Lecture 3: Domino problem and (a)periodicity.

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Outline of the talk.

Domino problem and periodicity on $\mathbb{Z}^2$ (I)

We can define two notions of **periodic** configuration:

- A configuration $x \in A^{\mathbb{Z}^2}$ is **weakly periodic** if its stabilizer is infinite.

  $\Leftrightarrow$ *x admits a non-trivial direction $\overrightarrow{u}$ of periodicity.*

- A configuration $x \in A^{\mathbb{Z}^2}$ is **strongly periodic** if its stabilizer is of finite index in $\mathbb{Z}^2$: $[\mathbb{Z}^2 : \mathrm{Stab}(x)] < \infty$.

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Proposition

On $\mathbb{Z}^2$, if an SFT contains a weakly periodic configuration, then it contains a strongly periodic one.

Proof: on the blackboard.

Domino problem and periodicity on $\mathbb{Z}^2$ (II)

Wang's conjecture (1961)

If a set of Wang tiles can tile the plane, then they can always be arranged to do so periodically.

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Suppose Wang's conjecture is true. Then you can decide **DP** !

Semi-algorithm 1:

1. gives a finite periodic pattern, if it exists
2. loops otherwise

Semi-algorithm 2:

1. gives an integer n so that there is no $[1; n] \times [1; n]$ locally admissible pattern, if it exists
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Consequence

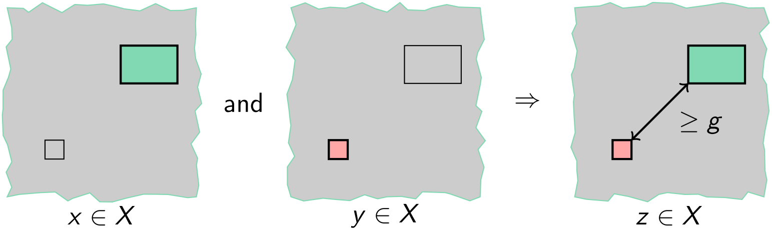
The undecidability of **DP** implies existence of an aperiodic SFT.

Outline of the talk.

Block gluing subshifts on $\mathbb{Z}^2$ (I)

A subshift $X \subset A^{\mathbb{Z}^2}$ is **block-gluing** with gap $g \in \mathbb{N}$ if for any two finite supports $S_1, S_2 \subset \mathbb{Z}^2$ at distance at least g, and for any $x, y \in X$

$$\text{there exists } z \in X \text{ s.t. } z_{|S_1} = x_{|S_1} \text{ and } z_{|S_2} = y_{|S_2}.$$



Remark: this is a **uniform** mixing condition.

Block gluing subshifts on $\mathbb{Z}^2$ (II)

Proposition (Folklore, written in Pavlov & Schraudner 2015)

A non-empty block-gluing SFT has a periodic configuration.

Proof: on the blackboard.

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Remark: Actually we prove something stronger: we can decide whether a locally admissible pattern is globally admissible (the language is decidable).

Outline of the talk.

Strongly aperiodic subshifts (I)

A subshift $X \subset A^G$ is **strongly aperiodic** if all its configurations have trivial stabilizer

$$\forall x \in X, \forall g \in G,\ \sigma^g(x) = x \Rightarrow g = 1_G.$$

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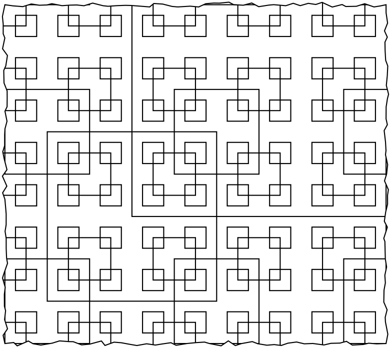
Example: Robinson's SFT is strongly aperiodic



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- groups $\mathbb{Z}^2 \rtimes H$ where H has decidable **WP** (Barbieri & Sablik, 2016).

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Question (simpler)

Does every f.g. group admit strongly aperiodic subshifts?

Strongly aperiodic subshifts (III)

Theorem (Gao, Jackson & Seward, 2009)

Every f.g. group G has a strongly aperiodic subshift on alphabet $\{0, 1\}$.

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Theorem (A. Barbieri & Thomassé, 2015)

Every f.g. group G has a strongly aperiodic subshift on alphabet $\{0, 1\}$.

Outline of the talk.

Lovász Local Lemma

(see Anton Chaplygin's talk yesterday)

$$\left.\begin{array}{l}(A_i)_{i=1\dots n} \textbf{ mutually independent} \\ \text{Each } A_i \text{ can be avoided}\end{array}\right\} \Rightarrow A_1,\dots,A_n \text{ can be avoided.}$$

Proposition

If events $A_1, \dots, A_n$ are mutually independent, then

$$Pr\left(\bigcap_{i=1}^{n} \bar{A}_i\right) = \prod_{i=1}^{n} (1 - Pr(A_i)) .$$

What about the dependent case ?

Lovász Local Lemma

(see Anton Chaplygin's talk yesterday)

$$\left.\begin{array}{l}(A_i)_{i=1\ldots n} \textbf{\textit{ not very dependent}} \\ \text{Each } A_i \text{ can be avoided}\end{array}\right\} \Rightarrow A_1, \ldots, A_n \text{ can be avoided.}$$

Lovász Local Lemma (1975)

Let $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$. For $A_i \in \mathcal{A}$, let $\Gamma(A_i)$ be the subset of $\mathcal{A}$ such that A_i is independent of the collection $\mathcal{A} \setminus (\{A_i\} \cup \Gamma(A_i))$. Suppose there are $x_i, \ldots, x_n$ such that $0 \leq x_i < 1$ and:

$$\forall A_i \in \mathcal{A} : Pr(A_i) \leq x_i \prod_{A_j \in \Gamma(A)} (1 - x_j)$$

then the probability of avoiding $A_1, A_2, \ldots, A_n$ is positive.

Lovász Local Lemma in Symbolic Dynamics (I)

How to use LLL in Symbolic Dynamics?

Suppose you want to prove that the subshift X is non-empty.

- Uniform Bernoulli measure on configurations space.
- Bad events $\approx$ forbidden patterns.
- Compactness + LLL (if applicable) show the non-emptiness of the subshift.

Lovász Local Lemma in Symbolic Dynamics (II)

Let G be a f.g. group, A a finite alphabet and μ the uniform Bernoulli probability measure on A^G.

A sufficient condition for being non-empty

Let $X \subset A^G$ be a subshift defined by $\mathcal{F} = \bigcup_{n \geq 1} \mathcal{F}_n$, where $\mathcal{F}_n \subset A^{B_n}$. Suppose that there exists a function $x : \mathbb{N} \times G \to (0,1)$ such that:

$$\forall n \in \mathbb{N}, g \in G,\ \mu(A_{n,g}) \leq x(n,g) \prod_{\substack{gS_n \cap hS_k \neq \emptyset \\ (k,h) \neq (n,g)}} (1 - x(k,h)),$$

where $A_{n,g} = \{x \in A^G : x|_{gS_n} \in \mathcal{F}_n\}$. Then the subshift X is non-empty.

Strong aperiodicity vs. the distinct neighborhood property

A subshift $X \subset A^G$ is **strongly aperiodic** if all its configurations have trivial stabilizer

$$\forall x \in X, \forall g \in G,\ \sigma^g(x) = x \Rightarrow g = 1_G.$$

Fix $A = \{0,1\}$.

A configuration $x \in \{0,1\}^G$ has the **distinct neighborhood property** if for every $h \in G \setminus \{1_G\}$, there exists a finite $T \subset G$ s.t.

$$\forall g \in G,\ x_{|ghT} \neq x_{|gT}.$$

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Proposition

If $x \in \{0,1\}^G$ has the distinct neighborhood property, then the subshift $\overline{Orb_\sigma(x)}$ is strongly aperiodic.

Proof: on the blackboard.

Distinct neighborhood property with LL

Proposition

Every infinite f.g. group G has a configuration $x \in \{0,1\}^G$ with the distinct neighborhood property.

Proof:

- Take $(s_i)_{i\in\mathbb{N}}$ an enumeration of G with $s_0 = 1_G$.
- Choose $(T_i)_{i\in\mathbb{N}}$ a sequence of finite sets of G s.t.

$$T_i \cap s_i T_i = \emptyset \text{ and } |T_i| = Ci \text{ for some constant } C.$$

- $A_{n,g} = \left\{x \in \{0,1\}^G \mid x_{|gT_n} = x_{|gs_nT_n}\right\}$.
- $x(n,g) = 2^{-\frac{Cn}{2}}$.

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Theorem

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An effectively closed strongly aperiodic subshift (I)

A subshift is G-**effectively closed** if it can be defined by a set of forbidden patterns recognizable by a Turing machine with oracle $\mathbf{WP}(G)$.

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Every finite graph with degree $\leq \Delta$ has a square-free coloring with $2e^{16}\Delta^2$ colors.

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Proposition

Let G a f.g. group and S a generating set. Then $\Gamma(G, S)$ has a square-free coloring with $2^{19}|S|^2$ colors.

An effectively closed strongly aperiodic subshift (II)

Theorem (A. Barbieri & Thomassé, 2015)

Every f.g. group G has a G-effectively closed strongly aperiodic subshift.

Sketch of the proof:

- Fix S and take $X \subset A^G$ be the subshift such that every square in $\Gamma(G, S)$ is forbidden.
- Let $g \in G$ such that $\sigma^g(x) = x$ for some $x \in X$.
- Factorize g as uwv with $u = v^{-1}$ and $|w|$ minimal (as a word on $(S \cup S^{-1})^*$). If $|w| = 0$, then $g = 1_G$.

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- If not, let $w = w_1 \dots w_n$ and consider the odd length walk $\pi = v_0 v_1 \dots v_{2n-1}$ on $\Gamma(G, S)$ defined by:

$$v_i = \begin{cases} 1_G & \text{if } i = 0 \\ w_1 \dots w_i & \text{if } i \in \{1, \dots, n\} \\ w w_1 \dots w_{i-n} & \text{if } i \in \{n+1, \dots, 2n-1\} \end{cases}$$

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- π is a path, and $x_{v_i} = x_{v_{i+n}} \Rightarrow g = 1_G$.

Conclusion

- Every one-ended f.g. group with decidable **WP** has strongly aperiodic SFTs?
- Does there exist G with decidable **DP** and strongly aperiodic SFTs?
- Does there exist G with undecidable **DP** and no strongly aperiodic SFT?

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Thank you for your attention !!