

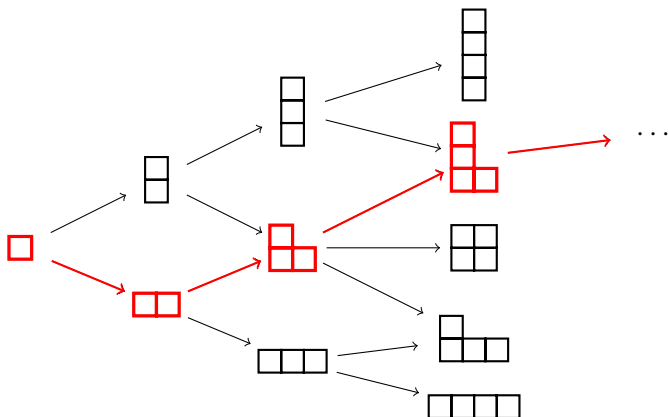
This presentation contains animations
which require PDF viewer which accepts
JavaScript.

For best results use Acrobat Reader.

Series of lectures:
jeu de taquin and asymptotic representation theory

Piotr Śniady

plan for this series of lectures:
representations of the symmetric groups
 $\mathfrak{S}_1 \subset \mathfrak{S}_2 \subset \mathfrak{S}_3 \subset \cdots$ and \mathfrak{S}_∞



plan for this series of lectures:

Lecture 1, August 30

what can we say about RSK
applied to random input?

Lecture 2, September 2

... and what does it tell us
about the **asymptotic representation theory**
of the symmetric groups \mathfrak{S}_n for $n \rightarrow \infty$ and \mathfrak{S}_∞ ?

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... and what does it tell us
about the **asymptotic representation theory**
of the symmetric groups \mathfrak{S}_n for $n \rightarrow \infty$ and \mathfrak{S}_∞ ?

Lecture 1A:

what can we say about RSK applied to random input?

Piotr Śniady

Polska Akademia Nauk

RSK is a bijection...

Input:

- word $\mathbf{w} = (w_1, \dots, w_n)$

Output:

- semistandard tableau P ,
- standard tableau Q ,

tableaux P and Q have
the same shape with n boxes

example:

 $\mathbf{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41)$

74	99		
23	53	70	
16	37	41	82

insertion tableau $P(\mathbf{w})$

8	9		
4	6	7	
1	2	3	5

recording tableau $Q(\mathbf{w})$

Lecture 1A: what can we say about RSK applied to random input?

RSK

RSK

RSK is a bijection...

RSK is a bijection...

Input:

• word $w = (w_1, \dots, w_n)$

Output:

• semistandard tableau P ,• standard tableau Q ,tableaux P and Q have
the same shape with n boxes

example:

 $w = (23, 53, 74, 18, 83, 71, 82, 37, 41)$

74	83	
23	53	78
38	37	41

insertion tableau $P(w)$

9	8	
4	8	7
2	2	3

recording tableau $Q(w)$

- start with empty tableaux $P := \emptyset$, $Q := \emptyset$;
- read the letters from the word w , one after another;
- for each LETTER:
 - iterate over the rows of the insertion tableau P , start from the first row;
 - insert the LETTER to some box in this row as far to the right as possible, so that the row remains increasing;
 - was this box empty?
 - NO the previous tenant must be bumped!
LETTER := bumped element;
proceed to the next row;
 - YES update information about the new box into the recording tableau Q ,
proceed to the next letter of the word;

Robinson-Schensted-Knuth algorithm — induction step

74	99			
23	53	70		
16	37	41	82	

insertion tableau $P(\mathbf{w})$

8	9			
4	6	7		
1	2	3	5	

recording tableau $Q(\mathbf{w})$

$$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41)$$

Robinson-Schensted-Knuth algorithm — induction step

74	99			
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insertion tableau $P(\mathbf{w})$

8	9			
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recording tableau $Q(\mathbf{w})$

$$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$$

Robinson-Schensted-Knuth algorithm — induction step

74	99			
23	53	70		
16	37	41	82	

insertion tableau $P(\mathbf{w})$

8	9			
4	6	7		
1	2	3	5	

recording tableau $Q(\mathbf{w})$

$$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$$

insertion tableau $P(\mathbf{w})$

recording tableau $Q(\mathbf{w})$

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The diagram illustrates the insertion of the element 37 into the insertion tableau $P(\mathbf{w})$. The tableau $P(\mathbf{w})$ is shown with three rows: the first row contains 74 and 99; the second row contains 23, 53, and 70; the third row contains 16, 37, 41, 82, and a red-shaded area representing the rest of the row. The element 37 is highlighted with a blue border and a red diagonal line, indicating it is the element being inserted. The recording tableau $Q(\mathbf{w})$ is shown to the right, with three rows: the first row contains 8 and 9; the second row contains 4, 6, and 7; the third row contains 1, 2, 3, and 5.

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8	9		
4	6	7	
1	2	3	5

recording tableau $Q(\mathbf{w})$

◀ ◻ ▶ ◀ ◻ ▶ ◀ ≡ ▶ ◀ ≡ ▶ ≡ ▶ ↺ 🔍 ↻

8	9		
4	6	7	
1	2	3	5

recording tableau $Q(\mathbf{w})$

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Robinson-Schensted-Knuth algorithm — induction step

74	99			
23	53	70		
16	18	41	82	

insertion tableau $P(\mathbf{w})$

8	9			
4	6	7		
1	2	3	5	

recording tableau $Q(\mathbf{w})$

$$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$$

The diagram illustrates the insertion of $w=74$ into a recording tableau $Q(w)$ to form an insertion tableau $P(w)$.

Insertion Tableau $P(w)$:

74	99		
23	37	70	
16	18	41	82

Recording Tableau $Q(w)$:

8	9		
4	6	7	
1	2	3	5

The insertion tableau $P(w)$ is shown with a blue box around the cell containing 74 and a red box around the cell containing 99, indicating the insertion process. The recording tableau $Q(w)$ is shown with a blue box around the cell containing 8, indicating the recording of the insertion.

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The diagram illustrates the insertion of element 74 into a recording tableau $Q(\mathbf{w})$ to form an insertion tableau $P(\mathbf{w})$.

Recording Tableau $Q(\mathbf{w})$: A Young diagram with three rows and four columns. The entries are:

8	9		
4	6	7	
1	2	3	5

Insertion Tableau $P(\mathbf{w})$: A Young diagram with three rows and four columns. The entries are:

74	99		
23	37	70	
16	18	41	82

The element 74 is shown being inserted into the first row, first column of $P(\mathbf{w})$, where it is crossed out with a red diagonal line. The cell is highlighted with a blue border. The rest of the first row is shaded red, indicating the insertion path.

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The diagram illustrates the insertion of $w=53$ into a recording tableau $Q(w)$ to form an insertion tableau $P(w)$.

Insertion Tableau $P(w)$:

53	99		
23	37	70	
16	18	41	82

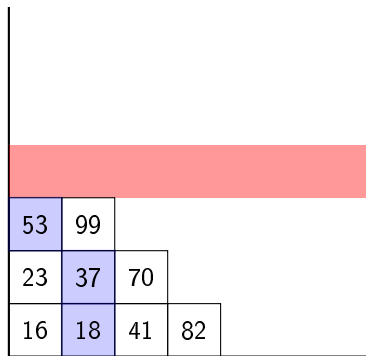
Recording Tableau $Q(w)$:

8	9		
4	6	7	
1	2	3	5

The insertion tableau $P(w)$ is shown with a blue box around the cell containing 53 and a red box around the cell containing 99, indicating the insertion process. The recording tableau $Q(w)$ is shown with a blue box around the cell containing 8, indicating the recording of the insertion.

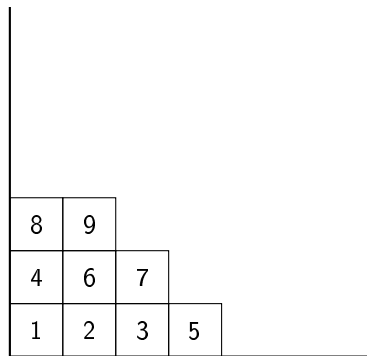
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Robinson-Schensted-Knuth algorithm — induction step



53	99		
23	37	70	
16	18	41	82

insertion tableau $P(\mathbf{w})$



8	9		
4	6	7	
1	2	3	5

recording tableau $Q(\mathbf{w})$

$$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$$

Robinson-Schensted-Knuth algorithm — induction step

74				
53	99			
23	37	70		
16	18	41	82	

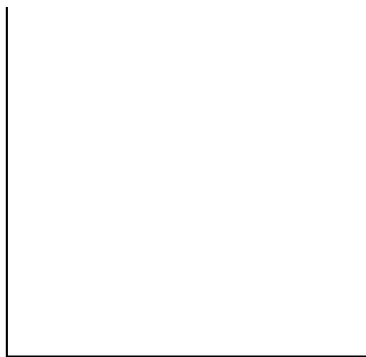
insertion tableau $P(\mathbf{w})$

10				
8	9			
4	6	7		
1	2	3	5	

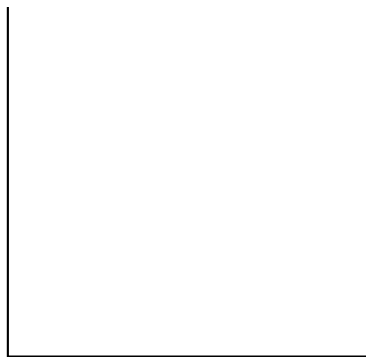
recording tableau $Q(\mathbf{w})$

$$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 18)$$

Robinson-Schensted-Knuth algorithm



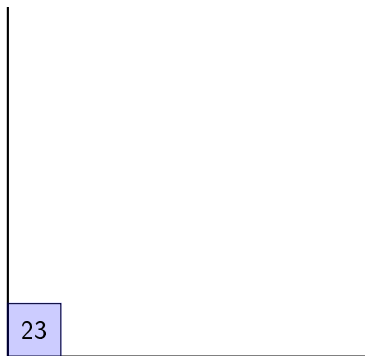
insertion tableau $P(\mathbf{w})$



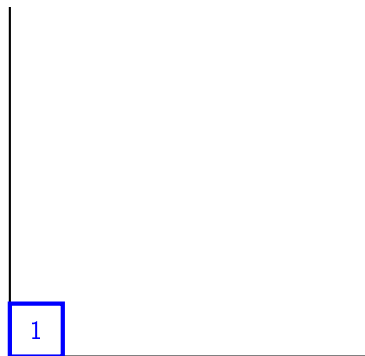
recording tableau $Q(\mathbf{w})$

$$\mathbf{w} = \emptyset$$

Robinson-Schensted-Knuth algorithm



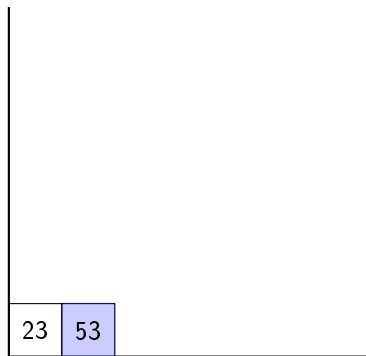
insertion tableau $P(\mathbf{w})$



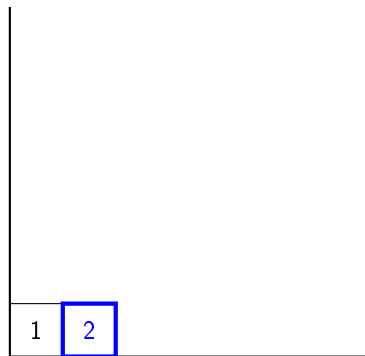
recording tableau $Q(\mathbf{w})$

$$\mathbf{w} = (23)$$

Robinson-Schensted-Knuth algorithm



insertion tableau $P(\mathbf{w})$



recording tableau $Q(\mathbf{w})$

$$\mathbf{w} = (23, 53)$$

Robinson-Schensted-Knuth algorithm

23	53	74
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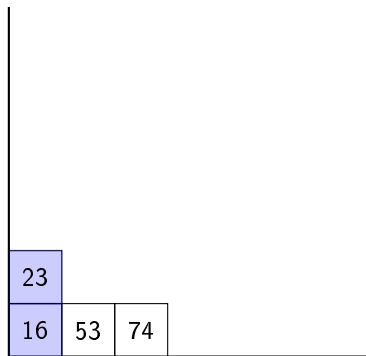
insertion tableau $P(\mathbf{w})$

1	2	3
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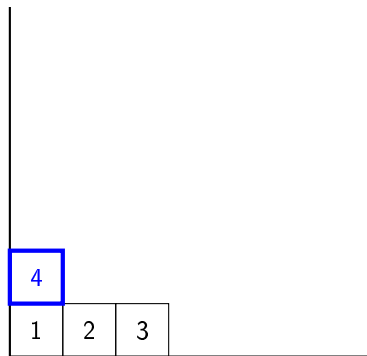
recording tableau $Q(\mathbf{w})$

$$\mathbf{w} = (23, 53, 74)$$

Robinson-Schensted-Knuth algorithm



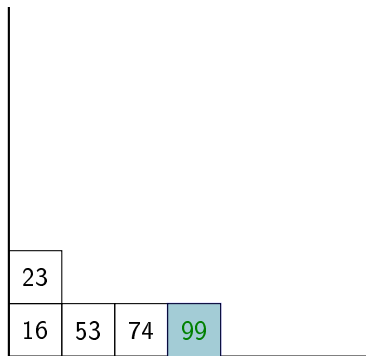
insertion tableau $P(\mathbf{w})$



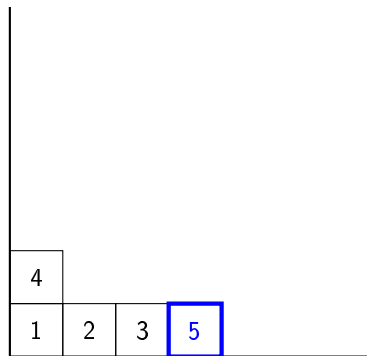
recording tableau $Q(\mathbf{w})$

$$\mathbf{w} = (23, 53, 74, 16)$$

Robinson-Schensted-Knuth algorithm



insertion tableau $P(\mathbf{w})$



recording tableau $Q(\mathbf{w})$

$$\mathbf{w} = (23, 53, 74, 16, 99)$$

Robinson-Schensted-Knuth algorithm

23	74		
16	53	70	99

insertion tableau $P(\mathbf{w})$

4	6		
1	2	3	5

recording tableau $Q(\mathbf{w})$

$$\mathbf{w} = (23, 53, 74, 16, 99, 70)$$

Robinson-Schensted-Knuth algorithm

23	74	99	
16	53	70	82

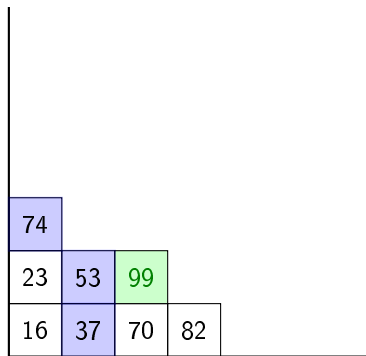
insertion tableau $P(\mathbf{w})$

4	6	7	
1	2	3	5

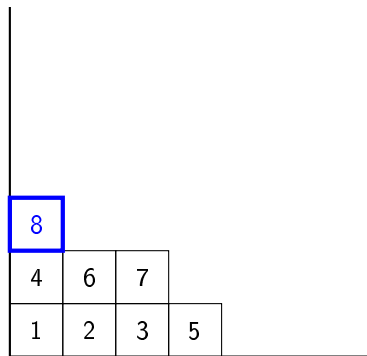
recording tableau $Q(\mathbf{w})$

$$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82)$$

Robinson-Schensted-Knuth algorithm



insertion tableau $P(\mathbf{w})$



recording tableau $Q(\mathbf{w})$

$$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82, 37)$$

Robinson-Schensted-Knuth algorithm

74	99			
23	53	70		
16	37	41	82	

insertion tableau $P(\mathbf{w})$

8	9			
4	6	7		
1	2	3	5	

recording tableau $Q(\mathbf{w})$

$$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41)$$

Robinson-Schensted-Knuth algorithm

74			
53	99		
23	37	70	82
16	34	41	73

insertion tableau $P(\mathbf{w})$

10			
8	9		
4	6	7	11
1	2	3	5

recording tableau $Q(\mathbf{w})$

$$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 34, 73)$$

Robinson-Schensted-Knuth algorithm

74				
53				
23	99			
16	37	70	82	
2	34	41	73	

insertion tableau $P(\mathbf{w})$

12				
10				
8	9			
4	6	7	11	
1	2	3	5	

recording tableau $Q(\mathbf{w})$

$$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 34, 73, 2)$$

Robinson-Schensted-Knuth algorithm

74				
53	99			
23	37			
16	34	70	82	
2	24	41	73	

insertion tableau $P(\mathbf{w})$

12				
10	13			
8	9			
4	6	7	11	
1	2	3	5	

recording tableau $Q(\mathbf{w})$

$$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 34, 73, 2, 24)$$

Robinson-Schensted-Knuth algorithm

74			
53			
37	99		
23	34		
16	24	70	82
2	17	41	73

insertion tableau $P(\mathbf{w})$

14			
12			
10	13		
8	9		
4	6	7	11
1	2	3	5

recording tableau $Q(\mathbf{w})$

$$\mathbf{w} = (23, 53, 74, 16, 99, 70, 82, 37, 41, 34, 73, 2, 24, 17)$$

the main problem

general problem

what can we say about RSK
applied to random input?

the main problem

general problem

what can we say about RSK
applied to random input?

concrete setup for today, version A

... if the word $\mathbf{w} = (w_1, \dots, w_n)$ is a random permutation from \mathfrak{S}_n ?

the main problem

general problem

what can we say about RSK
applied to random input?

concrete setup for today, version A

... if the word $\mathbf{w} = (w_1, \dots, w_n)$ is a random permutation from \mathfrak{S}_n ?

concrete setup for today, version B

... if $\mathbf{w} = (w_1, \dots, w_n)$ is a sequence of
iid (independent, identically distributed) random variables
with the uniform distribution $U(0, 1)$ on the interval $[0, 1]$?

the main problem

general problem

what can we say about RSK
applied to random input?

concrete setup for today, version A

... if the word $\mathbf{w} = (w_1, \dots, w_n)$ is a random permutation from \mathfrak{S}_n ?

concrete setup for today, version B

... if $\mathbf{w} = (w_1, \dots, w_n)$ is a sequence of
iid (independent, identically distributed) random variables
with the uniform distribution $U(0, 1)$ on the interval $[0, 1]$?

→ U_{LAM} 1963

Plancherel measure

exercise

if $\mathbf{w} = (w_1, \dots, w_n)$ is either

- a random permutation, or
- iid sequence $U(0, 1)$

then for each $\lambda \in \mathbb{Y}_n$

$$\mathbb{P}(\overbrace{\text{common shape of } P(\mathbf{w}) \text{ and } Q(\mathbf{w})}^{\text{RSKshape}(\mathbf{w})} \text{ is equal to } \lambda) = \frac{(\text{number of SYT of shape } \lambda)^2}{n!} = \frac{(\dim \rho_\lambda)^2}{n!}$$

*'random irreducible component
of the left regular representation $\ell^2(\mathfrak{S}_n)$
of the symmetric group'*

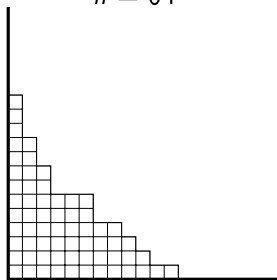
→ Lecture 2

limit shape for Plancherel measure

problem

what can we say about the common shape of $P(\mathbf{w})$ and $Q(\mathbf{w})$ when $n \rightarrow \infty$ and $\mathbf{w} = (w_1, \dots, w_n)$ is random?

$n = 64$

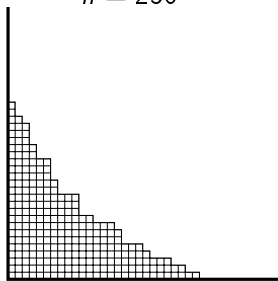


limit shape for Plancherel measure

problem

what can we say about the common shape of $P(\mathbf{w})$ and $Q(\mathbf{w})$ when $n \rightarrow \infty$ and $\mathbf{w} = (w_1, \dots, w_n)$ is random?

$n = 256$



Theorem (LOGAN&SHEPP, VERSHIK&KEROV 1977)

in the limit $n \rightarrow \infty$

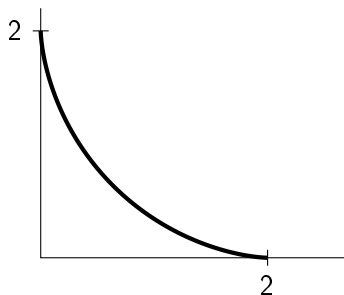
RSKshape(\mathbf{w}) (=the common shape of $P(\mathbf{w})$ and $Q(\mathbf{w})$)

after rescaling by the factor $\frac{1}{\sqrt{n}}$

becomes (with very high probability)

very close to some concrete limit shape

→lectures of Philippe Biane



key problem,
sloppy version


where
in the recording
tableau $Q(\mathbf{w})$
is located our favorite
number?

key problem, sloppy version

where
in the recording
tableau $Q(\mathbf{w})$
is located our favorite
number?

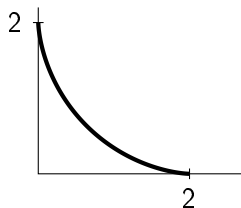
key problem, more specific

let $\mathbf{w} = (w_1, \dots, w_{n+1})$,
with w_1, \dots, w_n random, iid $U(0, 1)$
and w_{n+1} deterministic

what can we say
about the location of the box
 containing $n + 1$
in the recording tableau $Q(\mathbf{w})$?

key problem, sloppy version

where
in the recording
tableau $Q(\mathbf{w})$
is located our favorite
number?



key problem, more specific

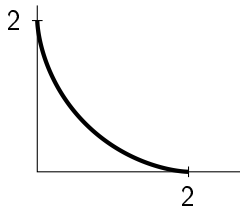
let $\mathbf{w} = (w_1, \dots, w_{n+1})$,
with w_1, \dots, w_n random, iid $U(0, 1)$
and w_{n+1} deterministic

what can we say
about the location of the box
 containing $n + 1$
in the recording tableau $Q(\mathbf{w})$?

silly answer:
it is somewhere
at the boundary of $\text{RSKshape}(\mathbf{w})$
which is $\approx \text{LSVK shape}$

key problem, sloppy version

where
in the recording
tableau $Q(\mathbf{w})$
is located our favorite
number?



key problem, more specific

let $\mathbf{w} = (w_1, \dots, w_{n+1})$,
with w_1, \dots, w_n random, iid $U(0, 1)$
and w_{n+1} deterministic

what can we say
about the location of the box
 containing $n + 1$
in the recording tableau $Q(\mathbf{w})$?

silly answer:
it is somewhere
at the boundary of $\text{RSKshape}(\mathbf{w})$
which is $\approx \text{LSVK shape}$

but where exactly?

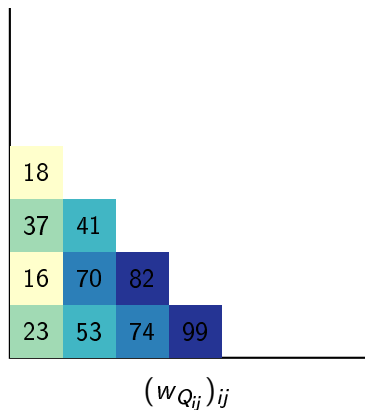
10				
8	9			
4	6	7		
1	2	3	5	

recording tableau $(Q_{ij})_{ij}$

18				
37	41			
16	70	82		
23	53	74	99	

 $(w_{Q_{ij}})_{ij}$

$$\begin{array}{ccccc}
 w_1 = 23, & w_2 = 53, & w_3 = 74, & w_4 = 16, & w_5 = 99, \\
 w_6 = 70, & w_7 = 82, & w_8 = 37, & w_9 = 41 &
 \end{array}$$



RSK
○○○

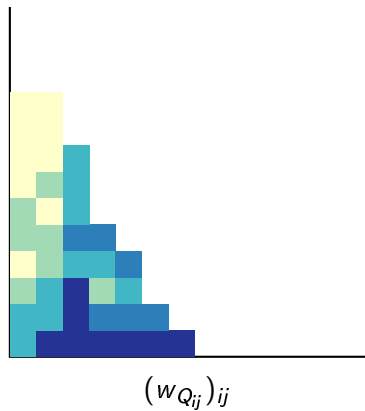
problems
○○

limit shape
○○

determinism of the last box
●●

bumping routes
○○○

hydrodynamics
○○○○○



RSK
○○○

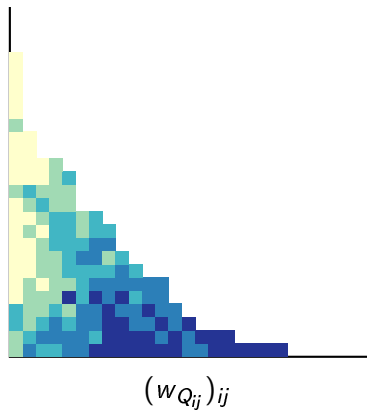
problems
○○

limit shape
○○

determinism of the last box
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bumping routes
○○○

hydrodynamics
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RSK
○○○

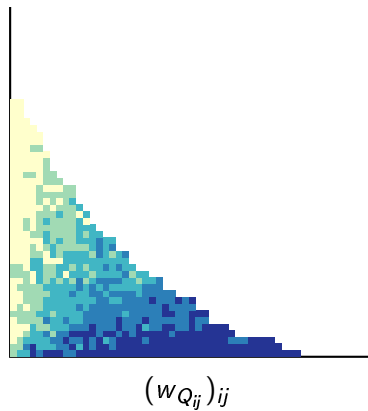
problems
○○

limit shape
○○

determinism of the last box
○●

bumping routes
○○○

hydrodynamics
○○○○○



RSK
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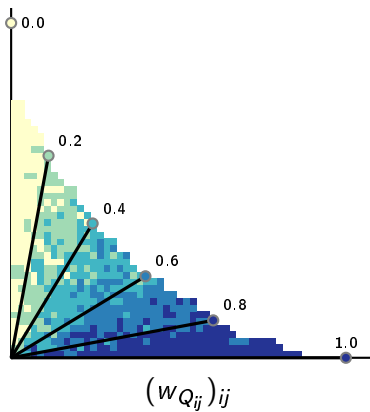
problems
○○

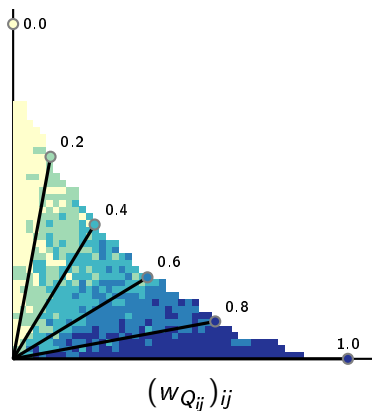
limit shape
○○

determinism of the last box
○●

bumping routes
○○○

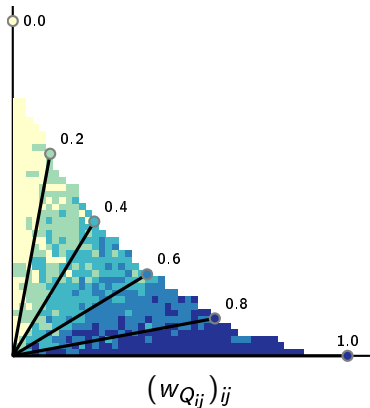
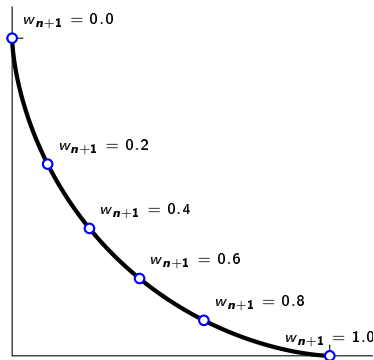
hydrodynamics
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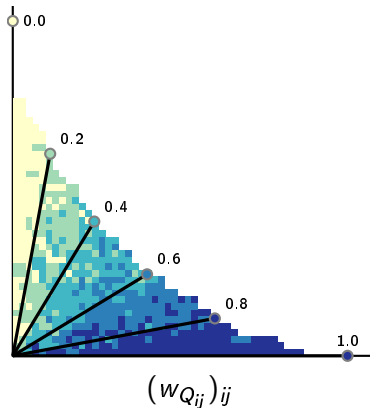
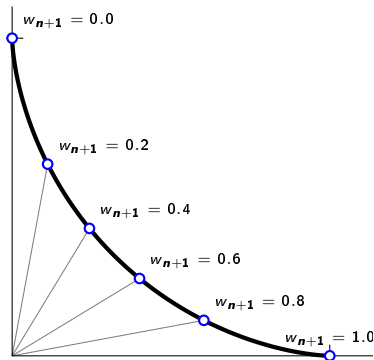
Theorem (ROMIK&ŚNIAŁDY 2015)

$$\left\| \frac{\boxed{}}{\sqrt{n}} - (\text{RSK} \cos w_{n+1}, \text{RSK} \sin w_{n+1}) \right\| \xrightarrow[n \rightarrow \infty]{\text{in probability}} 0$$



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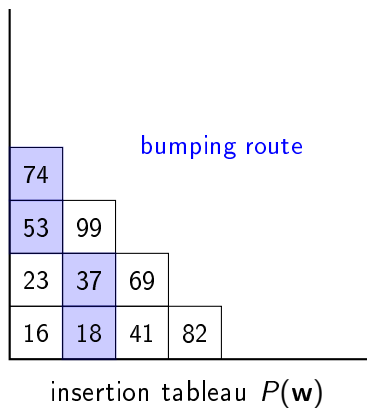
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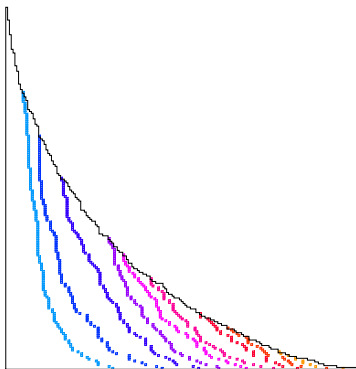
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bumping routes



$$\mathbf{w} = (23, 53, 74, 16, 99, 69, 82, 37, 41, \underbrace{18}_{w_n})$$

bumping routes



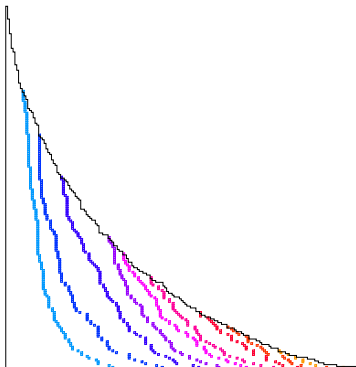
problem → MOORE 2006

what can we say about the
shapes of the bumping routes?

bumping routes

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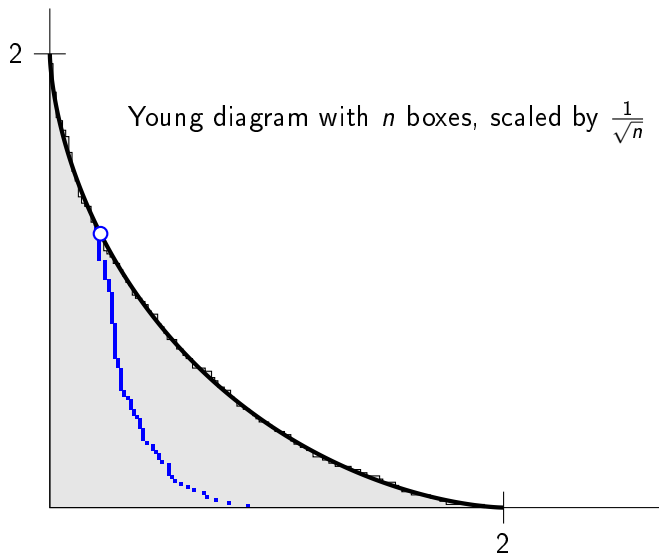
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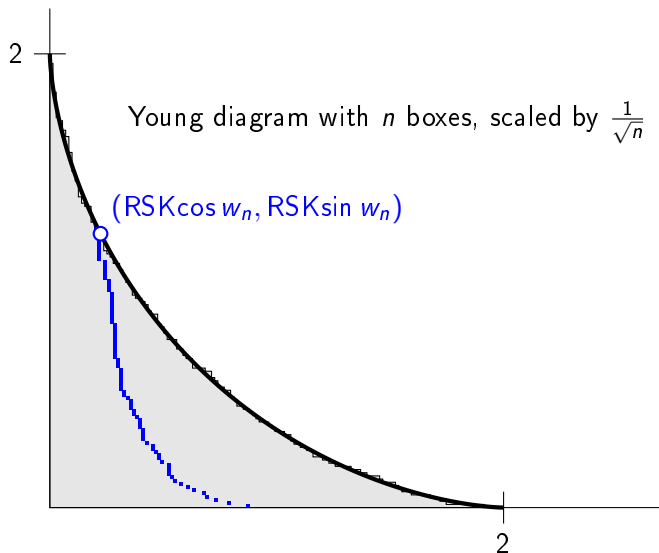
Theorem, ROMIK&ŚNIADY 2014

Bumping route (scaled by factor $\frac{1}{\sqrt{n} w_n}$)
obtained by adding entry w_n to the tableau P_{n-1}
converges in probability (as $n \rightarrow \infty$) to a deterministic curve G_T .

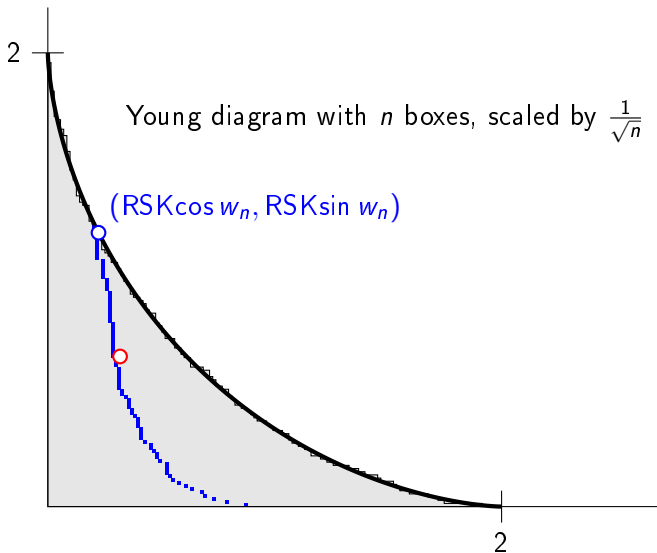
the key result explains the behavior of bumping routes



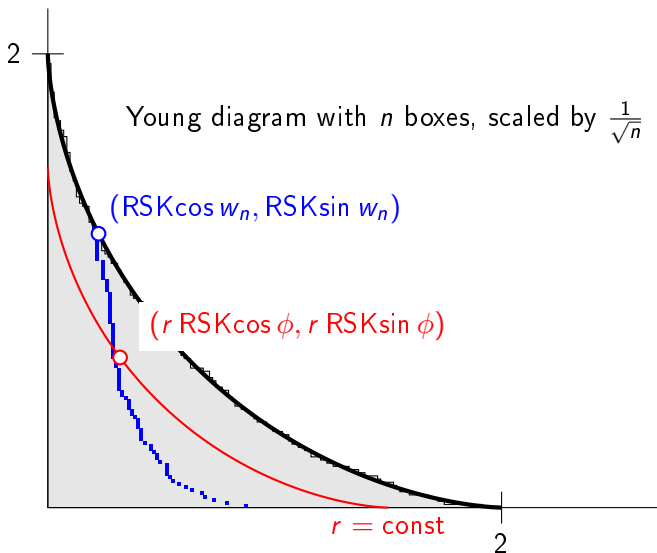
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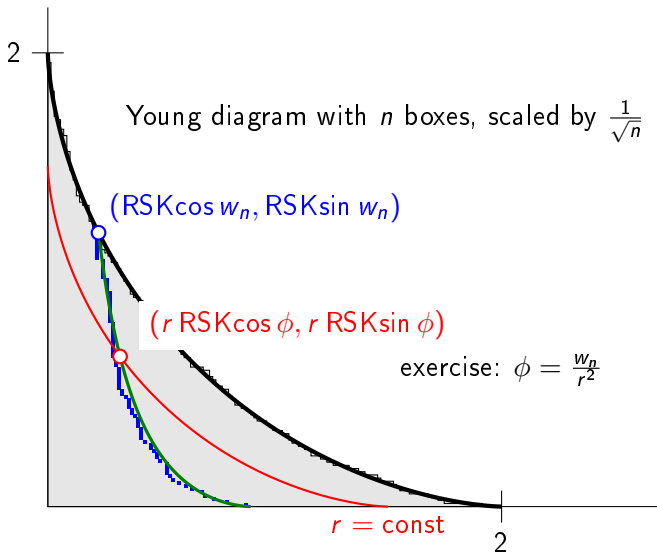
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diffusion of a box

- $\boxed{w_n}(P_m)$ denotes the location of the box containing w_n in the insertion tableau $P_m = P(w_1, \dots, w_m)$, for $m \geq n$;

problem

what can we say about the time evolution of $\boxed{w_n}(P_m)$ for $m = n, n+1, \dots$?

diffusion of a box

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Theorem (ŚNIADY, never published)

There exists an explicit function $G : \mathbb{R}_+ \rightarrow \mathbb{R}_+^2$ such that

$$\frac{\boxed{w_n}(P_{\lfloor ne^\tau \rfloor})}{\sqrt{n} w_n} \xrightarrow[n \rightarrow \infty]{\text{in probability}} G_\tau \quad \text{for } \tau \geq 0.$$

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exercise

prove this result using 'asymptotic determinism of last box insertion'

Hint: if \mathbf{w} is a permutation and $\text{RSK}(\mathbf{w}) = (P, Q)$ then $\text{RSK}(\mathbf{w}^{-1}) = (Q, P)$.

hydrodynamic limit of RSK

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exercise

- the above theorem concerns movement of a **single** particle; what can we say about **collective** movement of the fluid particles?
if we consider **transformations of the quarterplane** describing the time-evolution of the insertion tableau P : in which topology the convergence holds true?
- write a paper about it, add ŠNIADY as coauthor if you like,