Ubiquitous Doubling Algorithms, General Theory, and Applications

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Outline

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Simple and Yet Powerful Idea

Numerous approximations rely on simple iteration:

$$X_{j+1} = f(X_j) \text{ for } j = 0, 1, \ldots, \text{ gievn } X_0$$

to generate a sequence of approximations that hopefully converges to the desired target.

Often too slow, but can we make it go faster? Any ideas?

What about skipping all X_j but for $j = 2^i$?
Kind of like repeatedly squaring:

$$x, x^2, (x^2)^2 = x^4, (x^4)^2 = x^8, \ldots.$$

This is the basics of the *Doubling Algorithm* (DA).

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To accelerate $X_{j+1} = f(X_j)$ by computing X_j for $j = 2^i$ only. Make sense only if there is a cost-effective path $X_{2^{i-1}} \to X_{2^i}$.

No such a path guaranteed in general, unless f is linear: $f(X) = \Phi X$, and then

$$X_{2^i} = \Phi^{2^i} X_0 = \Phi^{2^{i-1}} X_{2^{i-1}}.$$

But still need to be able to square $\Phi^{2^{i-1}}$ efficiently.

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$X = A^T X(I + GX)^{-1}A + H$.

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Example (Discrete-time Algebraic Riccati Equation): $X = A^T X (I + GX)^{-1} A + H$. Naturally

$$X_{j+1} = f(X_j) := A^{\mathrm{T}} X_j (I + GX_j)^{-1} A + H, \quad \text{given } X_0$$

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Anderson (1978) did just that.

Go Forward

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three types nonlinear matrix equations (NMEs)

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This talk attempts to provide a coherent theory to go forward.

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Continuous-time Algebraic Riccati Equation (CARE) Type

$$XDX - AX - XB + C = 0.$$

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- H^*ARE from the Laplace transform inversion method in Markov modulated fluid flow [Ahn & Ramaswami (2004), Liu & Xue (2012), ...]

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- DARE $A^\top X(I+GX)^{-1}A-X+H=0$ from Discrete-time control system [...]

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- Solve Palindromic Quadratic Eigenvalue Problem (PQEP) $P(\lambda)z \equiv (\lambda^2 A^\mathsf{T} + \lambda Q + A)z = 0$ from high speed trains [Hilliges, Mehl, & V. Mehrmann (2004), ...]
- Compute Green function from quantum transport in nano research [S. Datta (2000), Guo & Lin (2010), ...]
- Solve QEP $P(\lambda)\boldsymbol{x} \equiv \left(\lambda^2 B + \lambda Q + A\right)\boldsymbol{x} = 0$ from Retarded Time-Delay System (TDS) [Jarlebring (2008)]
- $X + A^\mathsf{T} X^{-1} A = Q$ from surface acoustic wave simulation in telecommunication [Campbell (1998)]

Commonality

Let X be a solution to one of the equations. Then $\begin{array}{cc} & m \\ \begin{array}{c} m \\ n \end{array} & \begin{bmatrix} I \\ X \end{bmatrix} \end{array}$ is basis matrix of an eigenspace for $\mathscr{A} - \lambda\mathscr{B}$:

$$\mathscr{A}\begin{bmatrix} I \\ X \end{bmatrix} \equiv \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}\begin{bmatrix} I \\ X \end{bmatrix} M \equiv \mathscr{B}\begin{bmatrix} I \\ X \end{bmatrix} M,$$

where M is $m \times m$. As a consequence, $\mathrm{eig}(M) \subset \mathrm{eig}(\mathscr{A}, \mathscr{B})$.

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Blockwise we have

$$A_{11} + A_{12}X = (B_{11} + B_{12}X)M, \quad A_{21} + A_{22}X = (B_{21} + B_{22}X)M.$$

Assume $(B_{11} + B_{12}X)$ is invertible to get

$$A_{21} + A_{22}X = (B_{21} + B_{22}X) \underbrace{(B_{11} + B_{12}X)^{-1}(A_{11} + A_{12}X)}_{M}.$$

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$$A_{21} + A_{22}X = (B_{21} + B_{22}X)(B_{11} + B_{12}X)^{-1}(A_{11} + A_{12}X).$$

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$$A_{21} + A_{22}X = (B_{21} + B_{22}X)(B_{11} + B_{12}X)^{-1}(A_{11} + A_{12}X).$$

Case: $\mathscr{B} = I_{m+n}$:

$$A_{21} + A_{22}X = X(A_{11} + A_{12}X)$$

gives

$$-XA_{12}X + A_{22}X - XA_{11} + A_{21} = 0.$$

This is "CARE".

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$$A_{21} + A_{22}X = (B_{21} + B_{22}X)(B_{11} + B_{12}X)^{-1}(A_{11} + A_{12}X).$$

Case: $A_{12} = 0$, $A_{22} = I$, $B_{11} = I$, and $B_{21} = 0$, i.e.,

$$\mathscr{A} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & I \end{bmatrix}, \quad \mathscr{B} = \begin{bmatrix} I & B_{12} \\ 0 & B_{22} \end{bmatrix}.$$

We have $A_{21} + X = B_{22}X(I + B_{12}X)^{-1}A_{11}$, or equivalently

$$B_{22}X(I + B_{12}X)^{-1}A_{11} - A_{21} - X = 0.$$

This is “DARE”.

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$$A_{21} + A_{22}X = (B_{21} + B_{22}X)(B_{11} + B_{12}X)^{-1}(A_{11} + A_{12}X).$$

Case: $m = n$, $A_{12} = 0$, $A_{22} = I$, $B_{11} = 0$, $B_{12} = I$, and $B_{22} = 0$, i.e.,

$$\mathscr{A} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & I \end{bmatrix}, \quad \mathscr{B} = \begin{bmatrix} 0 & I \\ B_{21} & 0 \end{bmatrix}.$$

We have $A_{21} + X = B_{21}X^{-1}A_{11}$, or equivalently

$$X - B_{21}X^{-1}A_{11} = -A_{21}.$$

This is the NME.

Outline

Key ingredients

Three ingredients to DA:

- doubling transformation theorem
- initial setups (two standard forms)
- doubling iterative kernel

Outline

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Define $\widetilde{\mathscr{A}} = \mathscr{A}_\perp\mathscr{A}$ and $\widetilde{\mathscr{B}} = \mathscr{B}_\perp\mathscr{B}$. Then $\widetilde{\mathscr{A}} - \lambda\widetilde{\mathscr{B}}$ is regular.

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- keep "doubling" to quickly arrive at some null space, provided $\operatorname{eig}(M) \in \mathbb{D}_-$ (open unit disk);
- maintain special forms for $\mathscr{A}$- and $\mathscr{B}$-matrices to find Z in special form.

Outline

Road Map

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1) Transform $\mathscr{A} - \lambda\mathscr{B} \to \mathscr{A}' - \lambda\mathscr{B}'$ such that

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Road Map

$$\mathscr{A}\begin{bmatrix} I \\ X \end{bmatrix} \equiv \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}\begin{bmatrix} I \\ X \end{bmatrix} M \equiv \mathscr{B}\begin{bmatrix} I \\ X \end{bmatrix} M.$$

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One such a transformation is the *Möbius transformation*:

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3) Perform doubling iteration kernel.

First standard form (SF1)

$$\text{(SF1)} \qquad \mathscr{A}_0 = \begin{array}{c c} & \begin{array}{cc} m & n \end{array} \\ \begin{array}{c} m \\ n \end{array} & \begin{bmatrix} E_0 & 0 \\ -X_0 & I \end{bmatrix} \end{array}, \quad \mathscr{B}_0 = \begin{array}{c c} & \begin{array}{cc} m & n \end{array} \\ \begin{array}{c} m \\ n \end{array} & \begin{bmatrix} I & -Y_0 \\ 0 & F_0 \end{bmatrix} \end{array}.$$

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$$

Essentially Gaussian elimination (assuming involved inversions exist):

$$
\begin{aligned}
[\mathscr{A}' \mid \mathscr{B}'] = \left[\begin{array}{cc|cc} X & X & \color{red}{X} & X \\ X & X & X & X \end{array}\right] &\rightarrow \left[\begin{array}{cc|cc} X & X & I & X \\ X & X & \color{red}{X} & X \end{array}\right] \\
\rightarrow \left[\begin{array}{cc|cc} X & X & I & X \\ X & \color{red}{X} & 0 & X \end{array}\right] &\rightarrow \left[\begin{array}{cc|cc} X & \color{red}{X} & I & X \\ X & I & 0 & X \end{array}\right] \\
\rightarrow \left[\begin{array}{cc|cc} X & 0 & I & X \\ X & I & 0 & X \end{array}\right] &
\end{aligned}
$$

in SF1

Second standard form (SF2)

$$\text{(SF2)} \qquad \mathscr{A}_0 = \begin{array}{c c} & \begin{array}{cc} n & n \end{array} \\ \begin{array}{c} n \\ n \end{array} & \begin{bmatrix} E_0 & 0 \\ -X_0 & I \end{bmatrix} \end{array}, \quad \mathscr{B}_0 = \begin{array}{c c} & \begin{array}{cc} n & n \end{array} \\ \begin{array}{c} n \\ n \end{array} & \begin{bmatrix} -Y_0 & I \\ F_0 & 0 \end{bmatrix} \end{array}.$$

Second standard form (SF2)

$$\text{(SF2)} \qquad \mathscr{A}_0 = \begin{array}{c} \\ n \\ n \end{array}\!\!\begin{array}{c} \begin{array}{cc} n & n \end{array} \\ \begin{bmatrix} E_0 & 0 \\ -X_0 & I \end{bmatrix} \end{array}, \quad \mathscr{B}_0 = \begin{array}{c} \\ n \\ n \end{array}\!\!\begin{array}{c} \begin{array}{cc} n & n \end{array} \\ \begin{bmatrix} -Y_0 & I \\ F_0 & 0 \end{bmatrix} \end{array}.$$

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$$\to \left[\begin{array}{cc|cc} X & 0 & X & X \\ X & I & X & X \end{array}\right] \to \left[\begin{array}{cc|cc} X & 0 & X & I \\ X & I & X & X \end{array}\right]$$
$$\to \left[\begin{array}{cc|cc} X & 0 & X & I \\ X & I & X & 0 \end{array}\right]$$

in SF2

Outline

To keep in mind ...

To generate a sequence $\{\mathscr{A}_i - \lambda \mathscr{B}_i\}$ such that

- maintain respective standard form, and
- at the same time satisfy

$$\mathscr{A}_i \begin{bmatrix} I \\ X \end{bmatrix} = \mathscr{B}_i \begin{bmatrix} I \\ X \end{bmatrix} \mathscr{M}^{2^i} \quad \text{for } i = 0, 1, \ldots$$

When the spectral radius $\rho(\mathscr{M}) < 1$, $\{\mathscr{M}^{2^i}\}$ goes to 0 quadratically.

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When the spectral radius $\rho(\mathscr{M}) < 1$, $\{\mathscr{M}^{2^i}\}$ goes to 0 quadratically.

Guidance – doubling transformation theorem:

$$\operatorname{rank}([\mathscr{A}_\perp, \mathscr{B}_\perp]) = \ell, \quad [\mathscr{A}_\perp, \mathscr{B}_\perp] \begin{bmatrix} \mathscr{B} \\ -\mathscr{A} \end{bmatrix} = 0,$$

$$\widetilde{\mathscr{A}} := \mathscr{A}_\perp \mathscr{A}, \quad \widetilde{\mathscr{B}} := \mathscr{B}_\perp \mathscr{B},$$

$$\mathscr{A} Z = \mathscr{B} Z M \quad \Rightarrow \quad \widetilde{\mathscr{A}} Z = \widetilde{\mathscr{B}} Z M^2.$$

DA for SF1

To construct $\mathscr{A}_i - \lambda \mathscr{B}_i$ with

$$\mathscr{A}_i = \begin{array}{c c} & \begin{array}{cc} m & n \end{array} \\ \begin{array}{c} m \\ n \end{array} & \begin{bmatrix} E_i & 0 \\ -X_i & I \end{bmatrix} \end{array}, \quad \mathscr{B}_i = \begin{array}{c c} & \begin{array}{cc} m & n \end{array} \\ \begin{array}{c} m \\ n \end{array} & \begin{bmatrix} I & -Y_i \\ 0 & F_i \end{bmatrix} \end{array}.$$

Enough to know how to go from step i to step $i+1$.

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To find $\mathscr{A}_{i\perp}, \mathscr{B}_{i\perp} \in \mathbb{C}^{(m+n)\times(m+n)}$ such that

$$\text{rank}([\mathscr{A}_{i\perp}, \mathscr{B}_{i\perp}]) = m+n, \quad [\mathscr{A}_{i\perp}, \mathscr{B}_{i\perp}] \begin{bmatrix} \mathscr{B}_i \\ -\mathscr{A}_i \end{bmatrix} = 0.$$

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Can again perform Gaussian-like block eliminations

$$\begin{bmatrix} I_m & -Y_i \\ 0 & F_i \\ -E_i & 0 \\ X_i & -I_n \end{bmatrix} \xrightarrow{L_1} \begin{bmatrix} I_m & -Y_i \\ 0 & F_i \\ 0 & -E_i Y_i \\ 0 & -I_n + X_i Y_i \end{bmatrix} \xrightarrow{L_2} \begin{bmatrix} I_m & -Y_i \\ 0 & F_i \\ 0 & -E_i Y_i \\ 0 & -I_n \end{bmatrix} \xrightarrow{L_3} \begin{bmatrix} I_m & -Y_i \\ 0 & 0 \\ 0 & 0 \\ 0 & -I_n \end{bmatrix} \xrightarrow{L_4} \begin{bmatrix} I_m & -Y_i \\ 0 & -I_n \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

DA for SF1 (cont'd)

Post-process the last $m+n$ rows of $L_4L_3L_2L_1$ to give

$$\mathscr{A}_{i\perp} = \begin{bmatrix} E_i(I_m - Y_iX_i)^{-1} & 0 \\ -F_i(I_n - X_iY_i)^{-1}X_i & I_n \end{bmatrix}, \quad \mathscr{B}_{i\perp} = \begin{bmatrix} I_m & -E_i(I_m - Y_iX_i)^{-1}Y_i \\ 0 & -F_i(I_n - X_iY_i)^{-1} \end{bmatrix}.$$

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Finally

$$\mathscr{A}_{i+1} = \mathscr{A}_{i\perp}\mathscr{A}_i, \quad \mathscr{B}_{i+1} = \mathscr{B}_{i\perp}\mathscr{B}_i$$

to give

$$E_{i+1} = E_i(I_m - Y_iX_i)^{-1}E_i, \tag{1a}$$

$$F_{i+1} = F_i(I_n - X_iY_i)^{-1}F_i, \tag{1b}$$

$$X_{i+1} = X_i + F_i(I_n - X_iY_i)^{-1}X_iE_i, \tag{1c}$$

$$Y_{i+1} = Y_i + E_i(I_m - Y_iX_i)^{-1}Y_iF_i. \tag{1d}$$

DA for SF1 (cont'd)

Post-process the last $m+n$ rows of $L_4L_3L_2L_1$ to give

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At convergence, X_i goes to some solution X.

DA for SF2

To construct $\mathscr{A}_i - \lambda\mathscr{B}_i$ with

$$\mathscr{A}_i = \begin{array}{c c} & \begin{array}{cc} n & n \end{array} \\ \begin{array}{c} n \\ n \end{array} & \begin{bmatrix} E_i & 0 \\ -X_i & I \end{bmatrix} \end{array}, \quad \mathscr{B}_i = \begin{array}{c c} & \begin{array}{cc} n & n \end{array} \\ \begin{array}{c} n \\ n \end{array} & \begin{bmatrix} -Y_i & I \\ F_i & 0 \end{bmatrix} \end{array}.$$

Enough to know how to go from step i to step $i+1$.

To construct $\mathscr{A}_i - \lambda \mathscr{B}_i$ with

$$\mathscr{A}_i = \begin{array}{c c} & \begin{array}{cc} n & n \end{array} \\ \begin{array}{c} n \\ n \end{array} & \begin{bmatrix} E_i & 0 \\ -X_i & I \end{bmatrix} \end{array}, \quad \mathscr{B}_i = \begin{array}{c c} & \begin{array}{cc} n & n \end{array} \\ \begin{array}{c} n \\ n \end{array} & \begin{bmatrix} -Y_i & I \\ F_i & 0 \end{bmatrix} \end{array}.$$

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To find $\mathscr{A}_{i\perp}, \mathscr{B}_{i\perp} \in \mathbb{C}^{(m+n)\times(m+n)}$ such that

$$\operatorname{rank}([\mathscr{A}_{i\perp}, \mathscr{B}_{i\perp}]) = m+n, \quad [\mathscr{A}_{i\perp}, \mathscr{B}_{i\perp}] \begin{bmatrix} \mathscr{B}_i \\ -\mathscr{A}_i \end{bmatrix} = 0.$$

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To construct $\mathscr{A}_i - \lambda \mathscr{B}_i$ with

$$\mathscr{A}_i = \begin{array}{c c} & \begin{array}{cc} n & n \end{array} \\ \begin{array}{c} n \\ n \end{array} & \begin{bmatrix} E_i & 0 \\ -X_i & I \end{bmatrix} \end{array}, \quad \mathscr{B}_i = \begin{array}{c c} & \begin{array}{cc} n & n \end{array} \\ \begin{array}{c} n \\ n \end{array} & \begin{bmatrix} -Y_i & I \\ F_i & 0 \end{bmatrix} \end{array}.$$

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Can again perform Gaussian-like block eliminations

$$\begin{bmatrix} -Y_i & I_n \\ F_i & 0 \\ E_i & 0 \\ -X_i & I_n \end{bmatrix} \xrightarrow{L_1} \begin{bmatrix} X_i - Y_i & 0 \\ F_i & 0 \\ E_i & 0 \\ -X_i & I_n \end{bmatrix} \xrightarrow{L_2} \begin{bmatrix} I_n & 0 \\ F_i & 0 \\ E_i & 0 \\ -X_i & I_n \end{bmatrix} \xrightarrow{L_3} \begin{bmatrix} I_n & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & I_n \end{bmatrix} \xrightarrow{L_4} \begin{bmatrix} I_n & 0 \\ 0 & I_n \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

DA for SF2 (cont'd)

Post-process the last $2n$ rows of $L_4L_3L_2L_1$ to give

$$\mathscr{A}_{i\perp} = \begin{bmatrix} E_i(X_i - Y_i)^{-1} & 0 \\ -F_i(X_i - Y_i)^{-1} & I \end{bmatrix}, \quad \mathscr{B}_{i\perp} = \begin{bmatrix} I & E_i(X_i - Y_i)^{-1} \\ 0 & -F_i(X_i - Y_i)^{-1} \end{bmatrix}.$$

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Finally

$$\mathscr{A}_{i+1} = \mathscr{A}_{i\perp}\mathscr{A}_i, \quad \mathscr{B}_{i+1} = \mathscr{B}_{i\perp}\mathscr{B}_i$$

to give

$$E_{i+1} = E_i(X_i - Y_i)^{-1}E_i, \tag{2a}$$
$$F_{i+1} = F_i(Y_i - X_i)^{-1}F_i, \tag{2b}$$
$$X_{i+1} = X_i + F_i(X_i - Y_i)^{-1}E_i, \tag{2c}$$
$$Y_{i+1} = Y_i + E_i(Y_i - X_i)^{-1}F_i. \tag{2d}$$

DA for SF2 (cont'd)

Post-process the last $2n$ rows of $L_4L_3L_2L_1$ to give

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$$X_{i+1} = X_i + F_i(X_i - Y_i)^{-1}E_i, \tag{2c}$$
$$Y_{i+1} = Y_i + E_i(Y_i - X_i)^{-1}F_i. \tag{2d}$$

At convergence, X_i goes to some solution X.

Outline

Primary and Dual

In a DA, X_i, if it does, converges to a solution of "Primary equation". What about Y_i? "Dual" (to the primary equation) is the key for answering that.

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Framework:

$$\text{(primary)} \quad \mathscr{A}_0 \begin{bmatrix} I \\ X \end{bmatrix} = \mathscr{B}_0 \begin{bmatrix} I \\ X \end{bmatrix} \mathscr{M},$$

Primary and Dual

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$$\begin{aligned} &\text{(primary)} \quad \mathscr{A}_0 \begin{bmatrix} I \\ X \end{bmatrix} = \mathscr{B}_0 \begin{bmatrix} I \\ X \end{bmatrix} \mathscr{M}, \\ &\text{(dual)} \qquad \hat{\mathscr{A}}_0 \begin{bmatrix} I \\ Y \end{bmatrix} = \hat{\mathscr{B}}_0 \begin{bmatrix} I \\ Y \end{bmatrix} \mathscr{N}. \end{aligned}$$

Primary and Dual

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$$\text{(primary)} \quad \mathscr{A}_0 \begin{bmatrix} I \\ X \end{bmatrix} = \mathscr{B}_0 \begin{bmatrix} I \\ X \end{bmatrix} \mathscr{M},$$

$$\text{(dual)} \quad \hat{\mathscr{A}}_0 \begin{bmatrix} I \\ Y \end{bmatrix} = \hat{\mathscr{B}}_0 \begin{bmatrix} I \\ Y \end{bmatrix} \mathscr{N}.$$

Let $\Pi_{m,n} = \begin{bmatrix} 0 & I_n \\ I_m & 0 \end{bmatrix}$.

Case: SF1

$$\text{(SF1)} \quad \mathscr{A}_0 = \begin{array}{c c} & \begin{array}{cc} m & n \end{array} \\ \begin{array}{c} m \\ n \end{array} & \begin{bmatrix} E_0 & 0 \\ -X_0 & I \end{bmatrix} \end{array}, \quad \mathscr{B}_0 = \begin{array}{c c} & \begin{array}{cc} m & n \end{array} \\ \begin{array}{c} m \\ n \end{array} & \begin{bmatrix} I & -Y_0 \\ 0 & F_0 \end{bmatrix} \end{array}.$$

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$$

Define

$$
\hat{\mathscr{A}}_0 := \Pi_{m,n}^T \mathscr{B}_0 \Pi_{m,n}, \;\; \hat{\mathscr{B}}_0 := \Pi_{m,n}^T \mathscr{A}_0 \Pi_{m,n}
$$

to give

$$
\hat{\mathscr{A}}_0 = \begin{array}{c c} & \begin{array}{cc} n & m \end{array} \\ \begin{array}{c} n \\ m \end{array} & \begin{bmatrix} F_0 & 0 \\ -Y_0 & I \end{bmatrix} \end{array}, \quad \hat{\mathscr{B}}_0 = \begin{array}{c c} & \begin{array}{cc} n & m \end{array} \\ \begin{array}{c} n \\ m \end{array} & \begin{bmatrix} I & -X_0 \\ 0 & E_0 \end{bmatrix} \end{array}.
$$

Case: SF1

$$
\text{(SF1)}\quad \mathscr{A}_0 = \begin{array}{c c} & \begin{array}{cc} m & n \end{array} \\ \begin{array}{c} m \\ n \end{array} & \left[\begin{array}{cc} E_0 & 0 \\ -X_0 & I \end{array}\right] \end{array}, \quad \mathscr{B}_0 = \begin{array}{c c} & \begin{array}{cc} m & n \end{array} \\ \begin{array}{c} m \\ n \end{array} & \left[\begin{array}{cc} I & -Y_0 \\ 0 & F_0 \end{array}\right] \end{array}.
$$

Define

$$
\hat{\mathscr{A}}_0 := \Pi_{m,n}^T \mathscr{B}_0 \Pi_{m,n}, \quad \hat{\mathscr{B}}_0 := \Pi_{m,n}^T \mathscr{A}_0 \Pi_{m,n}
$$

to give

$$
\hat{\mathscr{A}}_0 = \begin{array}{c c} & \begin{array}{cc} n & m \end{array} \\ \begin{array}{c} n \\ m \end{array} & \left[\begin{array}{cc} F_0 & 0 \\ -Y_0 & I \end{array}\right] \end{array}, \quad \hat{\mathscr{B}}_0 = \begin{array}{c c} & \begin{array}{cc} n & m \end{array} \\ \begin{array}{c} n \\ m \end{array} & \left[\begin{array}{cc} I & -X_0 \\ 0 & E_0 \end{array}\right] \end{array}.
$$

reciprocal relationship in eigenvalues

If (λ, z) is an eigenpair of $\mathscr{A}_0 - \lambda\mathscr{B}_0$, then $(1/\lambda, \Pi_{m,n}^T z)$ is an eigenpair of $\hat{\mathscr{A}}_0 - \lambda\hat{\mathscr{B}}_0$ and vice versa.

Case: SF1 (cont'd)

$$\text{(primary)} \quad \mathscr{A}_0 \begin{bmatrix} I \\ X \end{bmatrix} = \mathscr{B}_0 \begin{bmatrix} I \\ X \end{bmatrix} \mathscr{M}; \qquad \text{(dual)} \quad \hat{\mathscr{A}}_0 \begin{bmatrix} I \\ Y \end{bmatrix} = \hat{\mathscr{B}}_0 \begin{bmatrix} I \\ Y \end{bmatrix} \mathscr{N}.$$

Case: SF1 (cont'd)

$$\text{(primary)} \quad \mathscr{A}_0 \begin{bmatrix} I \\ X \end{bmatrix} = \mathscr{B}_0 \begin{bmatrix} I \\ X \end{bmatrix} \mathscr{M}; \qquad \text{(dual)} \quad \hat{\mathscr{A}}_0 \begin{bmatrix} I \\ Y \end{bmatrix} = \hat{\mathscr{B}}_0 \begin{bmatrix} I \\ Y \end{bmatrix} \mathscr{N}.$$

Plug in $\mathscr{A}_0$ and $\mathscr{B}_0$ to get

$$\begin{aligned} E_0 &= (I - Y_0 X)\mathscr{M}, \quad & X - X_0 &= F_0 X \mathscr{M}, \\ F_0 &= (I - X_0 Y)\mathscr{N}, \quad & Y - Y_0 &= E_0 Y \mathscr{N}. \end{aligned}$$

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Eliminate $\mathscr{M}$ and $\mathscr{N}$ to get

$$\begin{aligned} \text{primary:} \quad X &= X_0 + F_0 X (I - Y_0 X)^{-1} E_0, \\ \text{dual:} \quad Y &= Y_0 + E_0 Y (I - X_0 Y)^{-1} F_0. \end{aligned}$$

Case: SF2

$$\text{(SF2)} \quad \mathscr{A}_0 = \begin{array}{c c} & \begin{array}{cc} n & n \end{array} \\ \begin{array}{c} n \\ n \end{array} & \begin{bmatrix} E_0 & 0 \\ -X_0 & I \end{bmatrix} \end{array}, \quad \mathscr{B}_0 = \begin{array}{c c} & \begin{array}{cc} n & n \end{array} \\ \begin{array}{c} n \\ n \end{array} & \begin{bmatrix} -Y_0 & I \\ F_0 & 0 \end{bmatrix} \end{array}.$$

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$$\text{(SF2)} \quad \mathscr{A}_0 = \begin{array}{cc} & \begin{array}{cc} n & n \end{array} \\ \begin{array}{c} n \\ n \end{array} & \left[\begin{array}{cc} E_0 & 0 \\ -X_0 & I \end{array} \right] \end{array}, \quad \mathscr{B}_0 = \begin{array}{cc} & \begin{array}{cc} n & n \end{array} \\ \begin{array}{c} n \\ n \end{array} & \left[\begin{array}{cc} -Y_0 & I \\ F_0 & 0 \end{array} \right] \end{array}.$$

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$$\hat{\mathscr{A}}_0 := \Pi_{n,n}^T \mathscr{B}_0, \ \hat{\mathscr{B}}_0 := \Pi_{n,n}^T \mathscr{A}_0$$

to give

$$\hat{\mathscr{A}}_0 = \begin{array}{cc} & \begin{array}{cc} n & n \end{array} \\ \begin{array}{c} n \\ n \end{array} & \left[\begin{array}{cc} F_0 & 0 \\ -Y_0 & I \end{array} \right] \end{array}, \quad \hat{\mathscr{B}}_0 = \begin{array}{cc} & \begin{array}{cc} n & n \end{array} \\ \begin{array}{c} n \\ n \end{array} & \left[\begin{array}{cc} -X_0 & I \\ E_0 & 0 \end{array} \right] \end{array}.$$

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If (λ, z) is an eigenpair of $\mathscr{A}_0 - \lambda\mathscr{B}_0$, then $(1/\lambda, z)$ is an eigenpair of $\hat{\mathscr{A}}_0 - \lambda\hat{\mathscr{B}}_0$ and vice versa.

$$\text{(primary)} \quad \mathscr{A}_0 \begin{bmatrix} I \\ X \end{bmatrix} = \mathscr{B}_0 \begin{bmatrix} I \\ X \end{bmatrix} \mathscr{M}; \qquad \text{(dual)} \quad \hat{\mathscr{A}}_0 \begin{bmatrix} I \\ Y \end{bmatrix} = \hat{\mathscr{B}}_0 \begin{bmatrix} I \\ Y \end{bmatrix} \mathscr{N}.$$

Case: SF2 (cont'd)

$$\text{(primary)} \quad \mathscr{A}_0 \begin{bmatrix} I \\ X \end{bmatrix} = \mathscr{B}_0 \begin{bmatrix} I \\ X \end{bmatrix} \mathscr{M}; \qquad \text{(dual)} \quad \hat{\mathscr{A}}_0 \begin{bmatrix} I \\ Y \end{bmatrix} = \hat{\mathscr{B}}_0 \begin{bmatrix} I \\ Y \end{bmatrix} \mathscr{N}.$$

Plug in $\mathscr{A}_0$ and $\mathscr{B}_0$ to get

$$E_0 = (X - Y_0)\mathscr{M}, \quad X - X_0 = F_0\mathscr{M},$$
$$F_0 = (Y - X_0)\mathscr{N}, \quad Y - Y_0 = E_0\mathscr{N}.$$

$$\text{(primary)} \quad \mathscr{A}_0 \begin{bmatrix} I \\ X \end{bmatrix} = \mathscr{B}_0 \begin{bmatrix} I \\ X \end{bmatrix} \mathscr{M}; \qquad \text{(dual)} \quad \hat{\mathscr{A}}_0 \begin{bmatrix} I \\ Y \end{bmatrix} = \hat{\mathscr{B}}_0 \begin{bmatrix} I \\ Y \end{bmatrix} \mathscr{N}.$$

Plug in $\mathscr{A}_0$ and $\mathscr{B}_0$ to get

$$\begin{aligned} E_0 &= (X - Y_0)\mathscr{M}, \quad & X - X_0 &= F_0\mathscr{M}, \\ F_0 &= (Y - X_0)\mathscr{N}, \quad & Y - Y_0 &= E_0\mathscr{N}. \end{aligned}$$

Eliminate $\mathscr{M}$ and $\mathscr{N}$ to get

$$\begin{aligned} \text{primary:} \quad X &= X_0 + F_0(X - Y_0)^{-1}E_0, \\ \text{dual:} \quad Y &= Y_0 + E_0(Y - X_0)^{-1}F_0. \end{aligned}$$

Case: SF2 (cont'd)

$$\text{(primary)} \quad \mathscr{A}_0 \begin{bmatrix} I \\ X \end{bmatrix} = \mathscr{B}_0 \begin{bmatrix} I \\ X \end{bmatrix} \mathscr{M}; \qquad \text{(dual)} \quad \hat{\mathscr{A}}_0 \begin{bmatrix} I \\ Y \end{bmatrix} = \hat{\mathscr{B}}_0 \begin{bmatrix} I \\ Y \end{bmatrix} \mathscr{N}.$$

Plug in $\mathscr{A}_0$ and $\mathscr{B}_0$ to get

$$E_0 = (X - Y_0)\mathscr{M}, \quad X - X_0 = F_0\mathscr{M},$$
$$F_0 = (Y - X_0)\mathscr{N}, \quad Y - Y_0 = E_0\mathscr{N}.$$

Eliminate $\mathscr{M}$ and $\mathscr{N}$ to get

$$\text{primary:} \quad X = X_0 + F_0(X - Y_0)^{-1}E_0,$$
$$\text{dual:} \quad Y = Y_0 + E_0(Y - X_0)^{-1}F_0.$$

self-dual-ness

"primary" is "dual".

DA on Primary and Dual

What happen if DA is applied to $\mathscr{A}_0 - \lambda \mathscr{B}_0$, generating

$$X_i,\ Y_i,\ E_i,\ F_i$$

and to $\hat{\mathscr{A}}_0 - \lambda \hat{\mathscr{B}}_0$, generating

$$\hat{X}_i,\ \hat{Y}_i,\ \hat{E}_i,\ \hat{F}_i?$$

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and to $\hat{\mathscr{A}}_0 - \lambda \hat{\mathscr{B}}_0$, generating

$$\hat{X}_i,\ \hat{Y}_i,\ \hat{E}_i,\ \hat{F}_i?$$

DA on primary and dual

For all $i \geq 0$,

$$\hat{X}_i = Y_i, \quad \hat{Y}_i = X_i,$$
$$\hat{E}_i = F_i, \quad \hat{F}_i = E_i.$$

Outline

General

When a DA is applied to both $\mathscr{A}_0 - \lambda\mathscr{B}_0$ and $\hat{\mathscr{A}}_0 - \lambda\hat{\mathscr{B}}_0$, two sequences $\{\mathscr{A}_i - \lambda\mathscr{B}_i\}_{i=0}^{\infty}$ and $\{\hat{\mathscr{A}}_i - \lambda\hat{\mathscr{B}}_i\}_{i=0}^{\infty}$ are produced, assuming no breakdown occurs.

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The sequences are essentially “identical”. Moreover,

$$\mathscr{A}_i \begin{bmatrix} I \\ X \end{bmatrix} = \mathscr{B}_i \begin{bmatrix} I \\ X \end{bmatrix} \mathscr{M}^{2^i}, \quad \hat{\mathscr{A}}_i \begin{bmatrix} I \\ Y \end{bmatrix} = \hat{\mathscr{B}}_i \begin{bmatrix} I \\ Y \end{bmatrix} \mathscr{N}^{2^i} \quad \text{for } i = 0, 1, \ldots.$$

General

When a DA is applied to both $\mathscr{A}_0 - \lambda\mathscr{B}_0$ and $\hat{\mathscr{A}}_0 - \lambda\hat{\mathscr{B}}_0$, two sequences $\{\mathscr{A}_i - \lambda\mathscr{B}_i\}_{i=0}^{\infty}$ and $\{\hat{\mathscr{A}}_i - \lambda\hat{\mathscr{B}}_i\}_{i=0}^{\infty}$ are produced, assuming no breakdown occurs.

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Certain conditions on $\rho(\mathscr{M})$ and $\rho(\mathscr{N})$ should be imposed.

General

When a DA is applied to both $\mathscr{A}_0 - \lambda\mathscr{B}_0$ and $\hat{\mathscr{A}}_0 - \lambda\hat{\mathscr{B}}_0$, two sequences $\{\mathscr{A}_i - \lambda\mathscr{B}_i\}_{i=0}^{\infty}$ and $\{\hat{\mathscr{A}}_i - \lambda\hat{\mathscr{B}}_i\}_{i=0}^{\infty}$ are produced, assuming no breakdown occurs.

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Certain conditions on $\rho(\mathscr{M})$ and $\rho(\mathscr{N})$ should be imposed.

DA computes special solutions $X = \Phi$ and $Y = \Psi$ to Primary and Dual.

Case: SF1

$$\mathscr{A}_i = \begin{array}{c c} & \begin{array}{cc} m & n \end{array} \\ \begin{array}{c} m \\ n \end{array} & \begin{bmatrix} E_i & 0 \\ -X_i & I \end{bmatrix} \end{array}, \quad \mathscr{B}_i = \begin{array}{c c} & \begin{array}{cc} m & n \end{array} \\ \begin{array}{c} m \\ n \end{array} & \begin{bmatrix} I & -Y_i \\ 0 & F_i \end{bmatrix} \end{array},$$

$$\hat{\mathscr{A}}_i = \begin{array}{c c} & \begin{array}{cc} n & m \end{array} \\ \begin{array}{c} n \\ m \end{array} & \begin{bmatrix} F_i & 0 \\ -Y_i & I \end{bmatrix} \end{array}, \quad \hat{\mathscr{B}}_i = \begin{array}{c c} & \begin{array}{cc} n & m \end{array} \\ \begin{array}{c} n \\ m \end{array} & \begin{bmatrix} I & -X_i \\ 0 & E_i \end{bmatrix} \end{array},$$

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$$E_i = (I - Y_i \Phi)\mathscr{M}^{2^i}, \quad \Phi - X_i = F_i \Phi \mathscr{M}^{2^i},$$

$$F_i = (I - X_i \Psi)\mathscr{N}^{2^i}, \quad \Psi - Y_i = E_i \Psi \mathscr{N}^{2^i}.$$

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$$\mathscr{A}_i = \begin{array}{c c} & \begin{array}{cc} m & n \end{array} \\ \begin{array}{c} m \\ n \end{array} & \left[\begin{array}{cc} E_i & 0 \\ -X_i & I \end{array}\right] \end{array}, \quad \mathscr{B}_i = \begin{array}{c c} & \begin{array}{cc} m & n \end{array} \\ \begin{array}{c} m \\ n \end{array} & \left[\begin{array}{cc} I & -Y_i \\ 0 & F_i \end{array}\right] \end{array},$$

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$$F_i = (I - X_i\Psi)\mathscr{N}^{2^i}, \quad \Psi - Y_i = E_i\Psi\mathscr{N}^{2^i}.$$

Convergence Theorem

Suppose that there are solutions $X = \Phi$ and $Y = \Psi$ to Primary and Dual such that $\rho(\mathscr{M}) \cdot \rho(\mathscr{N}) < 1$, and suppose that DA executes without any breakdown. Then X_i and Y_i converge to Φ and Ψ quadratically, and moreover,

$$\limsup_{i\to\infty} \|X_i - \Phi\|^{1/2^i} \le \rho(\mathscr{M}) \cdot \rho(\mathscr{N}), \quad \limsup_{i\to\infty} \|Y_i - \Psi\|^{1/2^i} \le \rho(\mathscr{M}) \cdot \rho(\mathscr{N}).$$

Case: SF2

$$\mathscr{A}_i = \begin{array}{c c} & \begin{array}{cc} n & n \end{array} \\ \begin{array}{c} n \\ n \end{array} & \begin{bmatrix} E_i & 0 \\ -X_i & I \end{bmatrix} \end{array}, \quad \mathscr{B}_i = \begin{array}{c c} & \begin{array}{cc} n & n \end{array} \\ \begin{array}{c} n \\ n \end{array} & \begin{bmatrix} -Y_i & I \\ F_i & 0 \end{bmatrix} \end{array},$$

$$\hat{\mathscr{A}}_i = \begin{array}{c c} & \begin{array}{cc} n & n \end{array} \\ \begin{array}{c} n \\ n \end{array} & \begin{bmatrix} F_i & 0 \\ -Y_i & I \end{bmatrix} \end{array}, \quad \hat{\mathscr{B}}_i = \begin{array}{c c} & \begin{array}{cc} n & n \end{array} \\ \begin{array}{c} n \\ n \end{array} & \begin{bmatrix} -X_i & I \\ E_i & 0 \end{bmatrix} \end{array},$$

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Outline

critical case

What if $\rho(\mathscr{M}) \cdot \rho(\mathscr{N}) = 1$, the critical case?

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Complicated but still possible to have linear convergence at the respectable rate $1/2$.

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More to come ...

Outline

CARE: $-XGX + A^{\mathrm{H}}X + XA + H = 0$

$$\mathscr{A} - \lambda\mathscr{B} := \mathscr{H} - \lambda I_{2n} \equiv \begin{bmatrix} A & -G \\ -H & -A^{\mathrm{H}} \end{bmatrix} - \lambda I_{2n},$$

$$\begin{bmatrix} \mathscr{A}' \\ \mathscr{B}' \end{bmatrix} = \begin{bmatrix} I_{2n} & -\gamma I_{2n} \\ I_{2n} & \gamma I_{2n} \end{bmatrix} \begin{bmatrix} \mathscr{H} \\ I_{2n} \end{bmatrix} \qquad (\gamma > 0),$$

$P(\mathscr{A}' - \lambda\mathscr{B}') = \mathscr{A}_0 - \lambda\mathscr{B}_0$ in SF1 and then apply DA.

(a) All $I - X_iY_i$ and $I - Y_iX_i$ are nonsingular;

(b) $0 \preceq X_0 \preceq X_i \preceq X_{i+1} \preceq \Phi$, $0 \succeq Y_0 \succeq Y_i \succeq Y_{i+1} \succeq \Psi$ and

$$\limsup_{i\to\infty} \|\Phi - X_i\|^{1/2^i} \le \rho(\mathscr{M})^2, \quad \limsup_{i\to\infty} \|\Psi - Y_i\|^{1/2^i} \le \rho(\mathscr{N})^2,$$

where

$\rho(\mathscr{M}) = \rho((I - Y_0\Phi)^{-1}E_0) = \rho(\mathscr{N}) = \rho((I - X_0\Psi)^{-1}E_0^{\mathrm{H}})$;

(c) $\mathrm{eig}(A - G\Phi) \subset \mathbb{C}_-$, $\mathrm{eig}(-H\Psi - A^{\mathrm{H}}) \subset \mathbb{C}_+$, and

$$\mathrm{eig}(\mathscr{H}) = \mathrm{eig}(A - G\Phi) \cup \mathrm{eig}(-H\Psi - A^{\mathrm{H}}).$$

Assume

$$\boxed{W = \begin{bmatrix} B & -D \\ -C & A \end{bmatrix} \text{ is a nonsingular or an irreducible singular } M\text{-matrix.}} \tag{3}$$

MARE with (3) *has a unique minimal nonnegative solution* Φ, *i.e.,*

$$0 \le \Phi \le X \quad \textit{for any other nonnegative solution } X\,.$$

Known in applied probability, stochastic fluid models, but rigorous matrix proof by C. Guo (2000):

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Known in applied probability, stochastic fluid models, but rigorous matrix proof by C. Guo (2000):

$$\mathscr{A} - \lambda\mathscr{B} := \mathscr{H} - \lambda I_{m+n} \equiv \begin{bmatrix} B & -D \\ C & -A^{\mathrm{H}} \end{bmatrix} - \lambda I_{m+n},$$

$$\begin{bmatrix} \mathscr{A}' \\ \mathscr{B}' \end{bmatrix} = \begin{bmatrix} I_{m+n} & -\beta I_{m+n} \\ I_{m+n} & \alpha I_{m+n} \end{bmatrix} \begin{bmatrix} \mathscr{H} \\ I_{m+n} \end{bmatrix} \qquad (\alpha \geq \max_i A_{ii},\ \beta \geq \max_i B_{ii}),$$

$P(\mathscr{A}' - \lambda\mathscr{B}') = \mathscr{A}_0 - \lambda\mathscr{B}_0$ in SF1 and then apply DA.

(a) All $I - X_i Y_i$ and $I - Y_i X_i$ are nonsingular M-matrices for all $i \geq 0$;

(b) $0 \leq X_0 \leq X_i \leq X_{i+1} \leq \Phi$, $0 \leq Y_0 \leq Y_i \leq Y_{i+1} \leq \Psi$ and

$$\limsup_{i\to\infty} \|\Phi - X_i\|^{1/2^i} \leq \rho(\mathscr{M})^2, \quad \limsup_{i\to\infty} \|\Psi - Y_i\|^{1/2^i} \leq \rho(\mathscr{N})^2,$$

where
$\rho(\mathscr{M}) = \rho((I - Y_0\Phi)^{-1}E_0) = \rho(\mathscr{N}) = \rho((I - X_0\Psi)^{-1}E_0^{\mathrm{H}})$;

(c) $\mathrm{eig}(B - D\Phi) \subset \mathbb{C}_-$, $\mathrm{eig}(-A - \Phi D) \subset \mathbb{C}_-$, and

$$\mathrm{eig}(\mathscr{H}) = \mathrm{eig}(B - D\Phi) \cup \mathrm{eig}(-A - \Phi D).$$

Moreover ...

Fast Train Eigenvalue Problem

$$P(\lambda)z \equiv (\lambda^2 A^\mathsf{T} + \lambda Q + A)z = 0.$$

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Framework of solvent approach:

1. Compute the stabilizing solution Φ of the matrix equation $X + A^\mathsf{T} X^{-1} A = Q$. Then

$$P(\lambda) = \lambda^2 A^\mathsf{T} + \lambda Q + A = (\lambda A^\mathsf{T} + X)X^{-1}(\lambda X + A).$$

2. Solve the (linear) eigenvalue problems for matrix pencils $\lambda A^\mathsf{T} + X$ and $\lambda X + A$.

Fast Train Eigenvalue Problem

$$
Q = \begin{array}{cc}
 & \begin{array}{ccccc} k & k & k & \cdots & k \end{array} \\
\begin{array}{c} k \\ k \\ k \\ \vdots \\ k \end{array} &
\begin{bmatrix}
H_0 & H_1^T & & & \\
H_1 & H_0 & H_1^T & & \\
 & H_1 & \ddots & \ddots & \\
 & & \ddots & \ddots & H_1^T \\
 & & & H_1 & H_0
\end{bmatrix}
\end{array},
\quad
A = \begin{array}{cc}
 & \begin{array}{cccc} k & \cdots & k & k \end{array} \\
\begin{array}{c} k \\ k \\ \vdots \\ k \end{array} &
\begin{bmatrix}
0 & \dots & 0 & H_1 \\
0 & \dots & 0 & 0 \\
\vdots & & \vdots & \vdots \\
0 & \dots & 0 & 0
\end{bmatrix}
\end{array}.
$$

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Numerous notorious numerical difficulties:

- most eigenvalues are 0 and ∞, $2(m-1)k$ in all;
- problem size n can range from 10^3 to 10^5
- most seriously, badly scaled finite eigenvalues: $10^{-50} \sim 10^{50}$ or to an even greater extreme
- all finite nonzero eigenvalues and eigenvectors are to be computed

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Cleverly implemented solvent approach based on DA gets it done!

Outline

MARE: $XDX - AX - XB + C = 0$

$$W = \begin{array}{cc} & \begin{array}{cc} m & n \end{array} \\ \begin{array}{c} m \\ n \end{array} & \begin{bmatrix} B & -D \\ -C & A \end{bmatrix} \end{array},$$

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$$B = \begin{bmatrix} 3 & -1 & & \\ & 3 & \ddots & \\ & & \ddots & -1 \\ -1 & & & 3 \end{bmatrix} \in \mathbb{R}^{n \times n}, \; C = 2I_n, \; A = \xi B, \; D = \xi C,$$

where $\xi > 0$ is a parameter.

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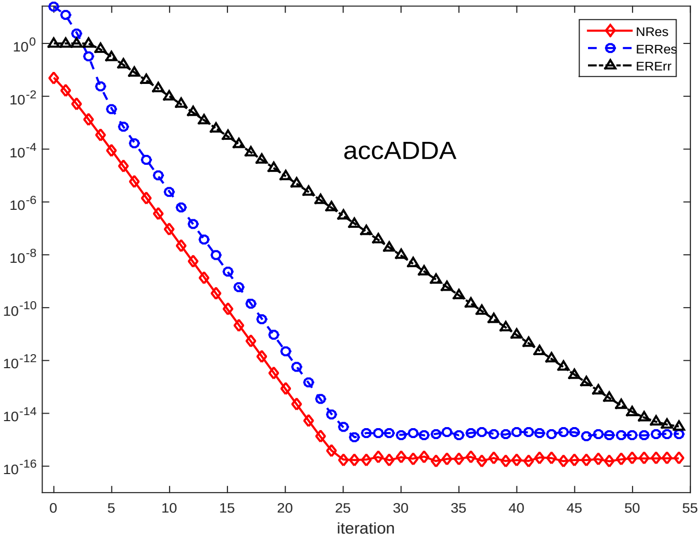
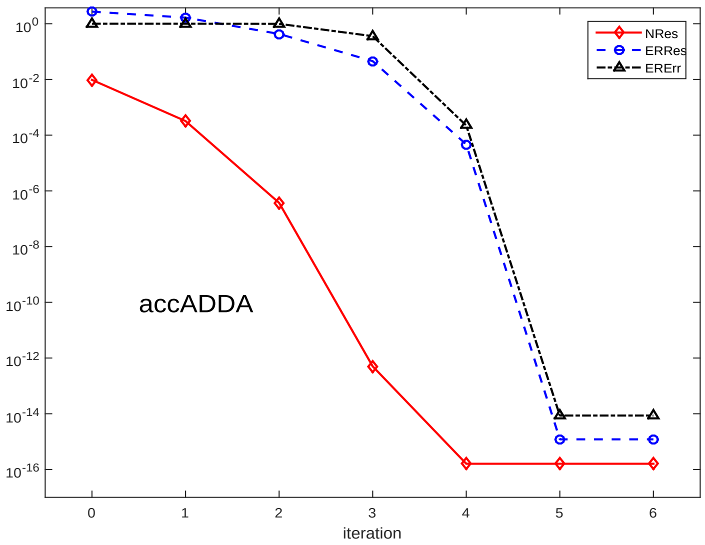
where $\xi > 0$ is a parameter.

Quadratic convergence for $\xi \neq 1$ and linear convergence otherwise.

MARE $\xi = 1$ (cont'd)



MARE $\xi = 10^4$ (cont'd)



Fast Train Eigenvalue Problem

$$P(\lambda)z \equiv (\lambda^2 A^{\mathsf{T}} + \lambda Q + A)z = 0, \quad \text{with}$$

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Can prove that it is equivalent to
$\widehat{P}(\hat{\lambda})y := (\hat{\lambda}^2 H_1^{\mathsf{T}} + \hat{\lambda} H_0 + H_1)y = 0$: $\lambda = \hat{\lambda}^m$.

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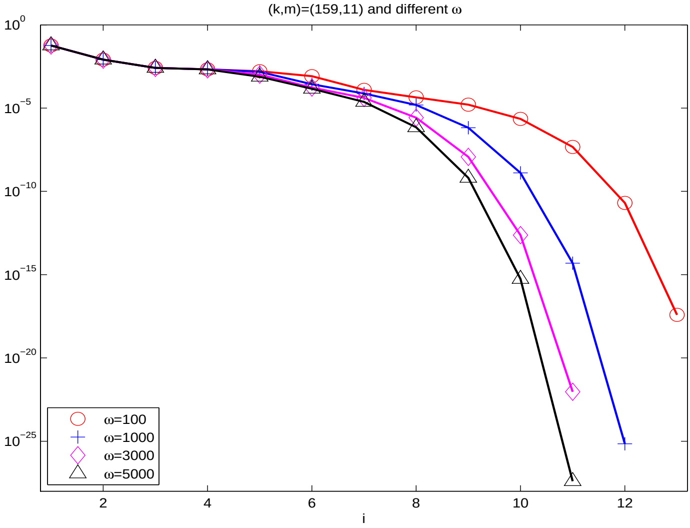
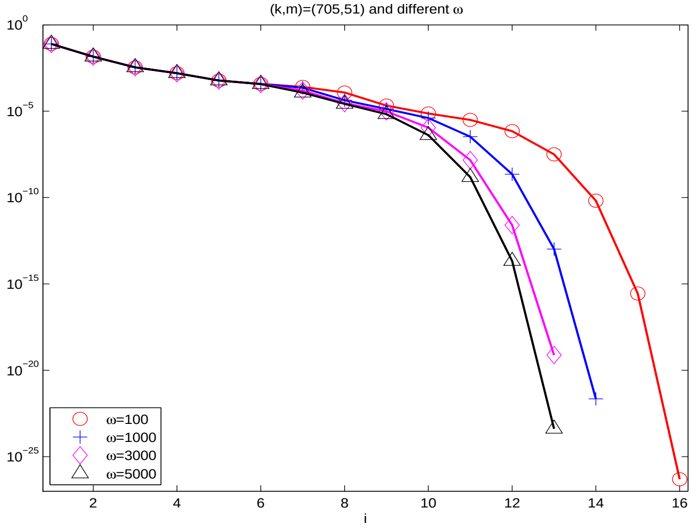
$\widehat{P}(\lambda) = (\lambda H_1^\mathsf{T} + \widehat{\Phi})\widehat{\Phi}^{-1}(\lambda\widehat{\Phi} + H_1)$ for a solution $\widehat{\Phi}$ of

$$\widehat{X} + H_1^\mathsf{T} \widehat{X}^{-1} H_1 = H_0$$

Fast Train Eigenvalue Problem (cont'd)



Fast Train Eigenvalue Problem (cont'd)



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