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Discrete two-sex age-structured models of population dynamics: stability, multistability, and chaos

<u>Revutskaya Oksana L.[1]</u>, Neverova G.P.[1;2], Kulakov M.P.[1], Frisman E.Ya.[1]

[1] Institute for Complex Analysis of Regional Problems, FEB RAS, Birobidzhan, Russia
[2] Institute for Automation and Control Processes, FEB RAS, Vladivostok, Russia

- In the case of polygamous species both the formation of the sex structure and the entire pattern of population dynamics are strongly related to the parameters determining the type of "mating relations" and the role of males in reproduction
- We suggest a simple mathematical model that enables the simultaneous observation of the formation of both age and sex structures and the explicit consideration of the asymmetry of the effects of females and males on population processes
- Analysis of the problem allows for the description of qualitative changes in population dynamics, which depends on the difference between the characteristics of sexes that determine survival and reproduction
- The purpose of this research is to investigate evolutionary scenarios of the oscillatory dynamics origination in populations with a simple age and sex structures and density-dependent regulation of the offspring survival

Model assumptions

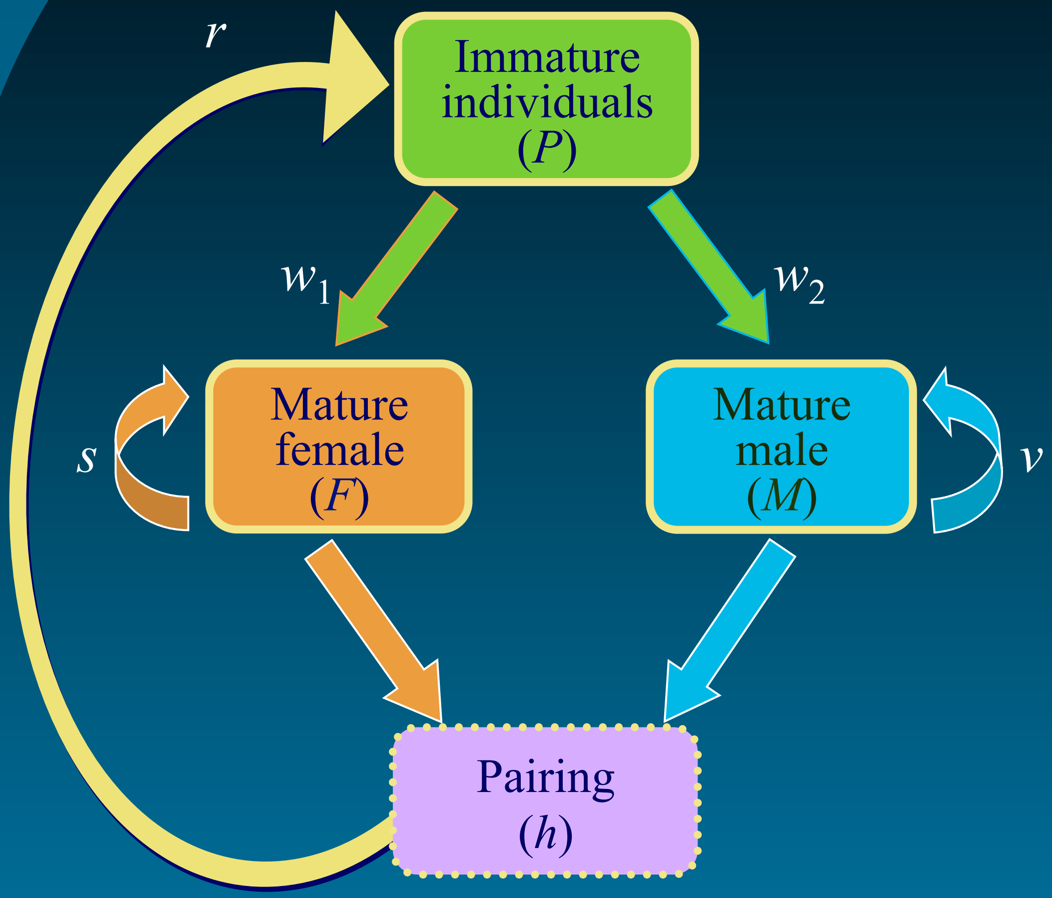


Fig. 1. Diagram of modelling population structure

- By the end of each reproduction season the population consists: juveniles (immature individuals) and two adult groups (mature females and males);
- The time between two reproduction seasons is sufficient for juveniles to become adults;
- A change in the number is determined by reproduction and death rate processes;
- Increase in the population size is regulated by density-dependent limitation of the juvenile survival rate.

The discrete-time model of population dynamics with age and sex structures

$$\begin{cases} P_{n+1} = r \cdot F_n \\ F_{n+1} = \delta w_1 P_n + sF_n \\ M_{n+1} = (1-\delta) w_2 P_n + vM_n \end{cases} \qquad \textbf{(1)}$$

n is a number of a reproduction season

δ is immature females quantity,

w_1 and w_2 are the survival rates of immature females and males,

s and v are the survival rates of mature females and males, respectively

The birth rate r is assumed to depend on the ratio between the numbers of males and females in the population

$$r_n = \frac{2aM_n}{(1/h)F_n + M_n}$$

a is the reproductive potential of the population (the maximum possible mean number of offspring per fertilized female)

h is a coefficient characterizing the type of mating relations in the population (h=1 corresponds to monogamy, h>1 to polygyny, h<1 to polyandry)

The discrete-time model of population dynamics with age and sex structures

The survival rates of immature females and males are assumed to be the population parameters most sensitive to the population density and to linearly depend on the population size:

$$w = w_1 = w_2 = 1 - \beta_1 P - \beta_2 (F + M),$$

where β_1 and β_2 are coefficients describing the intensity of intrapopulation competition. We further interpret negative values of survival rates as zeros.

Let us assume that equal numbers of females and males are born *(δ=0.5)*. Substitution of variables allows us to exclude the parameter β_2 and to write model (1) in terms of new variables, namely "relative" numbers

$$\begin{cases} p_{n+1} = 2af_n \dfrac{m_n}{(1/h)f_n + m_n} \\ f_{n+1} = 0.5(1 - p_n - \rho(f_n + m_n))p_n + sf_n \\ m_{n+1} = 0.5(1 - p_n - \rho(f_n + m_n))p_n + vm_n \end{cases}$$

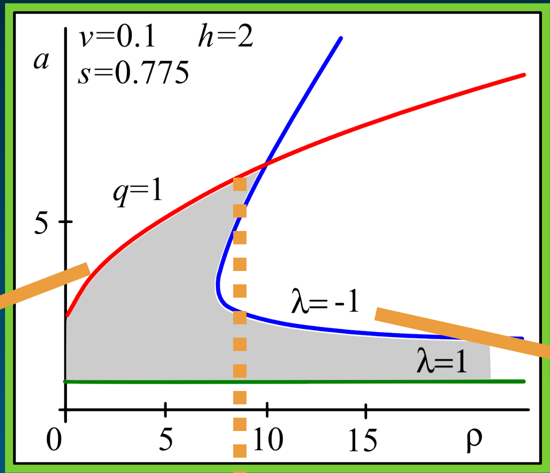
where $\rho = \beta_1 / \beta_2$ **(2)**

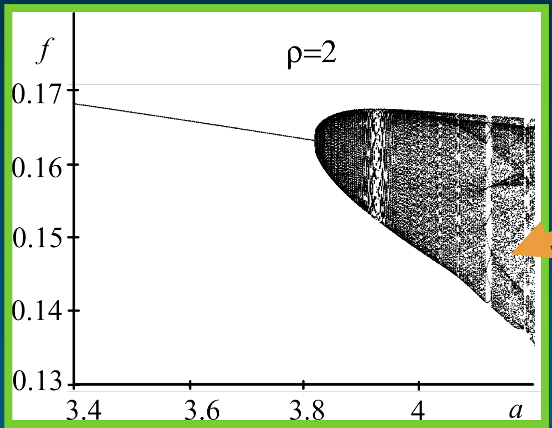
We have found the equilibrium points. The non-trivial solution stability is defined by eigenvalues, which are the solutions of the characteristic equation for the system (2).

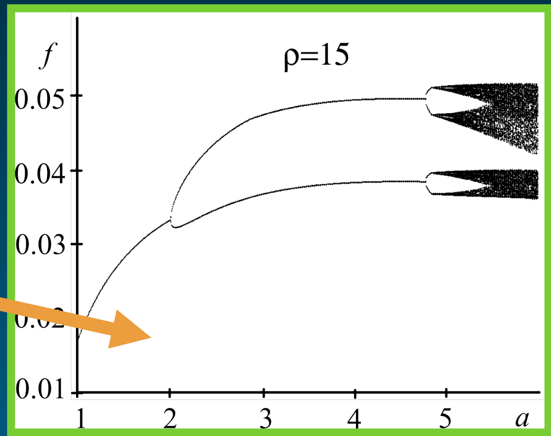
Classification of scenarios for stability loss by the non-zero solution of the system (2)

I. *The equilibrium loses its stability under the Neimark–Sacker bifurcation, and the dynamics of the population size demonstrates quasi-periodic fluctuations*

II. *The non-trivial solution loses its stability under the Feigenbaum scenario. The equilibrium stability loss is accompanied by a cascade of period-doubling bifurcations*







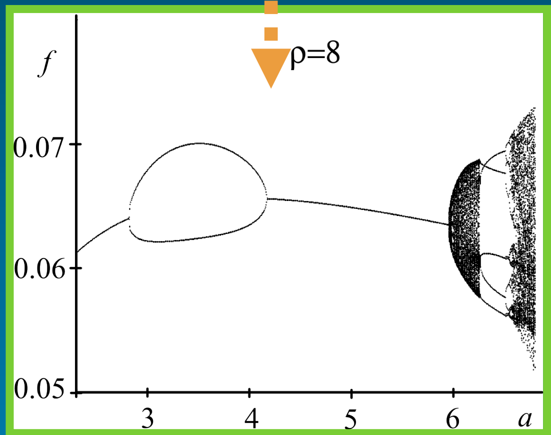


Fig. 2. Stability domain of the non-trivial solution and bifurcation diagrams of dynamic variable *f* of the system (2) dependent on the parameter *a*

III. *Within the range of the parameter ρ, two possible scenarios of stability loss may occur, namely, the emergence of a cycle length 2 or the emergence and destruction of the invariant curve*

Stability domain of the non-trivial stationary solution with a change of the model parameters

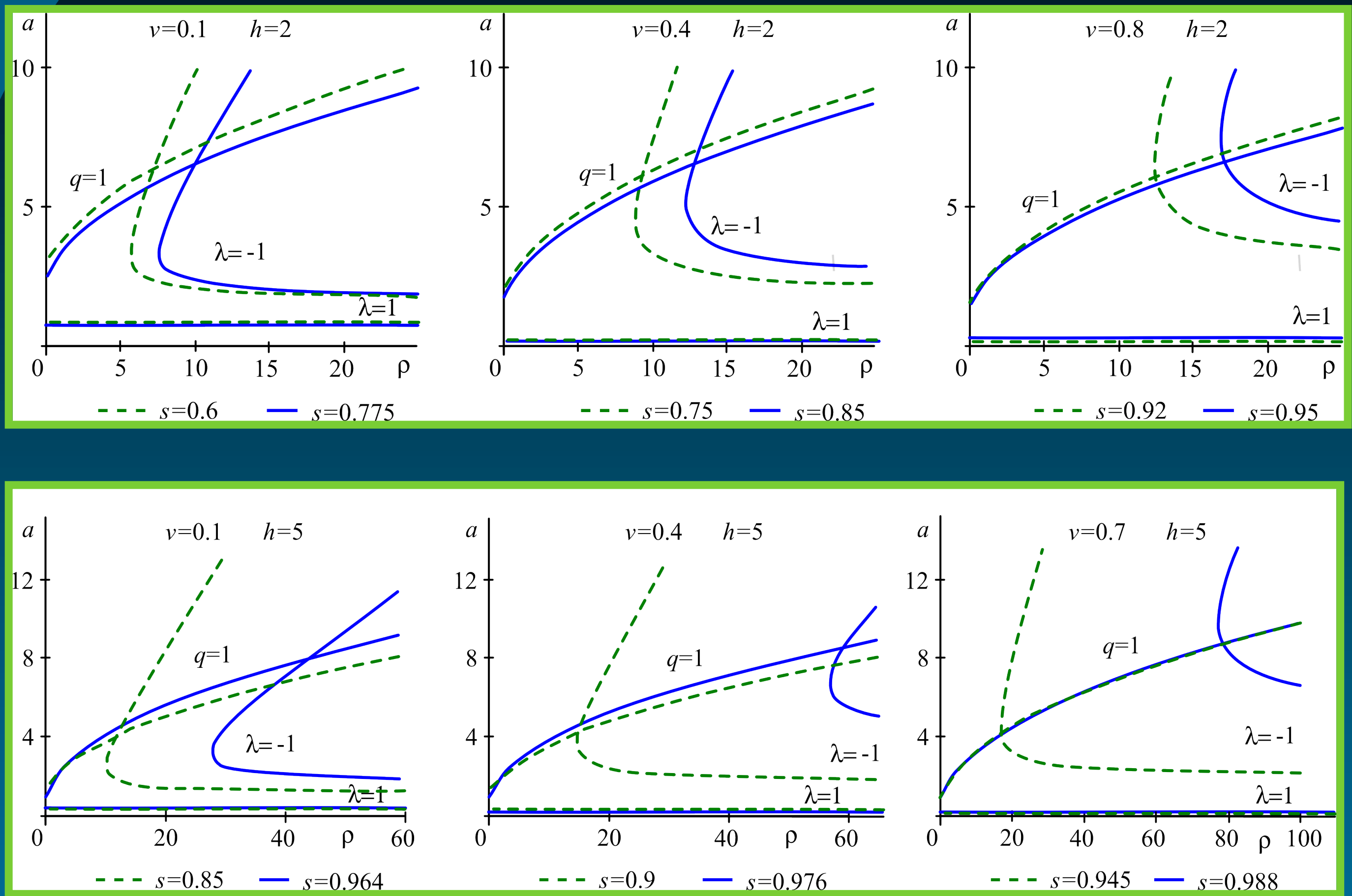
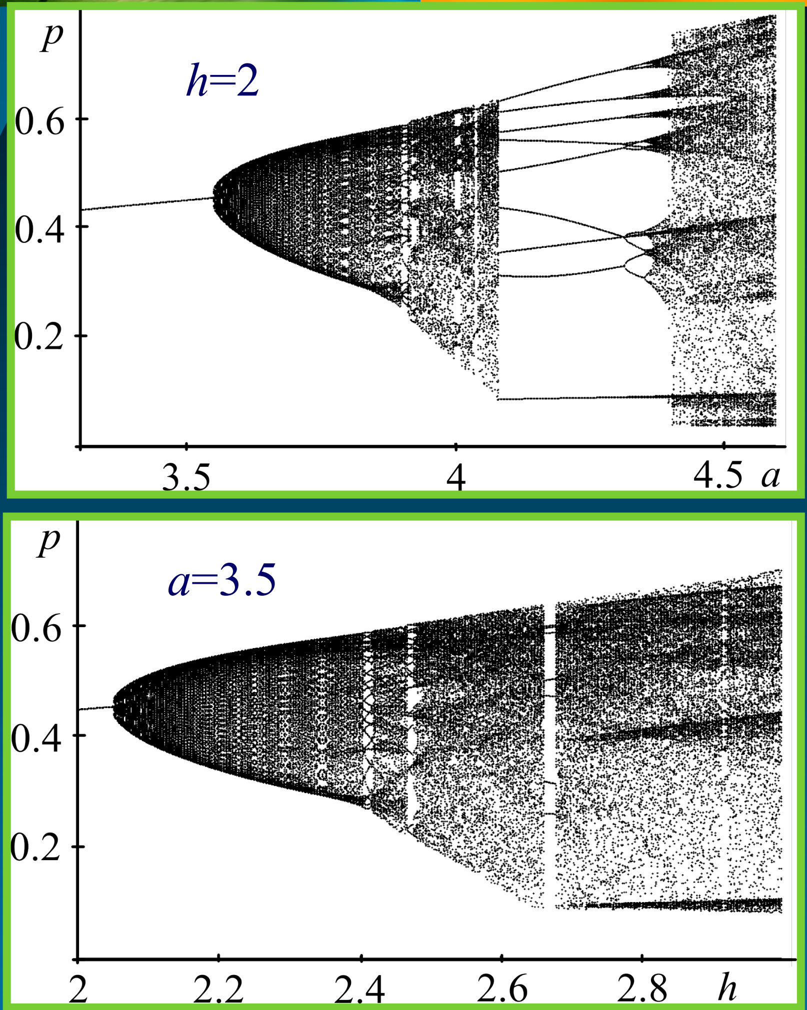


Fig. 3. Stability domain of the non-trivial solution with a change of the model parameters

Increase in the coefficients s and v (survival rates of female and males) leads to expansion of the non-trivial equilibrium stability domain.

Increase in the parameter h leads to decrease of stability domain.

The system dynamics depending on the reproductive potential a and parameter h



- *It is shown that an increase in the reproductive potential a leads to the loss of stability of the equilibrium solution of the system and appearance of oscillations.*
- *A increase in h, which characterizes average size of harem (in fact, decrease in the "role" of males or an increase in their sexual activity) also leads to the loss of stability of the equilibrium and transition to cyclic or chaotic modes.*

Fig. 4. The bifurcation diagrams of the dynamic variables p at change of the coefficients a and h at s=0.775, v=0.1, ρ=1.5

The system dynamics depending on the survival rates of females (s) and males (v)

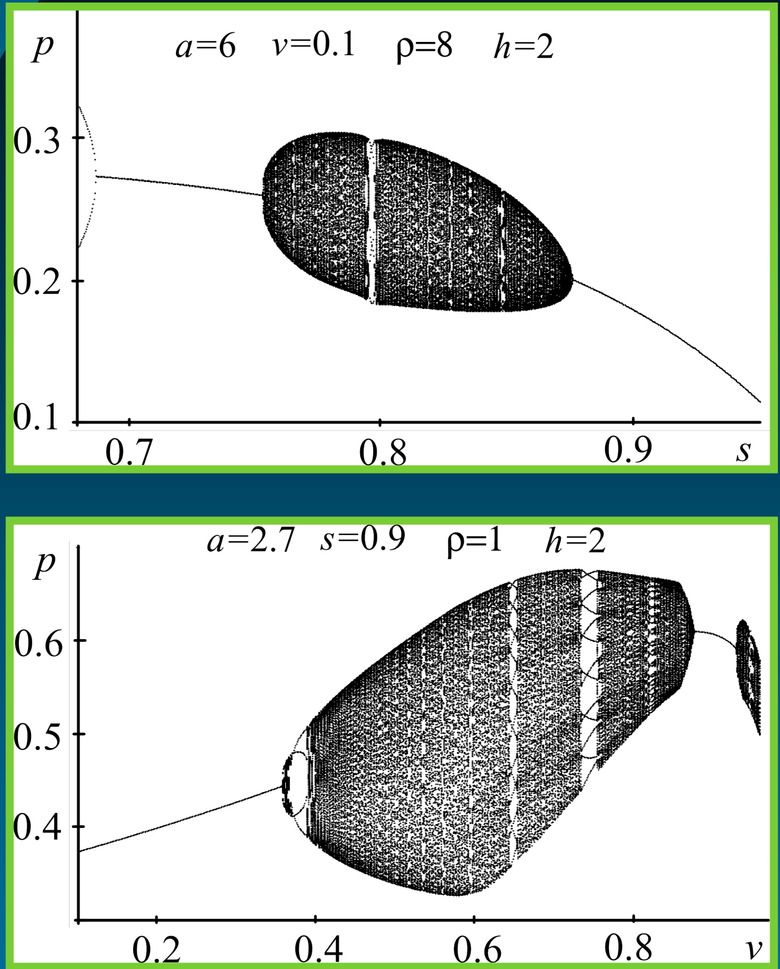
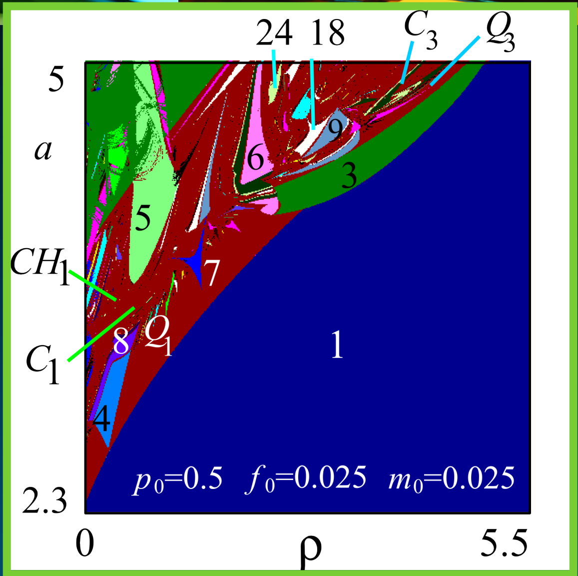


Fig. 5. The bifurcation diagrams of the dynamic variable *p* at change of the coefficients *s* and *v*

- *The exotic scenarios of population dynamics are observed at variation the parameters s and v. They are characterized by the appearance “bubbles of instability” limited stationary modes on the bifurcation diagrams*
- *The increase of survival rate leads to complex scenarios of modes change population dynamics, which consists in the repeated transitions from stationary or cyclic modes to chaos and back to fluctuation and stationary*

Dynamic regimes of the model (2)

Fig. 6. Dynamic mode map of the system (2)



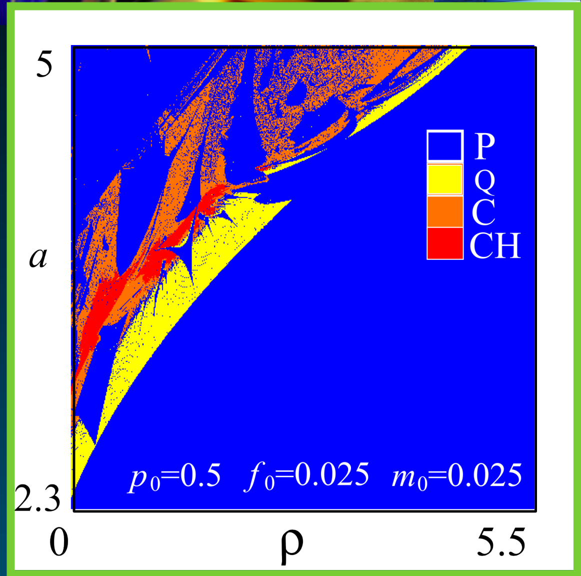
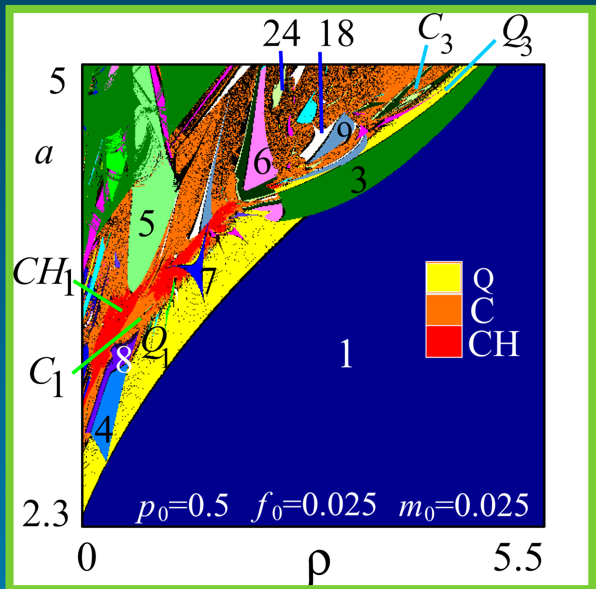


Fig. 7. Map of Lyapunov exponents for the system (2)

Fig. 8. Combined map of dynamic mode



h=2, v=0.1, s=0.775

The figures correspond to the period of observed cycles

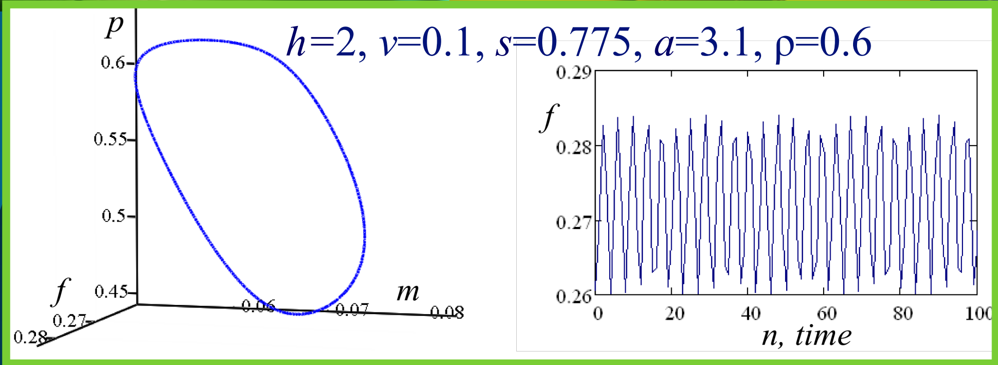
Types of dynamics modes:

P ($\lambda_3<\lambda_2<\lambda_1<0$) – periodic,

Q ($\lambda_3<\lambda_2<\lambda_1=0$) – quasi-periodic,

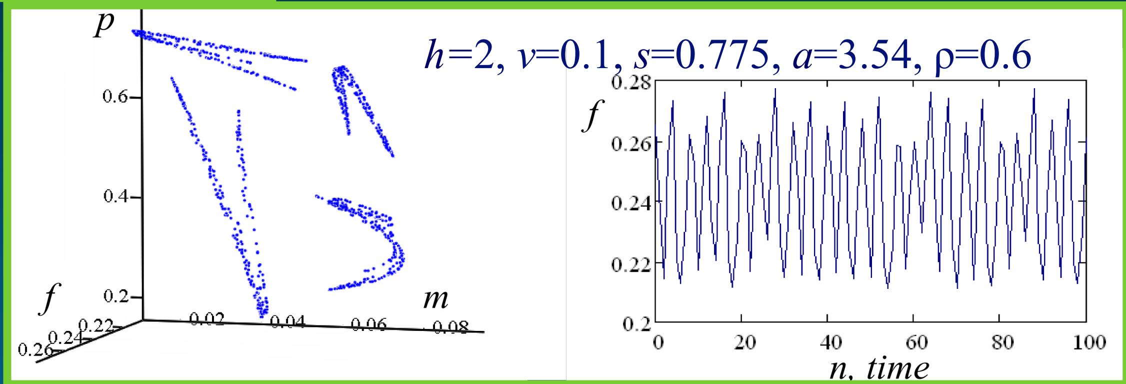
C ($\lambda_1>0>\lambda_2>\lambda_3$) –chaotic, CH ($\lambda_1>\lambda_2>0>\lambda_3$) – hyper-chaotic

Types of dynamic regimes for the model (2)



Type of dynamic modes is quasi-periodic ($\lambda_3<\lambda_2<\lambda_1=0$)

Type of dynamic modes is chaotic ($\lambda_1>0>\lambda_2>\lambda_3$)



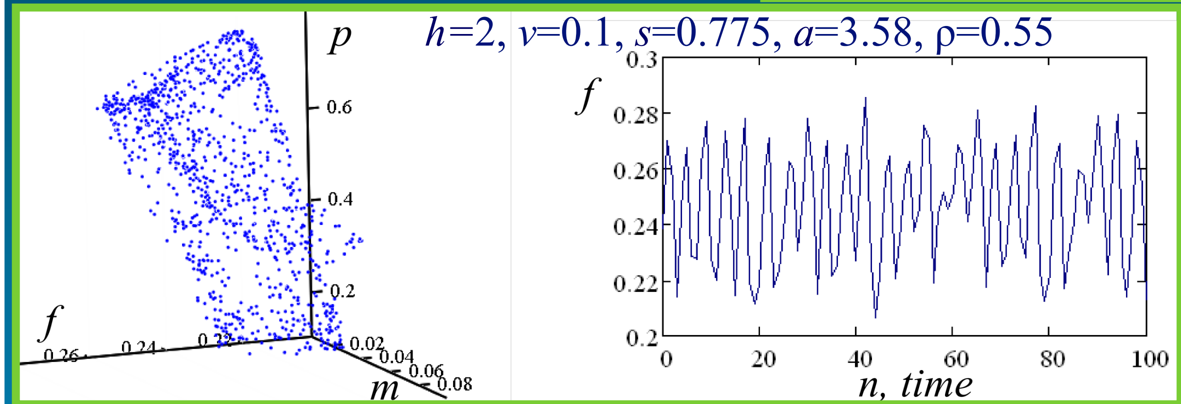


Fig. 9. Attractors of system (2) and changes in population size in time *n* (trajectories)

Type of dynamic modes is hyper-chaotic ($\lambda_1>\lambda_2>0>\lambda_3$)

Dynamic regimes of the model (2)

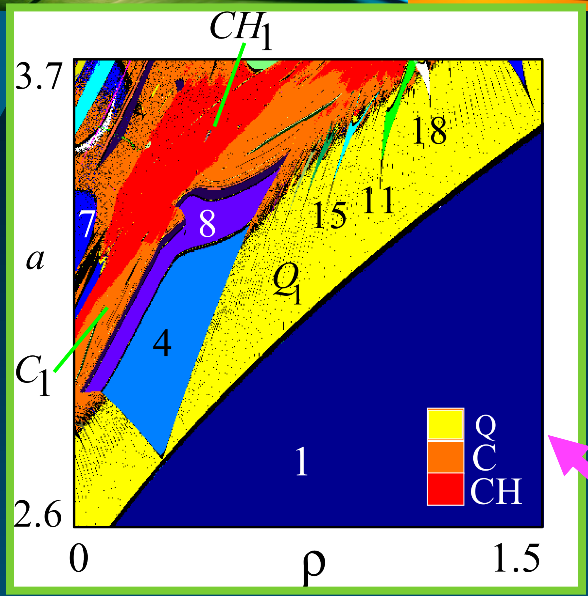
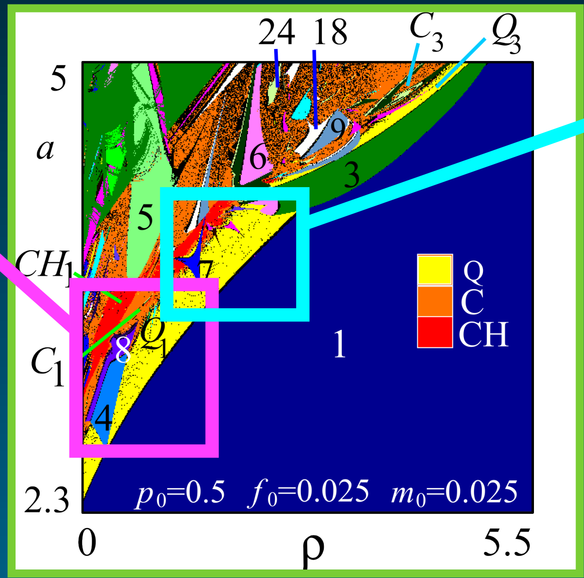
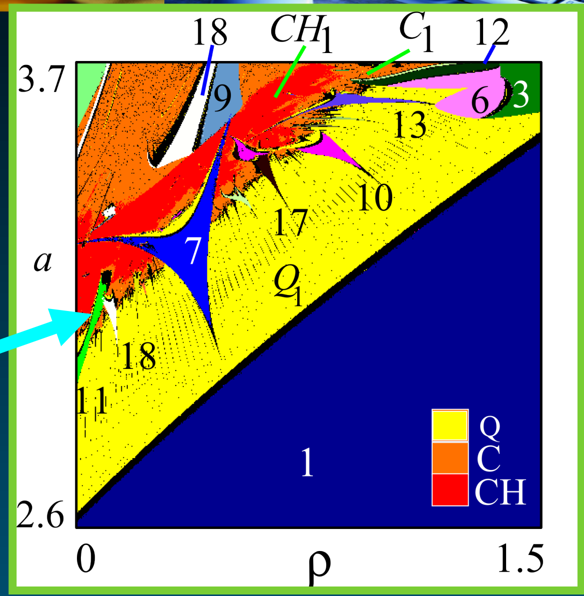


Fig. 10. Combined map of dynamic mode and its enlarged fragment at h=2, v=0.1, s=0.775





The maps shows the so-called Arnold tongues (resonance cycles). They are immersed in the area of quasi-periodic regimes.

With increase a, at first, there occurs chaos in the area of the tongues overlap, and later - hyper-chaos.

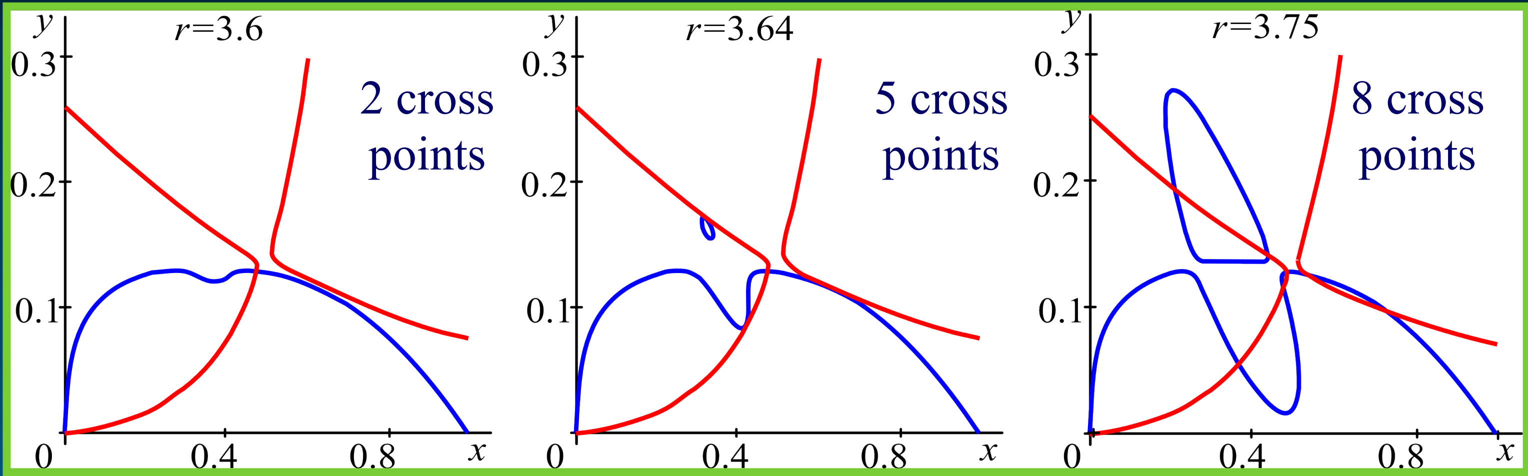
There exist the areas where resonance cycles of different lengths overlap. They correspond to a simultaneous realization of several modes (dependent on the initial conditions), or, in other words, to multistability.

The emergence of the cycle with length 3

At $r=a/(1+1/h)$,
$x=p$, $y=f+m$ the system (2) transforms to

$$\begin{cases} x_{n+1} = r \cdot y_n \\ y_{n+1} = (1 - x_n - \rho y_n)x_n + sy_n \end{cases} \qquad \textbf{(3)}$$

Fig. 11. Grapho-numerical solution of model equations (3) iterated three time at s=0.1 ρ=2.15



The system has a unique non-trivial solution, since the curves cross at two points

It is the process of semistable critical point emergence

and its desintegration on stable and unstable points

The cycle with period 3 emerges in the stability domain of non-trivial stationary point as a result of tangent bifurcation. Consequently, the population shift to equilibrium or to oscillations with period 3 depends on the initial condition.

Attraction basins of the model (3)

The attraction basins of stable equilibrium

The attraction basins of stable cycle of length 3

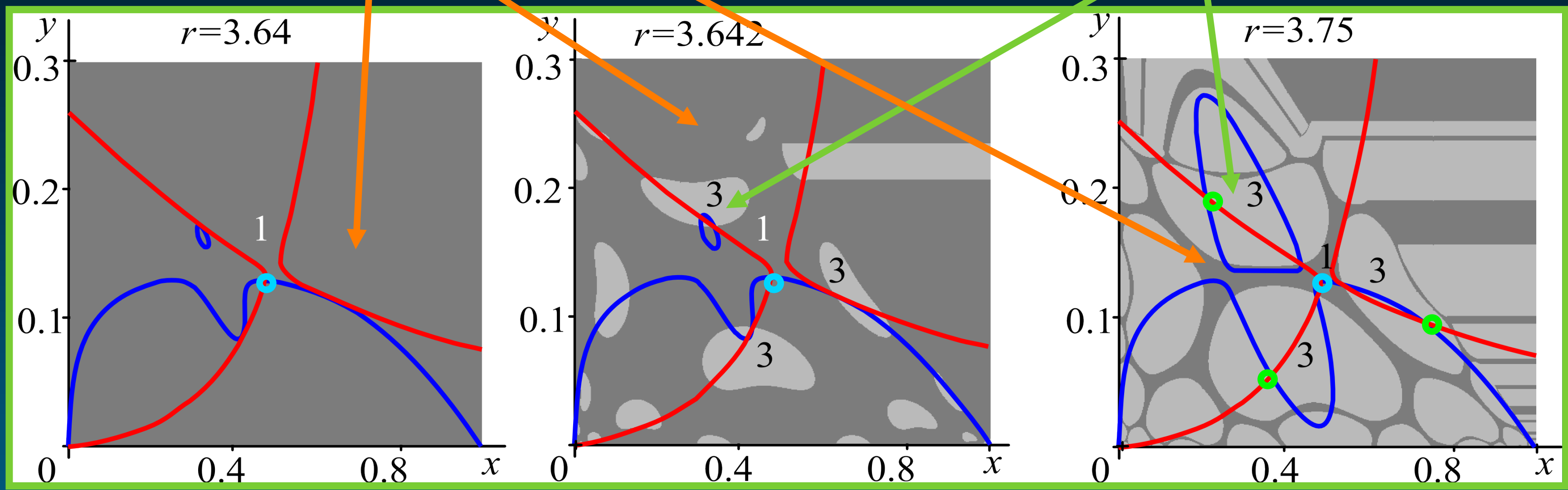


Fig. 12. The attraction basins of the model (2)
at s=v=0.1 ρ=2.15, h=1, $r=a/(1+1/h)$, $x=p$, $y=f+m$

The stable equilibrium

The stable cycle of length 3

The unstable cycle of length three is located on the borders of attraction basins. The stable one is "inside" its attraction basin, far from the basin of stable equilibrium.

Bifurcation diagrams of the model (3)

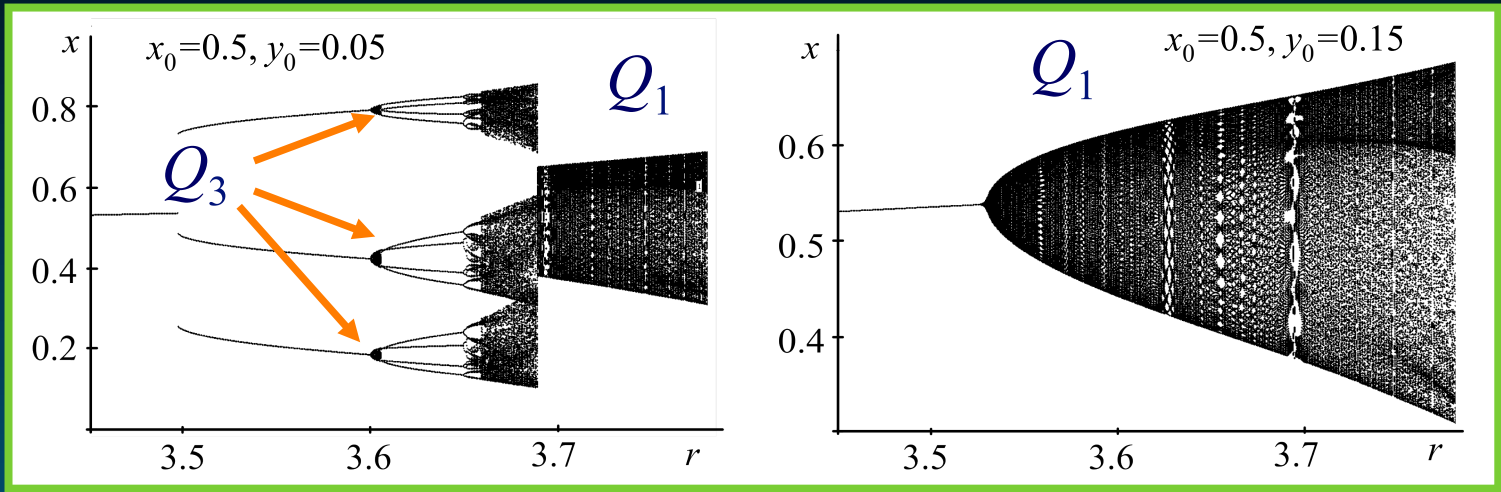


Fig. 13. The bifurcation diagram of dynamic variable x of the system (3) dependents on the parameter a at v=0.1 ρ=1.355 for different initial conditions

With the reproductive potential rise under some initial conditions, the cycle of length 3 emerges in the system.
At further increase of the parameter r, they lose stability and there appear three limiting invariant curves (Q_3).
At other initial values, the fixed point of the system (3) becomes unstable and the population size dynamics demonstrates quasi-periodic fluctuations (Q_1).

Conclusions

- We have demonstrated that an increase in birth and survival rates in the course of natural evolution in ecologically limited populations may result in instability and the appearance of chaotic attractors, the structures and dimensions of which change when model parameters are changed.
- The possibility of the appearance of chaotic population dynamic with an increase in the sexual potency of males (e.g., upon transition to polygamous reproduction) and a decrease in the proportion of males necessary for successful reproduction is showed.

Conclusions

- It is shown the offspring survival density-dependent regulation, which can lead to periodic and chaotic fluctuations in the number of population.
- Moreover, there exist the multistability areas where different dynamic regimes realize, dependent on the initial conditions. In particular, it is revealed coexistence of the stable non-zero fixed point and 3- length cycle. These aspects of dynamic behavior can explain a change in the oscillation period, appearance and disappearance of fluctuations in the population number.

Thank you for your attention!

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