Studying Affine Deligne Lusztig varieties via folded galleries in buildings

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Joint work with Elizabeth Milićević (Haverford College)
and Anne Thomas (University of Sydney)

CIRM Luminy
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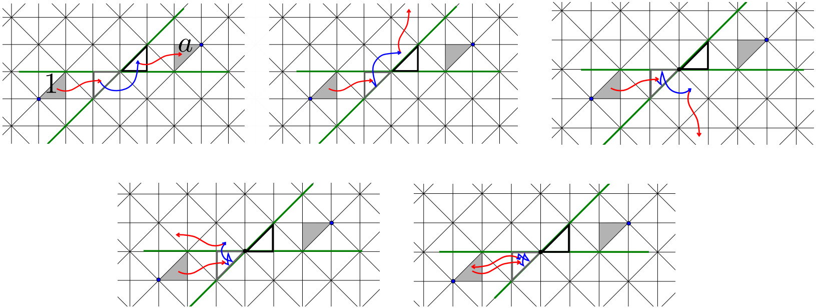
Explicit construction of a folded gallery

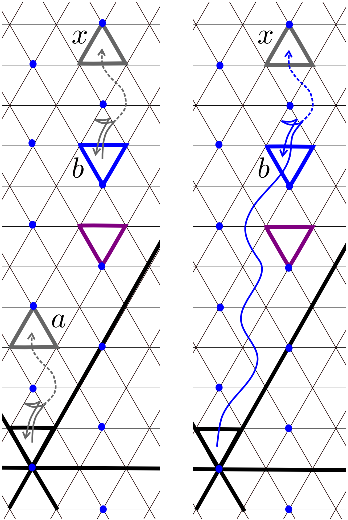
Construct a gallery from 1 to 1 of type $a = t^{2\rho} w_0$:



This gallery is positively folded for the right choice of a "sun".

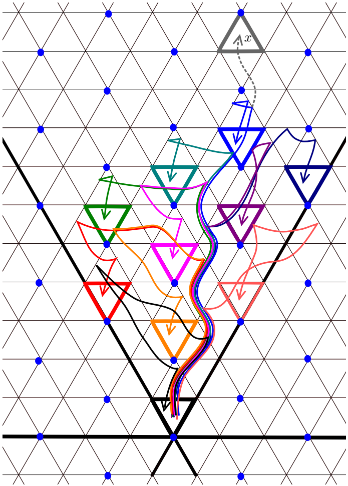
Manipulation of folded galleries

Translate and concatenate with a minimal gallery:

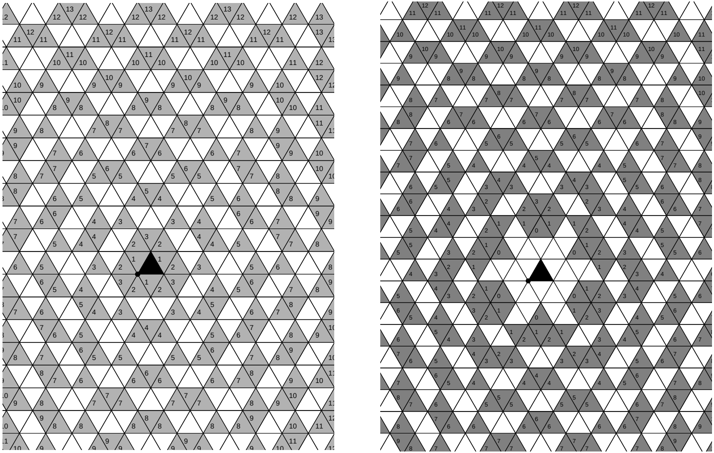


Root operators

Apply available root operators to the gallery constructed on the previous slide:



Folded galleries and ADLVs



pictures by Görtz–Haines–Kottwitz–Reuman, arXiv:0504443

Notation

Finite Fields:

- $\mathbb{F}_q$ finite field order $q = p^m$
- $k = \overline{\mathbb{F}}_q$ algebraic closure
- $\sigma : \overline{\mathbb{F}}_q \to \overline{\mathbb{F}}_q : a \mapsto a^q$ Frobenius automorphism

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Local Fields:

- $F = k((t))$ the field of Laurent series is a nonarchimedean local field of characteristic $p > 0$

$$F \ni \sum_{j \geq m} a_j t^j \quad \text{for some } m \in \mathbb{Z} \text{ and } a_j \in k$$

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- the Frobenius extends linearly to $\sigma : F \to F$
- $\mathcal{O} = k[[t]]$ power series, ring of integers of F

$$\mathcal{O} \ni \sum_{j \geq 0} a_j t^j \quad \text{with } a_j \in k$$

Affine flag variety

- $F = k((t))$ where $k = \overline{\mathbb{F}}_q$
- $\mathcal{O} = k[[t]]$
- project $\mathcal{O} \to k$ by setting $t = 0$, detects constant term a_0

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Recall:
The *affine flag variety* is the quotient $G(F)/I$, where we have

- G a split connected reductive group over $\mathbb{F}_q$,
- $B \subset G$ a Borel containing a split maximal torus T and
- I the *Iwahori subgroup* of $G(F)$ which is the inverse image of $B(k)$ under the projection $G(\mathcal{O}) \to G(k)$.

Definition of ADLVs

G split connected reductive over $\mathbb{F}_q$
I Iwahori subgroup
W the affine Weyl group
$k = \overline{\mathbb{F}}_q$, $F = k((t))$, σ the Frobenius map
$G(F) = \sqcup_{x \in W} IxI$

Definition

The *affine Deligne–Lusztig variety* $X_x(b) \subseteq G(F)/I$ is given by

$$X_x(b) = \{g \in G(F)/I \mid g^{-1} b\sigma(g) \in IxI\},$$

where $x \in W, b \in G(F)$.

ADLVs were introduced by Rapoport (2000).

Main Questions

Nonemptiness: For which $(x, b) \in W \times W$ is $X_x(b) \neq \emptyset$?

Dimension: What is the dimension of $X_x(b)$?

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In case b is *basic* these questions are solved:

- Beazley=Milićević, Görtz-Haines-Kottwitz-Reuman, Reuman, Görtz-He, He, ...
- Görtz, He and Nie (2012):
  nonemptiness pattern for all x and all basic b
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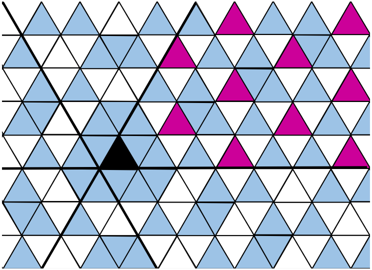
- Yang (2014): nonemptiness and dimension (x, b arbitrary) in case SL_3, that is type $\tilde{A}_2$.

The *basic* case

An element $b \in G(F)$ is basic if it is σ-conjugate to an element of length 0 in the extended affine Weyl group.

- All basic b in W are pairwise σ-conjugate.
- Dominant translations (pink) are not basic and pairwise not σ-conjugate.



basic elements (blue); translations in the dominant Weyl chamber (pink)

Our approach

In the following let $b = t^{\lambda}$ be a translation in W.
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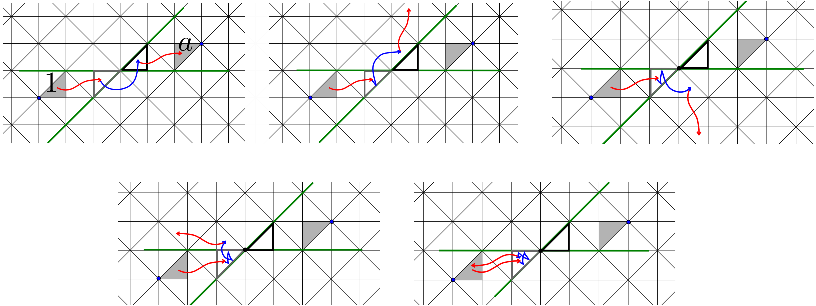
(3) Construct and manipulate such galleries using root operators, combinatorics in Coxeter complexes and explicit transformations.

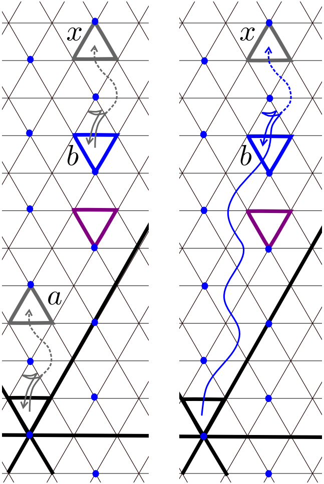
Explicit construction of a folded gallery

Construct a gallery from 1 to 1 of type $a = t^{2\rho} w_0$:



This implies that $X_a(1) \neq \emptyset$ and of dimension ≥ 7.

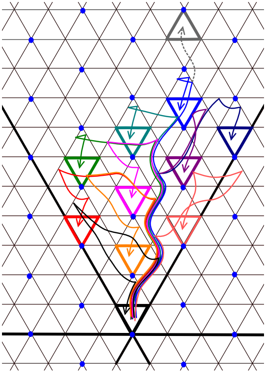
Manipulation of folded galleries



This implies $X_x(b) \neq \emptyset$ for $b = t^\mu$ dominant and close to x.

Root operators

Apply available root operators to the gallery constructed on the previous slide:



This implies $X_x(b) \neq \emptyset$ for most $b = t^\mu$ between 1 and x.

Theorem 1 (Milićević–S–Thomas)

Let $b = t^{\mu}$ be a pure translation and let $x = t^{\lambda} w \in W$. Assume that b is in the convex hull of x and the base alcove + two technical conditions on μ and λ. Then

$$X_x(1) \neq \emptyset \implies X_x(b) \neq \emptyset$$

and if $w = w_0$ then $X_x(1) \neq \emptyset$ and $X_x(b) \neq \emptyset$. If both varieties are nonempty then

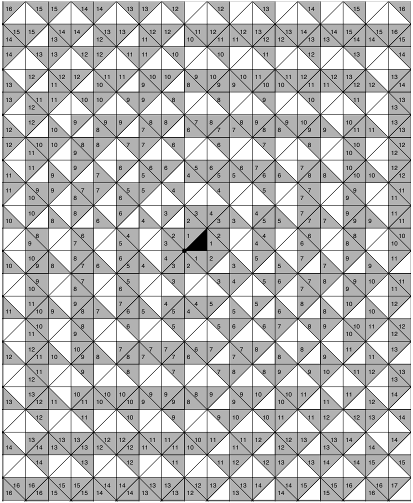
$$\dim X_x(b) = \dim X_x(1) - \langle \rho, \mu^{+} \rangle.$$

Precise assumptions:

- $t^{\lambda} w_0$ and $t^{-\mu} x$ are in the shrunken dominant Weyl chamber $\widetilde{\mathcal{C}}_f$
- b is in the convex hull of x and the base alcove
- μ lies in the negative cone based at $\lambda - 2\rho$.

Theorem 1 in type $\tilde{A}_2$

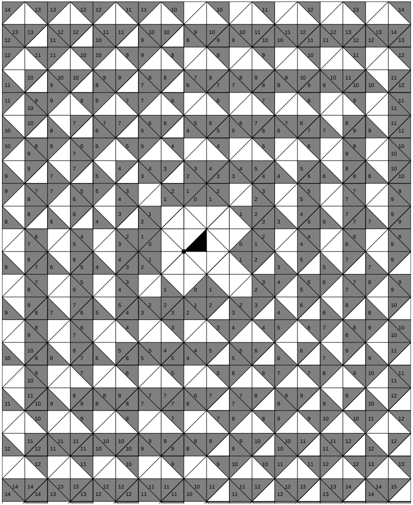
$b = 1$



picture: Görtz, Haines, Kottwitz and Reuman (2006)

Theorem 1 in type $\tilde{A}_2$

$b = t^{(1,0)}$



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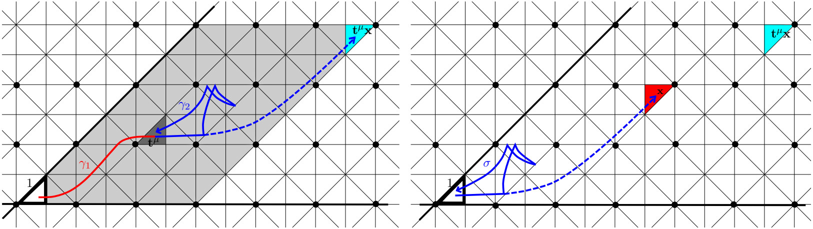
Geometric transformations of galleries



$$X_{t^\mu x}(t^\mu) \neq \emptyset \qquad \Longleftarrow \qquad X_x(1) \neq \emptyset$$

Theorem 2 (Milićević–S–Thomas)

If $b = t^\mu$ *is in the convex hull of* $t^\mu x$ *and the base alcove, then*

$$X_x(1) \neq \emptyset \implies X_{t^\mu x}(t^\mu) \neq \emptyset$$

and if nonempty then $\dim X_{t^\mu x}(t^\mu) \geq \dim X_x(1)$.

Thank you!

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Our preprint is available at `arxiv:1504.07076`.

More results

Conjugation

Theorem 3 (Milićević–S–Thomas)

Let $b = t^{\mu}$ *be a dominant pure translation and let* $x = t^{\lambda}w \in W$*. Assume that* $t^{\lambda}w_0$ *lies in the shrunken dominant Weyl chamber. Then for all* $u \in W_0$

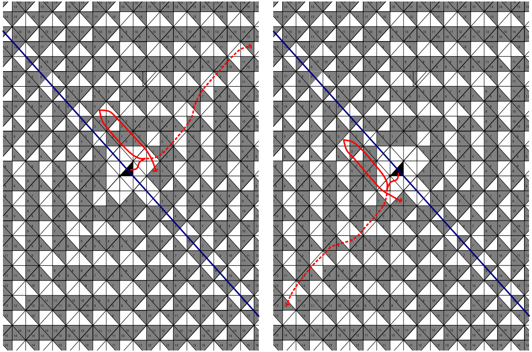
$$X_x(b) \neq \emptyset \implies X_{u^{-1}xu}(b) \neq \emptyset.$$

Moreover if these varieties are nonempty then

$$\dim X_{u^{-1}xu}(b) \geq \dim X_x(b) - \frac{1}{2}(\ell(u^{-1}xu) - \ell(x)).$$

Diagram automorphisms

Let g be an automorphism of the apartment induced by an automorphism of the diagram.



Theorem 4

$X_x(b) \neq \emptyset \iff X_{g(x)}(g(b)) \neq \emptyset$.
If both are not empty they have the same dimension.

Arbitrary translation alcoves

Theorem 5 (Milićević–S–Thomas)

Let $b = t^{\mu}$ be a pure translation and let $x \in W$. Assume that

- *b is in the convex hull of x and the base alcove*
- *x and $t^{-\mu}x$ lie in the same Weyl chamber*
- *if x is in a shrunken Weyl chamber then $t^{-\mu}x$ is in a shrunken Weyl chamber*

Then

$$X_x(1) \neq \emptyset \implies X_x(b) \neq \emptyset.$$

Moreover if these varieties are nonempty then

$$\dim X_x(b) \geq \dim X_x(1) - \langle \rho, \mu^+ \rangle - \langle \rho_{B^-}, \mu + \mu_{B^-} \rangle.$$

The p-adic setting

Theorem 6 (Milićević–S–Thomas)

Let b be a translation and let $x \in W$. There is a reasonable combinatorial definition of $\dim X_x(b)_{\mathbb{Q}_p}$, and using this definition

$$\dim X_x(b) = \dim X_x(b)_{\mathbb{Q}_p}.$$

Corollary

All previous Theorems hold in the p-adic setting.

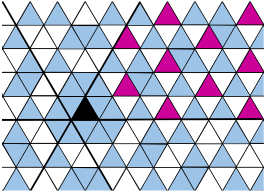
Basic case, previous results, conjectures

Previous results

Almost all previous results are in the basic case.

Common approach in the basic case:

- generalisation of classical Deligne–Lusztig theory,
- combinatorics on minimal length elements in conjugacy classes in the affine Weyl group W.



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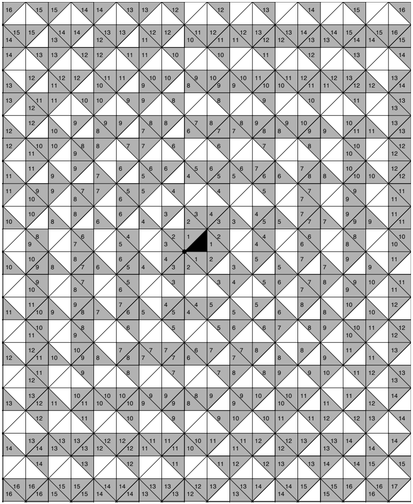
Computer experiments

In $G(F)/I$, Görtz, Haines, Kottwitz and Reuman (2006) conducted computer experiments in low rank; *e.g.* for $b = 1$:



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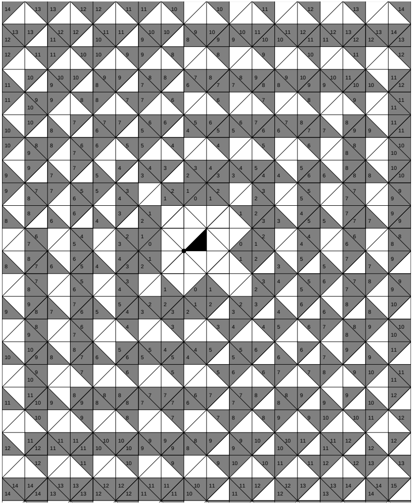


Conjecture for arbitrary b

Conjecture (Görtz-Haines-Kottwitz-Reuman 2010)

Let $b \in G(F)$. Then there exists $N_b \in \mathbb{N}$ such that for all $x \in \widetilde{W}$ with $\ell(x) > N_b$

$$X_x(b) \neq \emptyset \Longleftrightarrow X_x(\hat{b}) \neq \emptyset$$

and if both varieties are nonempty then

$$\dim X_x(b) = \dim X_x(\hat{b}) - \frac{1}{2}\left(\langle 2\rho, \nu_b\rangle + \mathrm{def}_G(b) - \mathrm{def}_G(\hat{b})\right).$$

Here $\hat{b}$ is an associated basic element, def_G is the defect, and ν_b is the Newton point parameterizing the σ-conjugacy class.

- (Yang 2014) This conjecture holds for $G = SL_3$.

Conjecture for b a translation

If $b = t^\mu$ is a pure translation, the previous conjecture simplifies:

Conjecture (Görtz-Haines-Kottwitz-Reuman 2010)

Let $b = t^\mu \in \widetilde{W}$ be a translation. There exists $N_b \in \mathbb{N}$ such that for all $x \in \widetilde{W}$ with $\ell(x) > N_b$

$$X_x(1) \neq \emptyset \iff X_x(b) \neq \emptyset,$$

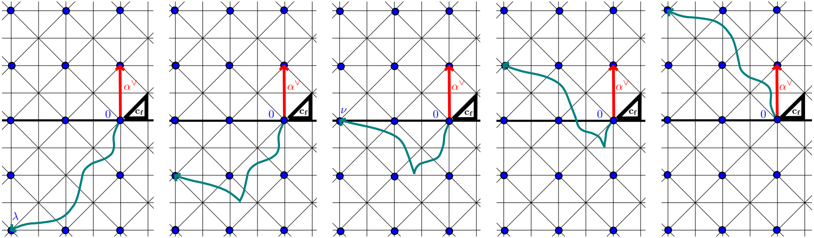
and if both varieties are nonempty then

$$\dim X_x(b) = \dim X_x(1) - \langle \rho, \mu^+ \rangle.$$

Root operators

Root operators

Root operators e_α, f_α for simple roots α were defined by Gaussent and Littelmann (2005). They act on sets of positively folded galleries of a fixed type.



$f_\alpha^2(\gamma)$ $f_\alpha(\gamma)$ γ $e_\alpha(\gamma)$ $e_\alpha^2(\gamma)$

Using properties of e_α and f_α we can easily control end-vertices and dimensions of galleries. If we work very carefully, we can also control start-alcoves and end-alcoves.
Availability of all root operators is crucial to our constructions.