) for the uniform convergence topology.*

Complexity restrictions

The complexity $p_X : \mathbb{N} \to \mathbb{N}$,

$$p_X(n) = \#\mathcal{L}_n(X) = \# \text{ words of length } n \text{ in } X.$$

Q: How the growth of the complexity restricts $\mathrm{Aut}(X, \sigma)$?

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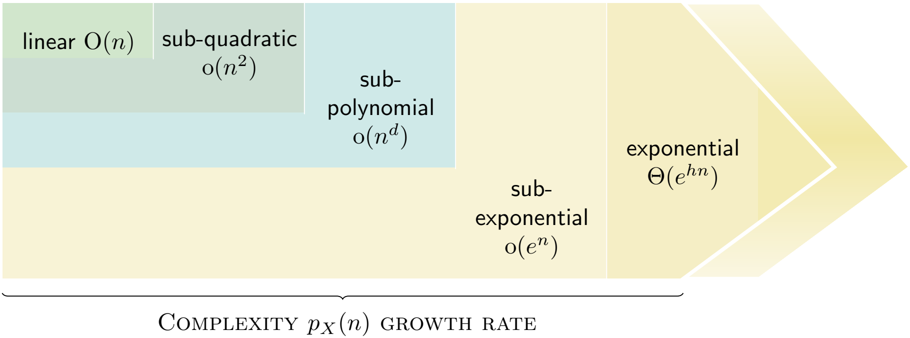

Plan

1. Automorphism of SFT
2. Automorphism of classical minimal systems
   a) Linear complexity case
   b) Toeplitz subshifts case
3. Automorphism for sub-exponential complexity subshifts and non embeddable groups.

Automorphism of SFT

Hedlund (69), Boyle, Lind & Rudolph (88): Let (X, σ) be an uncountable SFT. Then $\mathrm{Aut}(X, \sigma)$ contains

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Open problem: $\mathrm{Aut}(\{1, 2\}^{\mathbb{Z}}, \sigma) \simeq \mathrm{Aut}(\{1, 2, 3\}^{\mathbb{Z}}, \sigma)$?

- If (X, σ) is irreducible, $Z(\mathrm{Aut}(X, \sigma)) = \langle \sigma \rangle$ (Ryan, 72).

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- If (X, σ) is irreducible, $Z(\mathrm{Aut}(X, \sigma)) = \langle \sigma \rangle$ (Ryan, 72).

In this case:

$\mathrm{Aut}(X, \sigma)$ is not finitely generated, not amenable.

Embedding free products

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Study the action of $\langle \phi_1, \phi_2, \phi_3 \rangle$ on the the point $\cdots 000 * 000 \cdots$.
See it generates a group isomorphic to $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$.

E.g.

$$\phi_{\mathbf{1}} : \ \cdots 000 * 0000 \cdots \mapsto \cdots 000 * \mathbf{1}000 \cdots$$

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$$\phi_{\mathbf{3}}\phi_2\phi_1 : \cdots 000 * 0000 \cdots \mapsto \cdots 000 * 12\mathbf{3}0 \cdots$$

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Basic algebra shows it contains the free group on 2 generators, hence the free group with countably many generators.

Automorphism of SFT

A group G is residually finite if for any $1_G \neq g \in G$ there is a homomorphism $\varphi\colon G \to G_0$ onto a finite group G_0 such that $\varphi(g) \neq 1_{G_0}$.

Ex: finite group, $\mathbb{Z}^d$, free group, finitely generated linear group,...

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Since $\bigcup_n \mathrm{Per}_n$ is dense in X,

$$\varphi_n(\phi) = \mathrm{Id},\ \forall n \Rightarrow \phi = \mathrm{Id}.$$

Automorphism of SFT

Theorem (BLR)

For an SFT, the group $\mathrm{Aut}(X,\sigma)$ *is residually finite.*

Corollary

For an SFT, $\mathrm{Aut}(X,\sigma)$ *does not contains a divisible subgroup: For any* $\phi \in \mathrm{Aut}(X,\sigma) \setminus \{\mathrm{Id}\}$*, there exists* $n \in \mathbb{N}$ *s.t. the equation*

$$\psi^n = \phi$$

has no solution $\psi \in \mathrm{Aut}(X,\sigma)$*.*

Ex: $\mathrm{Aut}(X,\sigma)$ does not contains $\mathbb{Q}$.

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Ex: $\mathrm{Aut}(X, \sigma)$ does not contains $\mathbb{Q}$.
Open problem: is $\mathbb{Z}[1/p]$ contained in $\mathrm{Aut}(X, \sigma)$ for any prime p?

Automorphism of SFT

Theorem (BLR)

For an SFT, the group $\mathrm{Aut}(X, \sigma)$ *contains no finitely generated group with unsolvable word problem.*

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Proof. Given $\phi_1, \dots, \phi_\ell \in \mathrm{Aut}(X, \sigma)$, find a finite procedure to decide if

$$\psi = \phi_{i_1}^{\pm} \circ \cdots \circ \phi_{i_r}^{\pm} = \mathrm{Id}, \qquad i_1, \dots, i_r \in \{1, \dots, \ell\}.$$

By Curtys-Hedlund-Lyndon Theorem, it is enough to check if the block map of ψ with range $r_\psi = \mathrm{O}(r)$ satisfies

$$\hat{\psi}(x_{-r_\psi} \cdots x_{r_\psi}) = x_0.$$

Automorphism of $\mathbb{Z}^d$- SFT

Theorem (Hochman (10))

Let (X, σ) *be a* $\mathbb{Z}^d$*-SFT with* $h(X, \sigma) > 0$ *then* $\mathrm{Aut}(X, \sigma)$ *contains the direct sum of every countable collection of finite group.*

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Theorem (Hochman (10))

Let (X,σ) be a $\mathbb{Z}^d$-SFT with $h(X,\sigma) > 0$ such that the minimal orbits are dense (e.g. periodic orbits) then $\mathrm{Aut}(X,\sigma)$ conains a copy of $\mathrm{Aut}(\{1,\ldots,n\}^{\mathbb{Z}},\sigma)$ for any n.

References

- M. Boyle, D. Lind & D. Rudolph, *The automorphism group of a shift of finite type*. Trans. Amer. Math. Soc. 1988
- G.A. Hedlund, *Endomorphisms and automorphism of the shift dynamical system*. Math. Systems theory, 1969
- M. Hochman *On the automorphism groups of multidimensional shifts of finite type*. Egrodic Theory Dynam. Systems, 2010
- K.H. Kim & F. W. Roush, *On the automorphism groups of subshifts*. pure Math. Appl. Ser. B, 1990
- P. Ryan, *The shift and commutativity*. Math. Systems theory, 1972