Final size of an epidemic for a two group SIR model

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Kermack and McKendrick (1927)-(1933)

Epidemic model for a single population possibly with age of infection was introduced by **Kermack and McKendrick**. In this model, the population is decomposed into

- the class (S) of susceptible individuals
- the class (I) of infected individuals
- the class (R) of recovered (without reinfection)

Kermack and McKendrick without entering flux

This SIR model takes the following form

$$\left\{\begin{array}{l} \dfrac{dS(t)}{dt} = -\beta S(t) I(t) \\ \dfrac{dI(t)}{dt} = \beta S(t) I(t) - \eta I(t) \\ \dfrac{dR(t)}{dt} = \eta I(t) \end{array}\right. \tag{1}$$

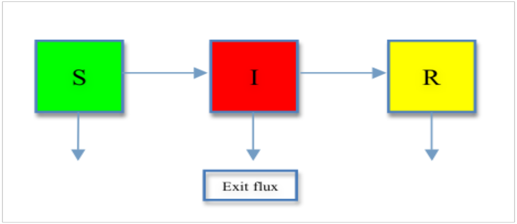


Figure: Diagram flux.

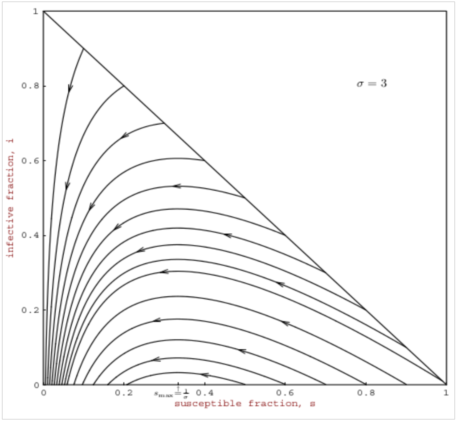


Figure: When $\eta > 0$ some susceptible can escape the epidemic.

We have the conservation formula

$$\frac{d}{dt}\left[S(t)+I(t)-\frac{\eta}{\beta}\ln(S(t))\right]=0. \tag{2}$$

By exploiting this conservation formula Hethcote (1976) proved.

Theorem

Let $S(0)>0$ and $I(0)>0$. If $R_0:=\beta S_0/\eta \leq 1$, then $I(t)$ decreases to zero as $t \to +\infty$. If $R_0:=\beta S_0/\eta > 1$, then $I(t)$ first increases up to a maximum value

$$I_{max}=S_0+I_0-\frac{\eta}{\beta}\ln(S_0)-\frac{\eta}{\beta}+\frac{\eta}{\beta}\ln(\frac{\eta}{\beta})$$

and then decreases to zero as $t \to +\infty$. The susceptible $S(t)$ is a decreasing function and the limiting value $S(+\infty)$ is the unique root in $\left(0,\frac{\eta}{\beta}\right)$ of the equation

$$S(+\infty)-\frac{\eta}{\beta}\ln(S(+\infty))=S_0+I_0-\frac{\eta}{\beta}\ln(S_0)$$

or equivalently

$$\ln\left(\frac{S(+\infty)}{S_0}\right)=R_0\left(\frac{S(+\infty)}{S_0}-1\right)-\frac{R_0}{S_0}I_0.$$

Two groups Kermack and McKendrick without entering flux

$$\begin{cases} \dfrac{dS(t)}{dt} = -\text{diag}\left(S(t)\right) BI(t) \\ \dfrac{dI(t)}{dt} = \text{diag}\left(S(t)\right) BI(t) - EI(t) \\ \dfrac{dR(t)}{dt} = EI(t) \end{cases} \tag{3}$$

The recovery of individuals (or quarantine of infectious) is described by the matrix

$$E = \begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix}$$

while the transmission of pathogen is described by the matrix

$$B = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}.$$

Examples of two groups diseases

- Vector born diseases (i.e. diseases transmitted by a parasite)
- Mosquito-Borne Diseases (Malaria, Chikungunya etc)
- Two groups populations with asymmetric transmission probability or susceptibility
- Hospital-acquired infection where the probability of transmission from the health care worker and the patients are not symmetric
- Co-infection (ex. HIV and tuberculosis)
- Super-spreaders in infectious diseases (see Stein *Inter. J. of Infectious Diseases* (2011))

Diagram flux

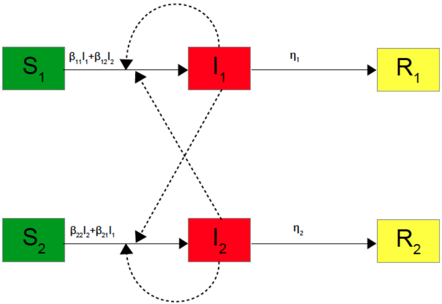


Figure: The figure represents a transfer diagram of the individual fluxes of the system. In this diagram each solid arrow represents a flux of individuals, while the dashed arrows represent the influence of either infectious of sub-population 1 or infectious of sub-population 2.

Assumption

We assume that

(i) B *is a non negative matrix irreducible;*

(ii) $\eta_1 > 0$ *and* $\eta_2 > 0$.

Remark

One may observe that B irreducible is equivalent to assume that

$$\beta_{12} > 0 \text{ and } \beta_{21} > 0.$$

Conservation formula

We obtain

$$\frac{d}{dt}\left[BE^{-1}(S+I)(t) - \ln(S(t))\right] = 0, \ \forall t \geq 0$$

and since $I(+\infty) = 0$ we obtain

$$BE^{-1}S(+\infty) - \ln(S(+\infty)) = BE^{-1}(S+I)(0) - \ln(S(0)).$$

Reformulation of the conservation formula

The conservation formula can be rewritten as the fixed point problem

$$\begin{cases} S_1(+\infty) = S_1(0)\exp\left(\frac{\beta_{11}}{\eta_1}\left[S_1(+\infty) - V_1\right] + \frac{\beta_{12}}{\eta_2}\left[S_2(+\infty) - V_2\right]\right) \\ S_2(+\infty) = S_2(0)\exp\left(\frac{\beta_{21}}{\eta_1}\left[S_1(+\infty) - V_1\right] + \frac{\beta_{22}}{\eta_2}\left[S_2(+\infty) - V_2\right]\right). \end{cases}$$

where

$$V := (S + I)(0).$$

The equation can be rewritten as

$$S(+\infty) = T\left(S(+\infty)\right)$$

Assuming that $S(0) \gg 0$ then by using the fact that T is monotone increasing and that

$$0 \ll T(0) \leq T(S(0)) \leq S(0)$$

We deduce that the following limits exit

$$0 \ll S^- := \lim_{n \to +\infty} T^n(0) \leq S^+ := \lim_{n \to +\infty} T^n(S(0)) \leq S(0).$$

The map T is monotone increasing and convex.

Theorem

The map T has at most two equilibrium. More precisely we have the following alternative either

(i) $S^- = S^+$ *and* T *has only one equilibrium in* $[0, S(0)]$

or

(ii) $S^- \ll S^+$ *and the only equilibrium of* T *in* $[0, S(0)]$ *are* S^- *and* S^+.

Final size of the epidemic

The main result is this work is the following theorem.

Theorem

Let $S(0) = S_0 \gg 0$ and $I(0) = I_0 > 0$. Then the final size of an epidemic of model is given by

$$\lim_{t \to +\infty} S(t) = S^-, \quad \lim_{t \to +\infty} I(t) = 0 \text{ and } \lim_{t \to +\infty} R(t) = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} - S^-.$$

Remark

Due to the above theorem and due the approximation formula $S^- = \lim_{n \to +\infty} T^n(0)$, we can compute numerically the finale size of the epidemic.

Numerical simulations

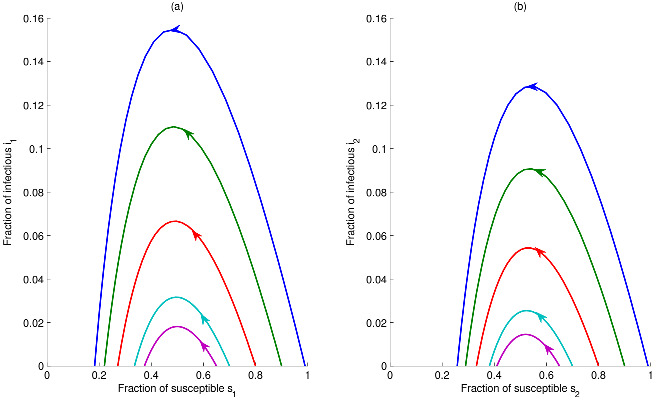


Figure: Figure (a) (resp. (b)) represents the evolution of the fraction of susceptible s_1 of sub-population 1 (resp. s_2 of sub-population 2) with respect to the fraction of infectious i_1 of sub-population 1 (resp. i_2 of sub-population 2).

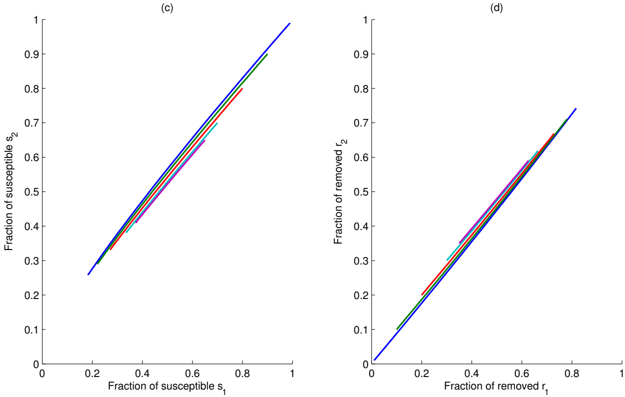
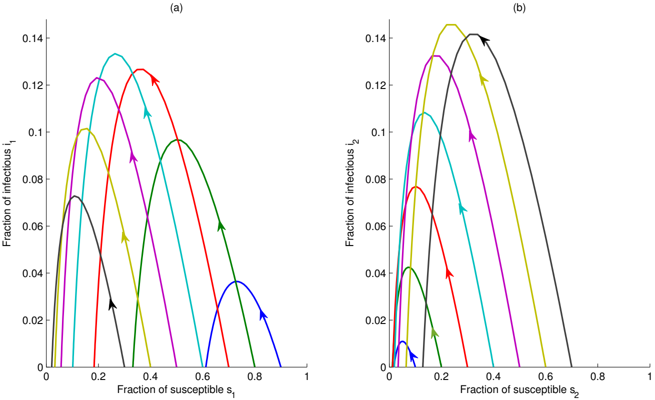


Figure:

Figure (c) (resp. (d)) represents the evolution of the fraction of susceptible s_2 (resp. removed r_2) of sub-population 2 with respect to the fraction of susceptible s_1 (resp. removed r_1) of sub-population 1.



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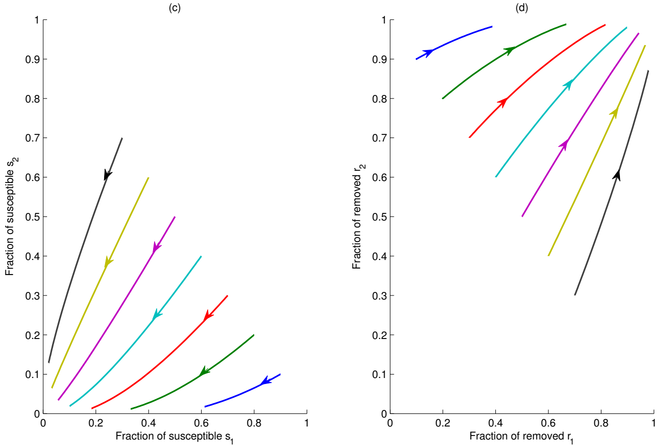


Figure: Figure (c) (resp. (d)) represents the evolution of the fraction of susceptible s_2 (resp. removed r_2) of sub-population 2 with respect to the fraction of susceptible s_1 (resp. removed r_1) of sub-population 1.

Basic reproduction number

We define R_0 as the spectral radius of the matrix

$$\operatorname{diag}\left(S_0\right) B E^{-1} = \begin{pmatrix} \dfrac{S_{10}\beta_{11}}{\eta_1} & \dfrac{S_{10}\beta_{12}}{\eta_2} \\ \dfrac{S_{20}\beta_{21}}{\eta_1} & \dfrac{S_{20}\beta_{22}}{\eta_2} \end{pmatrix}.$$

Behaviour of the infectious classes

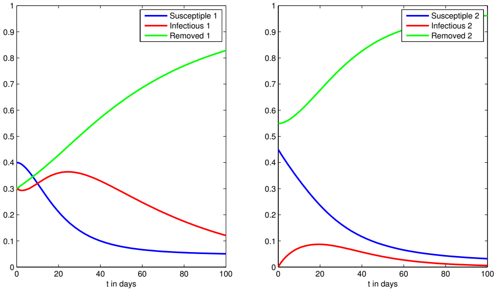


Figure: In this figure we plot the fraction of susceptible (blue line), the fraction of infectious (red line) and the fraction of removed (green line) for system. The sub-population 1 is represented on the left side and the sub-population 2 is represented on the right side. We fix $\widehat{\beta}_{11} = \widehat{\beta}_{22} = 0$; $\widehat{\beta}_{12} = 0.5$; $\widehat{\beta}_{21} = 0.1$; $\eta_1 = 0.02$; $\eta_2 = 0.1$; $s_{10} = 0.4$; $i_{10} = 0.3$; $r_{01} = 0.3$; $s_{20} = 0.45$; $i_{20} = 0.001$; $r_{20} = 0.549$. Here $R_0 = 2.1213 > 1$. The map $i_2(t)$ is decreasing, then increasing and finally decreases to 0. The kind of behavior does exit for a single population model.

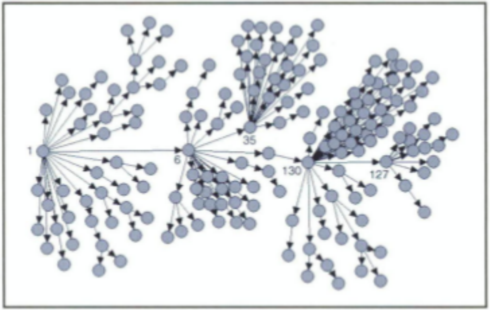
The Role of Super Spreaders in the 2003 SARS Epidemic in Singapore

SARS = Severe Acute Respiratory Syndrome

We will subdivide the population into two classes the super spreader individuals and the non super spreader individuals. In the context of epidemiology the super spreader individuals are known as 20/80 rule (i.e. 20% of the individuals within any given population are thought to contribute at least 80% to the transmission potential of a pathogen).

Network of transmission

FIGURE 2. Probable cases of severe acute respiratory syndrome, by reported source of infection* — Singapore, February 25–April 30, 2003



* Patient 1 represents Case 1; Patient 6, Case 2; Patient 35, Case 3; Patient 130, Case 4; and Patient 127, Case 5. Excludes 22 cases with either no or poorly defined direct contacts or who were cases translocated to Singapore and the seven contacts of one of these cases.

Reference: Bogatti SP. Netdraw 1.0 Network Visualization Software. Harvard, Massachusetts: Analytic Technologies, 2002.

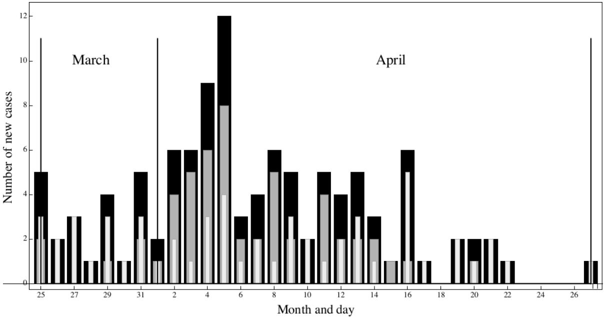


Figure: Case data from March 25, 2003 to April 27, 2003: Centers for Disease Control and Prevention (CDC), Severe Acute Respiratory Syndrome in Singapore, 2003, Morbidity and Mortality Weekly Report, Vol. 52, No. 18, May 9, 2003. Light gray bars: new I_1 cases (outside hospital); Dark gray bars: new I_2 cases (inside hospital); Black bars: total new cases.

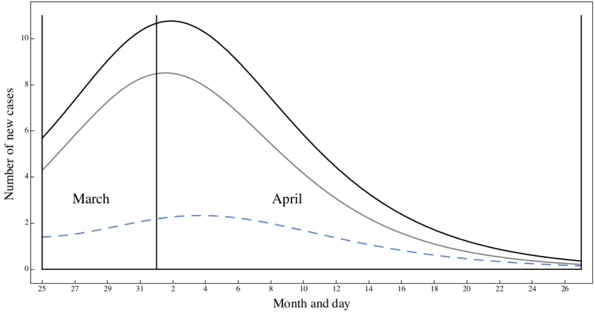


Figure: New cases from March 25, 2003 to April 27, 2003. Gray dashed graph: new I_1 cases (outside hospital); Gray solid graph: new I_2 cases (inside hospital); Black graph: total new cases. The simulation aligns with the data in the CDC report.

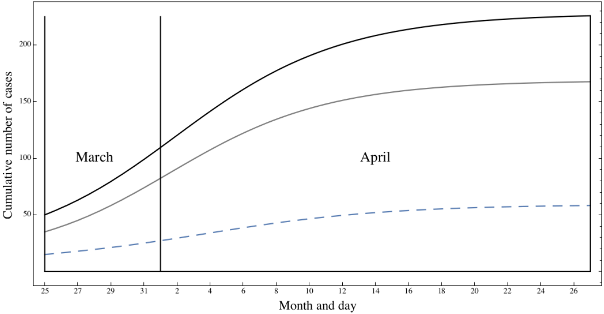


Figure: Cumulative cases from March 25, 2003 to April 27, 2003. Gray dashed graph: cumulative I_1 cases (outside hospital); Gray solid graph: cumulative I_2 cases (inside hospital); Black graph: total cumulative cases. The simulation aligns with the data in the CDC report.

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