Periodic solutions for a model of rotational stocking in a seasonally driven grazing system

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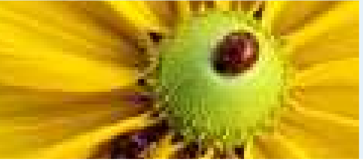
Introduction . . . and why seasonality

Preliminary

. . . bit of fluff today . . .

. . . more on Holistic Management$^{\mathrm{TM}}$ . . .

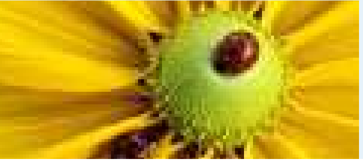
. . . lots of plots . . .



Oscillatory drivers in biology

But seriously, folks:

oscillatory drivers may be far more important to biology than I used to think



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Not only because they are ubiquitous, but also because novel and unexpected things may result from oscillatory driving; I was very happy hear about Parondo's paradox from Fabio—which exists in a chemistry lab as well as mathematically.

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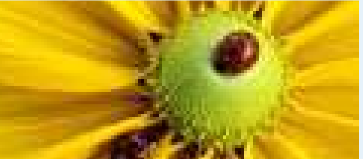
Not only because they are ubiquitous, but also because novel and unexpected things may result from oscillatory driving; I was very happy hear about Parondo's paradox from Fabio—which exists in a chemistry lab as well as mathematically.

So—here are some features of a model of seasonally driven plant-animal dynamics.

Savanna plant-animal dynamics



The basics of the Owen-Smith model

$$\frac{dV}{dt} = w(t)rV\left(1 - \frac{V}{K}\right) - \frac{i_m(V - v_u)H}{b_i + V - v_u}$$

$$\frac{dH}{dt} = \frac{ci_m(V - v_u)H}{b_g + V - v_u} - H\left(m_p + q_s - q_0 + qm_p\frac{b_g + V - v_u}{ci_m(V - v_u)}\right)$$

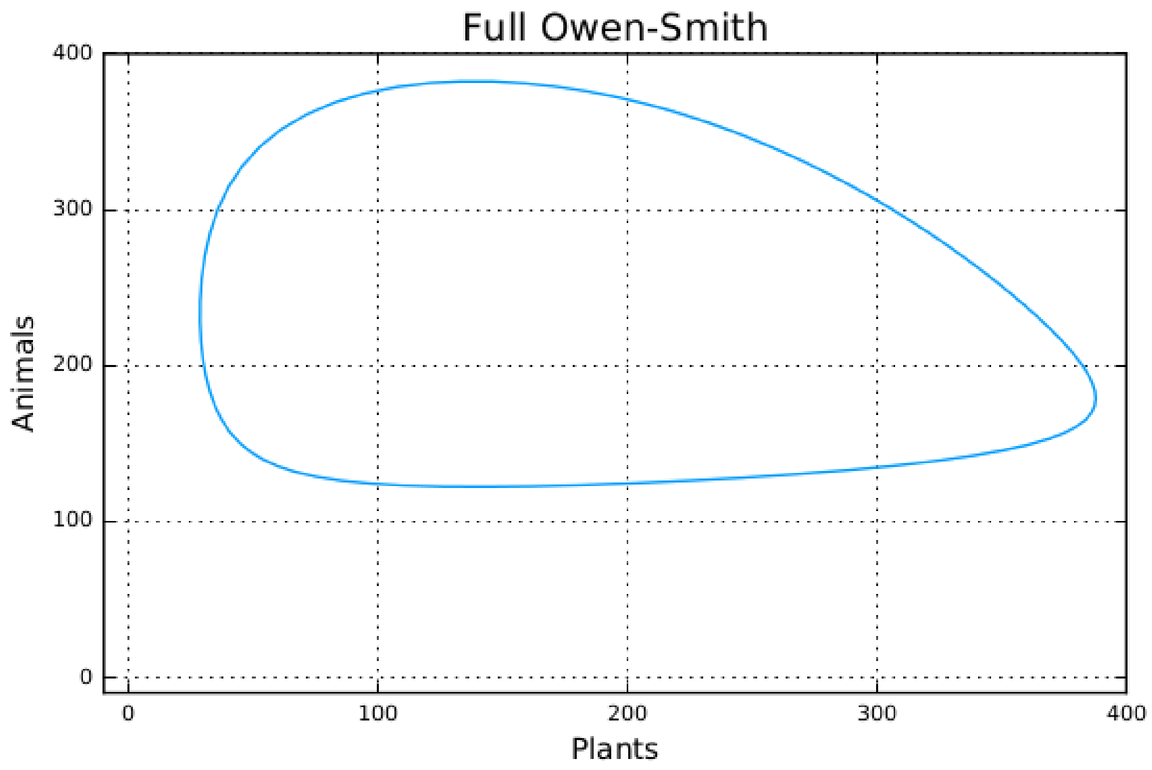
where, for year length T,

$$w(t) = \begin{cases} 0 & \text{if} \quad t/T - \lfloor t/T \rfloor < 1 - \theta \\ 1/\theta & \text{otherwise} \end{cases}$$

Thus the mean of $w(t)$ over one year is 1. Norman uses $\theta = 0.5$.

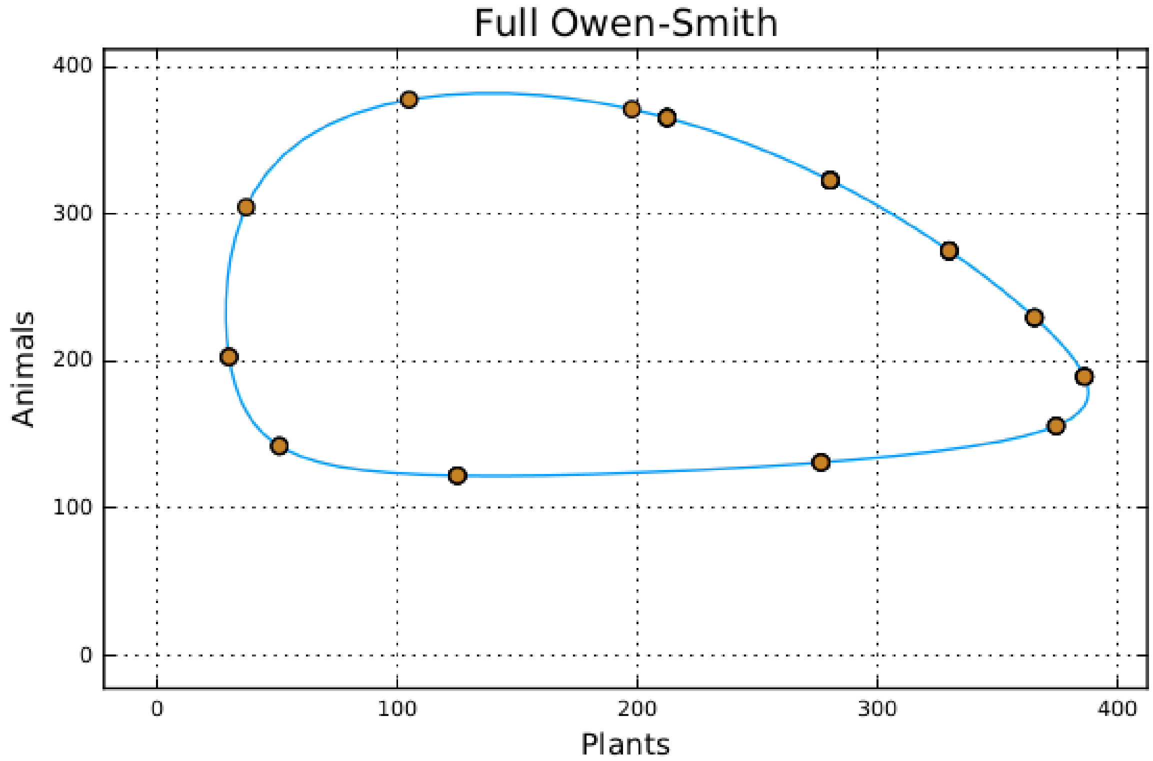
. . . Rosenzweig-Mcarthur, interpreted according to Getz' "metaphysiology" plus seasonally on-off plant growth, a starvation effect in the rate of biomass loss, and a subtle twist on the saturation effect of animal intake.

Without seasonality, stable coexistence, either at fixed values or one a limit cycle

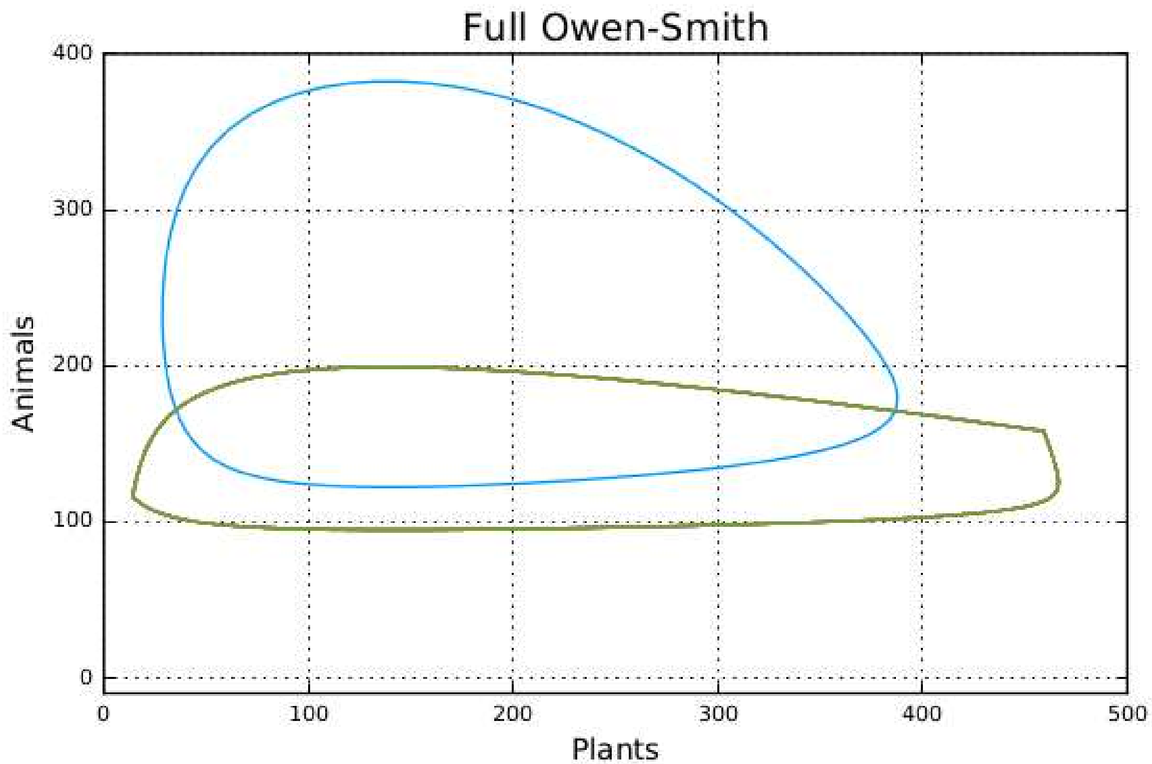


Without seasonality, stable coexistence, either at fixed values or one a limit cycle

This cycle, for (one set of) Norman's parameter values, is shorter than a year.



Norman's "stabilisation": reduced amplitude of fluctuation in animal biomass density

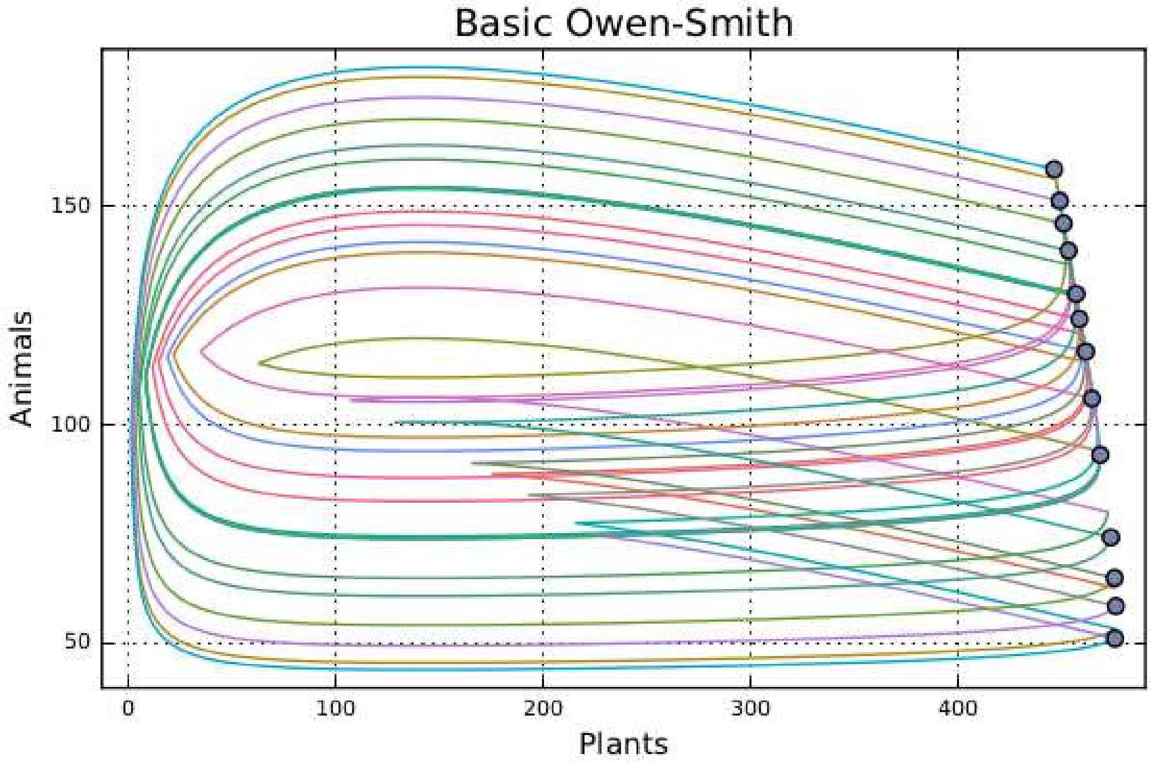


$r = 0.3$

$b_i = 75.0$

$c = 0.7$

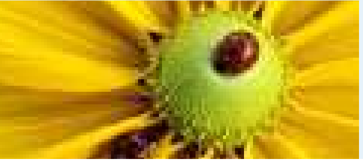
However, for parameter values not far from these, chaotic solutions are possible:



$r = 0.25$

$b_i = 50.0$

$c = 0.62$



Seasonal behaviours

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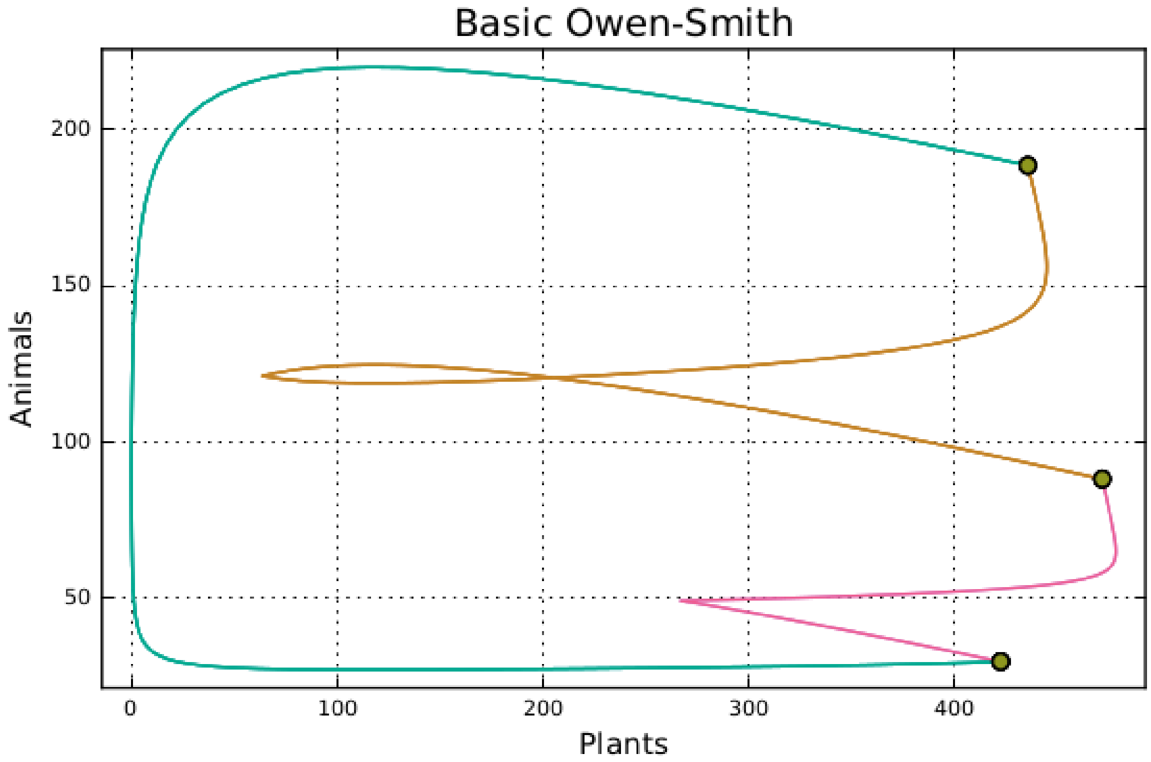

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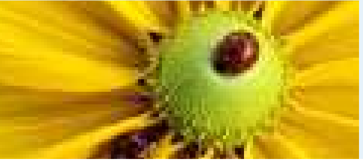
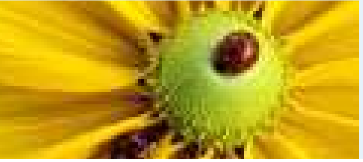
Rotational stocking: the basic ideas



GMU = Grazing Management Unit

It has a *stocking rate* total animal biomass divided by the area.

It has *paddocks*, wherein animals are confined eg. by fences



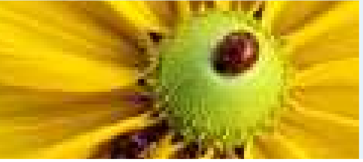
Concentration factor

For *rotational stocking* as used here, the animals are confined to one of k paddocks.

Thus the concentration factor is k: for a GMU stocking rate of h grams per metre^2, the occupied paddock has a density of kh.

Target stocking rate strategy: say GMU target is $\tilde{h}$, impose it at the start of each movement to a new occupied paddock. Achieve it via "put-and-take"—that is, the net change in animal biomass during a grazing period could be negative (resulting in put) or positive (resulting in take).

Rotational stocking for improving plant cover: a static model



Cover dynamics

ϕ is the fraction of soil covered by plants — to be interpreted as the mean field value (so no actual spatial structure).

Loss rate $l(h)$ and recovery rate $r(h)$ (to model animal impact on cover) give

$$\dot{\phi} = -l(h)\phi + r(h)(1-\phi) = r(h) - [l(h) + r(h)]\phi$$

$\dot{\phi}$ is the rate of change in the cover fraction.

Fixed point at $\phi^* = r(h)/(l(h) + r(h))$. This is large (i.e. near 1) if $\alpha \gg 1$.

No seasonality here: if $\phi < r/(l + r)$ then cover increases, come rain or drought.

Savory hypothesis: high, brief animal densities facilitate (re)vegetation

For animals to facilitate (re)vegetation, the effect of h must be to increase the recovery rate r relative to the loss rate l. One easy way to formulate that is

$$l(h) = l_0; \qquad r(h) = r_0 + r_2 h^{\theta}$$

.

Decoupling of cover from population dynamics

When h is independent of ϕ, the model decouples.

Examples: h is constant, or h is a prescribed function of time.

In particular: rotation through k paddocks with constant GMU stocking rate $\tilde{h}$. Then, modulo shifts in time, in every paddock we have (here n is an integer that counts years):

$$h(t) = \begin{cases} k\tilde{h} & \text{for} \quad nT \le t < nT + T/k \\ 0 & \text{for } nT + T/k \le t < (n+1)T \end{cases}$$

Analytical solution of the reduced model

The maths gets slightly crowded, but in terms of ODEs, it is quite simple:
— a linear equation
— with integrable coefficients, moreover
— periodic coefficients.

Hence the solution exists uniquely, is periodic, and easily written down up to quadrature.

Constant GMU stocking rate: explicit solution

During the occupied period of duration T/k cover grows at the rate $l_0 + r_0 + r_2(kh)^\theta$ towards an upper fixed point $\widehat{\phi}_v$ given by

$$\widehat{\phi}_v = p_A^*(1 - e^{-\lambda_2(T-T/k)}) + p_B^*(e^{-\lambda_2(T-T/k)-e^{\lambda T}})$$

where $p_A^* = \frac{r_0 + r_2\tilde{h}^\theta k^\theta}{l_0 + r_0 + r_2\tilde{h}^\theta k^\theta}, \quad p_B^* = \frac{r_0}{l_0 + r_0}$

while $\lambda_1 = l_0 + r_0 + r_2\tilde{h}^\theta k^\theta, \quad \lambda_2 = l_0 + r_0, \quad \lambda = l_0 + r_0 + r_2\tilde{h}^\theta k^{(\theta-1)}$.

During the rest period, cover grows at the rate $-(l_0 + r_0)$ towards a lower fixed point $\widetilde{\phi}$

$$\widetilde{\phi}_v = p_B^*(1 - e^{-\lambda_1 T/k}) + p_A^*(e^{\lambda_1 T/k - e^{\lambda T}})$$

Using a matrix formulation it is not hard to show that the periodic solution is a globally attracting limit cycle.

Rotational stocking for improving plant cover: with NOS dynamics

Joint model

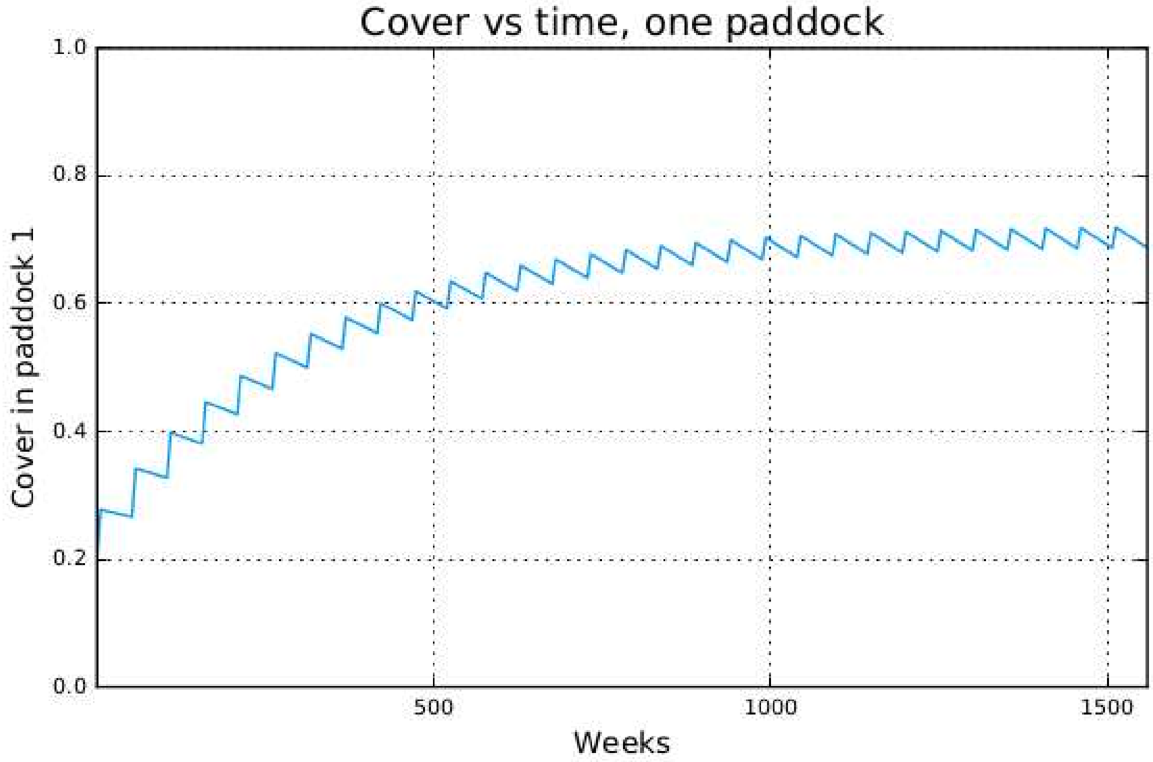
$$
\begin{aligned}
\frac{d\phi_v}{dt} &= -l(h)\phi_v + r(h)(1-\phi_v) \\
\frac{dv}{dt} &= w(t)\rho_0\left(1-\frac{v}{K}\right) - \frac{i\phi_v(v/v_{1/2})}{1+\phi_v(v/v_{1/2})}h \\
\frac{dh}{dt} &= \frac{ci\phi_v(v/v_{1/2})}{1+\phi_v(v/v_{1/2})}h - \mu h.
\end{aligned}
$$

where $l(h) = l_0$ and $r(h) = r_0 + r_2 h^\theta$.

No analytical results, only computations. Had hoped that if $\theta = 1$, the fixed target stocking rate strategy results in dynamics that are independent of k, but no.
Nevertheless (as I showed in Niteroi), when θ is close to 1, the value of k makes very little difference.

Can cherry-pick parameters to fit the HM^{TM} claims

The Holistic Management (^{TM}) idea is that grazing herds can be used as ecosystem engineers, for example to increase cover and yield.
My choice of $l(h)$ and $r(h)$ was specifically designed to achieve that in the ideal world of a model.



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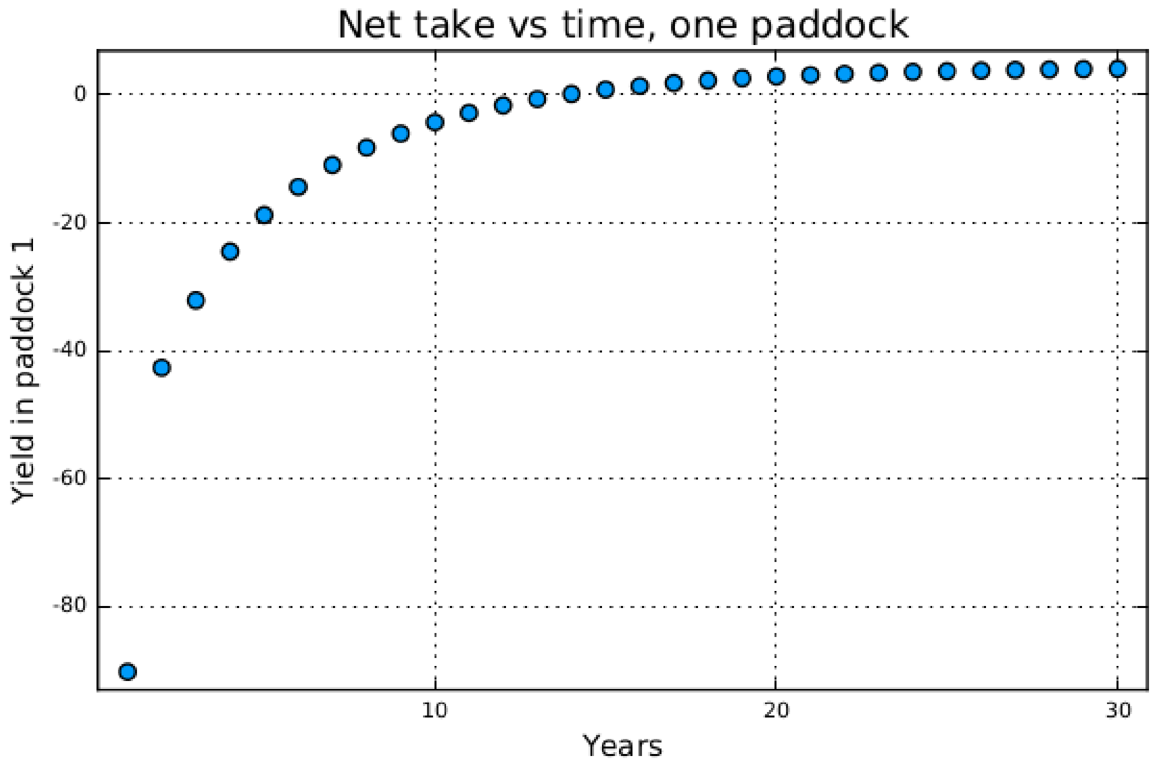

Rotational stocking suppresses chaos

Recall that the basic NOS model is capable of chaotic behaviour. However,

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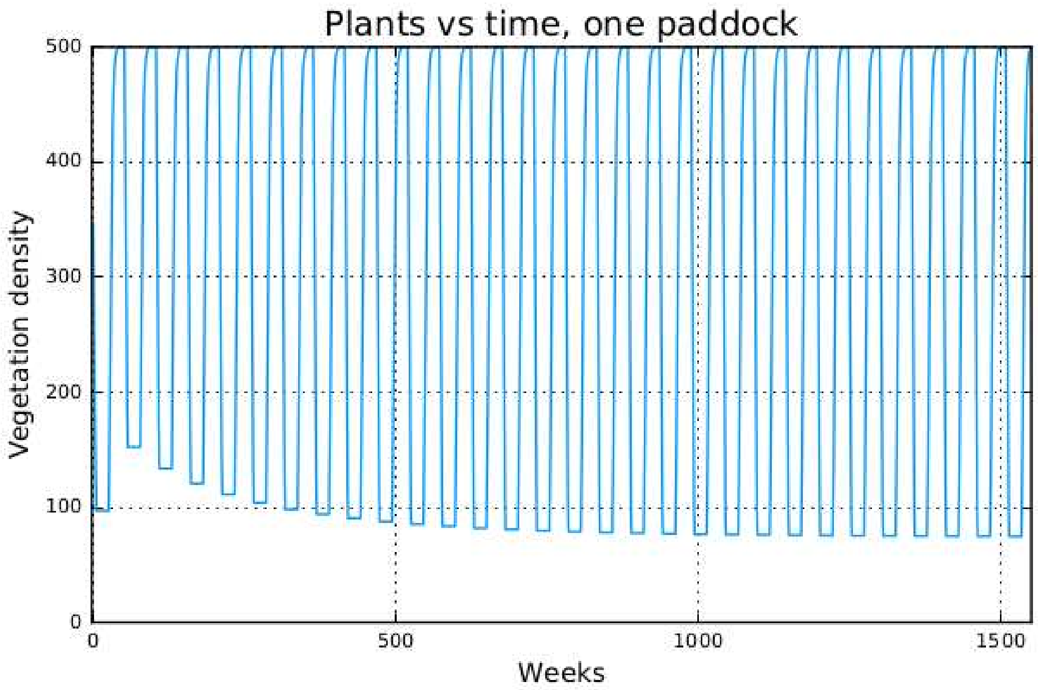

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for the same parameters in a 10-paddock GMU.

THANK YOU!