Center of Walled Brauer algebra and the Supersymmetric polynomials

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Jucys-Murphy elements in Symmetric groups

- $\mathfrak{S}_r$: symmetric group of r letters.
- $\mathbb{C}[\mathfrak{S}_r]$: the Group algebra of $\mathfrak{S}_r$ over $\mathbb{C}$.
- (a, b) : the transposition exchanging a and b $(1 \leq a, b \leq r)$.
- The elements

$$L_k := \sum_{j=1}^{k-1} (j, k) \quad (1 \leq k \leq r)$$

are called the *Jucys-Murphy elements* of $\mathbb{C}[\mathfrak{S}_r]$.

Properties of Jucys-Murphy elements in $\mathbb{C}[\mathfrak{S}_r]$

1. $L_k L_{k'} = L_{k'} L_k$ for $1 \leq k, k' \leq r$.
2. $\{p(L_1, \ldots, L_r) \mid p$ is a symmetric polynomial in $X_1, \ldots, X_r\}$ is equal to the center $Z(\mathbb{C}[\mathfrak{S}_r])$ of $\mathbb{C}[\mathfrak{S}_r]$.
3. On a simple module $S(\lambda)$ of $\mathbb{C}[\mathfrak{S}_r]$ corresponding to a partition λ of r, the central element $p(L_1, \ldots, L_r)$ acts by a scalar multiplication given by the *content evaluation*:

$$p(L_1, \ldots, L_r) = p(\mathrm{cont}(\lambda, 1), \ldots, \mathrm{cont}(\lambda, r))$$

on $S(\lambda)$, where $\mathrm{cont}(\lambda, k)$ denotes the *content of λ at i*.

Properties of Jucys-Murphy elements in $\mathbb{C}[\mathfrak{S}_r]$

Recall that, for a tableau $\mathfrak{t}$ of a shape λ, $c_{\mathfrak{t}}(k)$ is defined by the integer $u - v$, when k is located at the position (u, v) on λ.
Set $\mathrm{cont}(\lambda, k) := c_{\mathfrak{t}^\lambda}(k)$, where $\mathfrak{t}^\lambda$ is the standard tableau such that the entries $1, 2, \ldots, |\lambda|$ appear in increasing order from left to right along successive rows.

Example:

$$\lambda = (4, 2, 1), \qquad \mathfrak{t}^\lambda := \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array} .$$

We have
$\mathrm{cont}(\lambda, 1) = 0$, $\mathrm{cont}(\lambda, 2) = 1$, $\mathrm{cont}(\lambda, 3) = 2$, $\mathrm{cont}(\lambda, 4) = 3$,
$\mathrm{cont}(\lambda, 5) = -1$, $\mathrm{cont}(\lambda, 6) = 0$,
$\mathrm{cont}(\lambda, 7) = -2$.

$$\begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 3 \\ \hline -1 & 0 \\ \cline{1-2} -2 \\ \cline{1-1} \end{array}$$

Aim of talk

- We are interested in the *walled Brauer algebra* $B_{r,s}(\delta)$, which can be regraded as a generalization of $\mathbb{C}[\mathfrak{S}_r]$ in the context of Schur-Weyl duality:

  - $\mathbb{C}[\mathfrak{S}_r]$ is the centralizer of the action of $\mathfrak{gl}_n$ on the tensor space $(\mathbb{C}^n)^{\otimes r}$

  - $B_{r,s}(n)$ is the centralizer of the action of $\mathfrak{gl}_n$ on the mixed tensor space $(\mathbb{C}^n)^{\otimes r} \otimes ((\mathbb{C}^n)^*)^{\otimes s}$.

- We will introduce the *Jucys-Murphy elements* for $B_{r,s}(\delta)$ which generalize the ones for $\mathbb{C}[\mathfrak{S}_r]$, and will explain a relation between these elements and the center of $B_{r,s}(\delta)$ via the *supersymmetric polynomials*.

Walled Brauer diagrams

Let r and s be nonnegative integers. An (r, s)-*walled Brauer diagram* is a graph consisting of two rows with $r + s$ vertices in each row such that the following conditions hold:

(1) Each vertex is connected by a strand to exactly one other vertex.

(2) There is a vertical wall which separates the first r vertices from the last s vertices in each row.

(3) There are two kinds of strands: A *vertical strand* connects a vertex on the top row with one on the bottom row, and it cannot cross the wall. A *horizontal strand* connects vertices in the same row, and it must cross the wall.

Fix a complex number δ.

$B_{r,s}(\delta) :=$ the $\mathbb{C}$-vector space spanned by the basis consisting of all the (r, s)-walled Brauer diagrams.

Walled Brauer algebra

Multiplication of (r, s)-walled Brauer diagrams:

For (r, s)-walled Brauer diagrams d_1, d_2, put d_1 under d_2 and identify the top vertices of d_1 with the bottom vertices of d_2.

Remove the loops in the middle row, if there exist.

Then thus obtained diagram, denoted by $d_1 * d_2$, is again an (r, s)-walled Brauer diagram.

We define the multiplication of d_1 by d_2

$$d_1 d_2 := \delta^n \, d_1 * d_2 \in B_{r,s}(\delta),$$

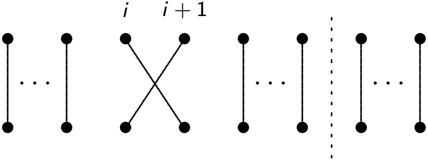
where n denotes the number of loops we removed in the middle row.

Extending this multiplication by linearity, we obtain a multiplication on $B_{r,s}(\delta)$, which can be easily seen to be associative.

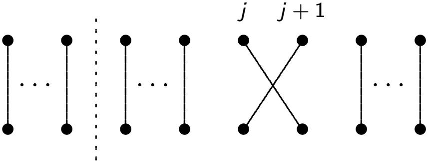
We call thus obtained $\mathbb{C}$-algebra the *walled Brauer algebra*.

Generators of $B_{r,s}(\delta)$

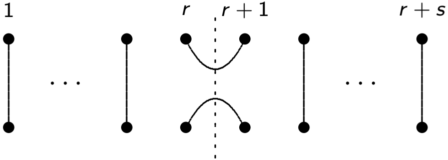
For $1 \leq i \leq r-1$ and $r \leq j \leq r+s-1$:

$s_i :=$ 

,

$s_j :=$ 

,

$e_{r,r+1} :=$ 

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Presentation of $B_{r,s}(\delta)$

The algebra $B_{r,s}(\delta)$ is generated by s_i $(1 \leq i \leq r-1)$, s_j $(r \leq j \leq r+s-1)$, and $e_{r,r+1}$ with the following defining relations:

$$\begin{aligned}
&s_i^2 = 1, \\
&s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_i s_j = s_j s_i \;\; (|i-j|>1), \\
&e_{r,r+1}^2 = \delta e_{r,r+1}, \quad e_{r,r+1} s_j = s_j e_{r,r+1} \;\; (j \neq r-1, r+1), \\
&e_{r,r+1} = e_{r,r+1} s_{r-1} e_{r,r+1} = e_{r,r+1} s_{r+1} e_{r,r+1}, \\
&e_{r,r+1} s_{r-1} s_{r+1} e_{r,r+1} s_{r-1} = e_{r,r+1} s_{r-1} s_{r+1} e_{r,r+1} s_{r+1}, \\
&s_{r-1} e_{r,r+1} s_{r-1}^{-1} s_{r+1} e_{r,r+1} = s_{r+1} e_{r,r+1} s_{r-1} s_{r+1} e_{r,r+1}.
\end{aligned}$$

Semisimplicity criterion

Proposition. (Cox-De Visscher-Doty-Martin, 2008) The walled Brauer algebra $B_{r,s}(\delta)$ is semisimple if and only if one of the followings holds:

(1) $r = 0$ or $s = 0$,

(2) $\delta \notin \mathbb{Z}$,

(3) $|\delta| > r + s - 2$,

(4) $\delta = 0, \quad \text{and} \quad (r, s) \in \{(1, 2), (1, 3), (2, 1), (3, 1)\}$.

In particular, for a fixed pair (r, s), $B_{r,s}(\delta)$ is semisimple except for finitely many values $\delta \in \mathbb{C}$.

Cell modules

For a partition $\mu = (\mu_1, \mu_2, \cdots)$, set $|\mu| := \sum_{i \geq 1} \mu_i$ and set $\ell(\mu) := |\{i \geq 1 \mid \mu_i \neq 0\}|$.
Let us denote Λ the set of pair of partitions (bipartitions). Set

$$\Lambda_{r,s}^t := \left\{ (\lambda^L, \lambda^R) \in \Lambda \mid |\lambda^L| = r - t,\ |\lambda^R| = s - t \right\},$$

$$\Lambda_{r,s} := \bigsqcup_{t=0}^{\min(r,s)} \Lambda_{r,s}^t,$$

$$\dot{\Lambda}_{r,s} = \begin{cases} \Lambda_{r,s} & \text{if } \delta \neq 0 \text{ or } r \neq s \text{ or } r = s = 0, \\ \Lambda_{r,s} - \{(\emptyset, \emptyset)\} & \text{if } \delta = 0 \text{ and } r = s \neq 0. \end{cases}$$

The elements in $\dot{\Lambda}_{r,s}$ are called the *weights* of $B_{r,s}(\delta)$.

Cell modules

For $\lambda \in \dot{\Lambda}_{r,s}$, there is an indecomposable module, $C_{r,s}(\lambda)$, called the *cell module*, which has an irreducible head $D_{r,s}(\lambda)$ and the family

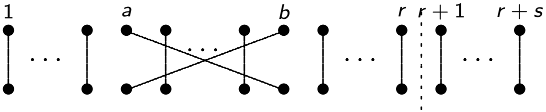
$$\left\{ D_{r,s}(\lambda) \mid \lambda \in \dot{\Lambda}_{r,s} \right\}$$

is the complete set of mutually non-isomorphic simple modules over $B_{r,s}(\delta)$ (Cox-De Visscher-Doty-Martin, 2008).

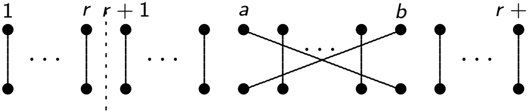
Jucys-Murphy elements

Conventions:

$(a, b) = (b, a) =$

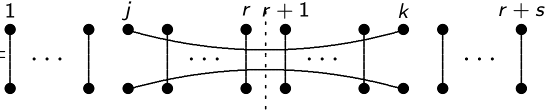


if $1 \leq a < b \leq r$,



if $r + 1 \leq a < b \leq r + s$,

and

$e_{j,k} =$ 

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Jucys-Murphy elements

Definition

For each $1 \leq k \leq r+s$, we define

$$L_k := \begin{cases} \sum_{j=1}^{k-1} (j,k) & \text{if } 1 \leq k \leq r, \\ -\sum_{j=1}^{r} e_{j,k} + \sum_{j=r+1}^{k-1} (j,k) + \delta & \text{if } r+1 \leq k \leq r+s. \end{cases}$$

We call these L_k's the *Jucys-Murphy elements of* $B_{r,s}(\delta)$.

Proposition

The elements L_k*'s are commuting to each other.*

Remark

The above Jucys-Murphy elements are similar to those in Brundan-Stroppel (2012) and those in Sartori-Stroppel (2015), but different. Still, we were strongly motivated by their works.

Jucys-Murphy elements

The Jucys-Murphy elements are designed to satisfy the following:

Proposition

For each $k \in \mathbb{Z}_{\geq 0}$, the element

$$L_1^k + \cdots + L_r^k + (-1)^{k+1}(L_{r+1}^k + \cdots + L_{r+s}^k)$$

belongs to the center of $B_{r,s}(\delta)$.

The elements in the above is known as
the *k-th power sum supersymmetric polynomial* in $L_1, \ldots, L_{r+s}$.

Supersymmetric polynomials

Definition

Let r, s be nonnegative integers. We say that a polynomial p in $\mathbb{C}[x_1, \ldots, x_r, y_1 \ldots, y_s]$ is *supersymmetric* if

(1) p is *doubly symmetric*; i.e, it is symmetric in $x_1, \ldots, x_r$ and $y_1, \ldots, y_s$ separately,

(2) p satisfies the *cancellation property*; i.e., the substitution $x_r = -y_1 = t$ yields a polynomial in $x_1, \ldots, x_{r-1}, y_2, \ldots, y_s$ which is independent of t.

We denote $S_{r,s}[x; y]$ the set of supersymmetric polynomials in $x_1, \ldots, x_r, y_1, \ldots, y_s$, which is a subalgebra of the polynomial ring $\mathbb{C}[x_1, \ldots, x_r, y_1, \ldots, y_s]$.

Supersymmetric polynomials

Example. In $S_{2,2}[x; y]$

1. $p_1 := x_1 + x_2 + y_1 + y_2$.
2. $p_2 := x_1^2 + x_2^2 - (y_1^2 + y_2^2)$.
3. $e_2 := x_1 x_2 + (x_1 + x_2)(y_1 + y_2) + y_1 y_2 + y_1^2 + y_2^2$.
   Indeed, it is doubly symmetric, and the substitution
   $x_2 = t, y_1 = -t$ gives
   $x_1 t + (x_1 + t)(-t + y_2) + (-t)y_2 + (-t)^2 + y_2^2 = x_1 y_2 + y_2^2$,
   which is independent to t.

Power sum supersymmetric polynomials

For $k \geq 1$, the *k-th power sum supersymmetric polynomial* is given by

$$p_k(x_1, \ldots, x_r, y_1, \ldots, y_s) := x_1^k + \cdots + x_r^k + (-1)^{k+1}(y_1^k + \cdots + y_s^k).$$

It is known that $S_{r,s}[x; y]$ is generated by $\{p_k \mid k \geq 1\}$ (Stembridge(1985)).

Corollary

For every supersymmetric polynomial p in $S_{r,s}[x; y]$, the element

$$p(L_1, \ldots, L_{r+s})$$

belongs to the center $Z(B_{r,s}(\delta))$ of $B_{r,s}(\delta)$.

Elementary supersymmetric polynomials

The *elementary supersymmetric polynomials*

$$e_k(x_1, \ldots, x_r, y_1, \ldots, y_s) \quad (k \in \mathbb{Z}_{\geq 0})$$

are given by the generating function

$$\sum_{k=0}^{\infty} e_k(x_1, \ldots, x_r, y_1, \ldots, y_s) z^k = \frac{\prod_{i=1}^{r}(1 + x_i z)}{\prod_{j=1}^{s}(1 - y_j z)}.$$

It is also known that $\{e_k \mid k \in \mathbb{Z}_{\geq 0}\}$ generates the ring of supersymmetric polynomials (Stembridge, 1985). Then the next lemma follows immediately.

Elementary supersymmetric polynomials

Lemma

Let $(a_1, \ldots, a_r, b_1, \ldots, b_s)$ and $(c_1, \ldots, c_r, d_1, \ldots, d_s)$ be elements in $\mathbb{C}^{r+s}$. Then the followings are equivalent:

(1) *For every supersymmetric polynomial $p \in S_{r,s}[x; y]$, we have*

$$p(a_1, \ldots, a_r, b_1, \ldots, b_s) = p(c_1, \ldots, c_r, d_1, \ldots, d_s).$$

(2) *We have an equality*

$$\frac{\prod_{i=1}^{r}(1 + a_i z)}{\prod_{j=1}^{s}(1 - b_j z)} = \frac{\prod_{i=1}^{r}(1 + c_i z)}{\prod_{j=1}^{s}(1 - d_j z)}$$

of rational functions in z.

Contents evaluation of supersymmetric polynomials

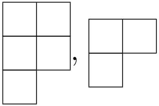
For a partition μ, let $\mathfrak{t}^\mu$ be the standard tableau such that the entries $1, 2, \ldots, |\mu|$ appear in increasing order from left to right along successive rows. Define

$$\mathrm{cont}(\mu, i) := c_{\mathfrak{t}^\mu}(i), \quad \text{the content of } \mathfrak{t}^\mu \text{ at } i.$$

Note that the multiset $\{\mathrm{cont}(\mu, i) \mid 1 \le i \le |\mu|\}$ determines the Young diagram $[\mu]$ and hence the partition μ.
For each $\lambda \in \Lambda^t_{r,s}$, set

$$c(\lambda, i) := \begin{cases} \mathrm{cont}(\lambda^L, i) & \text{if } 1 \le i \le r - t, \\ 0 & \text{if } r - t + 1 \le i \le r + t, \\ \mathrm{cont}(\lambda^R, i - r - t) + \delta & \text{if } r + t + 1 \le i \le r + s. \end{cases}$$

Contents evaluation of supersymmetric polynomials

Example. $r = 6,\ s = 4,\ t = 1.$ $\lambda = \Big($  $\Big) \in \Lambda^1_{6,4}$. Then

$$c(\lambda, 1) = 0,\ c(\lambda, 2) = 1,\ c(\lambda, 3) = -1,\ c(\lambda, 4) = 0,\ c(\lambda, 5) = -2,$$
$$c(\lambda, 6) = 0,\ c(\lambda, 7) = 0,$$
$$c(\lambda, 8) = 0 + \delta,\ c(\lambda, 9) = 1 + \delta,\ c(\lambda, 10) = -1 + \delta.$$

Contents evaluation of supersymmetric polynomials

Lemma

Assume that $B_{r,s}(\delta)$ is semisimple. If $\lambda \neq \mu$ for $\lambda, \mu \in \dot{\Lambda}_{r,s}$, then there is a supersymmetric polynomial $p^{\lambda,\mu}$ such that

$$p^{\lambda,\mu}((c(\lambda,i))_{1\le i\le r+s}) \neq p^{\lambda,\mu}((c(\mu,i))_{1\le i\le r+s}).$$

Proof. Let $\lambda \in \Lambda^{t}_{r,s}$, $\mu \in \Lambda^{t'}_{r,s}$ for some $0 \le t, t' \le \min(r,s)$.
It is enough to check that for each case when $B_{r,s}(\delta)$ is semisimple,

$$\frac{\prod_{i=1}^{r-t}(1+\mathrm{cont}(\lambda^L,i)z)}{\prod_{j=1}^{s-t}(1-(\mathrm{cont}(\lambda^R,j)+\delta)z)} = \frac{\prod_{i=1}^{r-t'}(1+\mathrm{cont}(\mu^L,i)z)}{\prod_{j=1}^{s-t'}(1-(\mathrm{cont}(\mu^R,j)+\delta)z)}.$$

Implies $\lambda = \mu$. One can check it for the cases (1) - (4) in the semisimplicity criterion. □

Contents evaluation of supersymmetric polynomials

Proposition

For $\lambda \in \Lambda^t_{r,s}$ and for $k \geq 0$, we have

$$\begin{aligned} & p_k(L_1, \ldots, L_r, L_{r+1}, \ldots, L_{r+s}) \\ &= L_1^k + \cdots + L_r^k + (-1)^{k+1}(L_{r+1}^k + \cdots + L_{r+s}^k) \\ &= \sum_{i=1}^{r-t} \mathrm{cont}(\lambda^L, i)^k + (-1)^{k+1} \sum_{i=1}^{s-t} (\mathrm{cont}(\lambda^R, i) + \delta)^k \\ &= p_k(c(\lambda, 1), \ldots, c(\lambda, r+s)) \end{aligned}$$

on the cell module $C_{r,s}(\lambda)$.

Corollary

Let f be a supersymmetric polynomial in $S_{r,s}[x; y]$. Then we have

$$f(L_1, \ldots, L_{r+s}) = f(c(\lambda, 1), \ldots, c(\lambda, r+s))$$

on $C_{r,s}(\lambda)$ for every $\lambda \in \Lambda_{r,s}$.

Supersymmetric polynomials in JM elements

Theorem

If the walled Brauer algebra $B_{r,s}(\delta)$ is semisimple, then the supersymmetric polynomials in $L_1, \dots, L_{r+s}$ generate the center of $B_{r,s}(\delta)$.

It can be proved by combining two facts in the previous pages:

1. If $\lambda \neq \mu$ for $\lambda, \mu \in \dot{\Lambda}_{r,s}$, then there is a supersymmetric polynomial $p^{\lambda,\mu}$ such that

$$p^{\lambda,\mu}((c(\lambda, i))_{1 \leq i \leq r+s}) \neq p^{\lambda,\mu}((c(\mu, i))_{1 \leq i \leq r+s}).$$

2. A supersymmetric polynomial p in Jucys-Murphy elements acts on the cell module $C_{r,s}(\lambda)$ by the contents evaluation $p(c(\lambda, i)_{1 \leq i \leq r+s})$.

Supersymmetric polynomials in JM elements

Conjecture

The supersymmetric polynomials in $L_1, \ldots, L_{r+s}$ generate the center of $B_{r,s}(\delta)$ for any choice of δ.

Definition (Kosuda-Murakami (1993), Leduc (1994))

Let r and s be nonnegative integers. Let R be an integral domain and let q, ρ be elements in R such that q^{-1}, ρ^{-1} and $\delta := \frac{\rho-\rho^{-1}}{q-q^{-1}}$ belongs to R. We denote by $H^R_{r,s}(q,\rho)$ the associative algebra over R generated by $S_1, \dots, S_{r-1}$, $S_{r+1}, \dots, S_{r+s-1}$, $E_{r,r+1}$ with the defining relations

$$
\begin{aligned}
&(S_i - q)(S_i + q^{-1}) = 0,\\
&S_i S_{i+1} S_i = S_{i+1} S_i S_{i+1}, \quad S_i S_j = S_j S_i \ \ (|i-j|>1),\\
&E_{r,r+1}^2 = \delta E_{r,r+1}, \quad E_{r,r+1} S_j = S_j E_{r,r+1} \ \ (j \neq r-1, r+1),\\
&\rho E_{r,r+1} = E_{r,r+1} S_{r-1} E_{r,r+1} = E_{r,r+1} S_{r+1} E_{r,r+1},\\
&E_{r,r+1} S_{r-1}^{-1} S_{r+1} E_{r,r+1} S_{r-1} = E_{r,r+1} S_{r-1}^{-1} S_{r+1} E_{r,r+1} S_{r+1},\\
&S_{r-1} E_{r,r+1} S_{r-1}^{-1} S_{r+1} E_{r,r+1} = S_{r+1} E_{r,r+1} S_{r-1}^{-1} S_{r+1} E_{r,r+1}.
\end{aligned}
$$

Definition

Set

$$\widetilde{L}_1 := 0, \quad \widetilde{L}_{r+1} := \rho\Big(\sum_{j=1}^{r} -E_{j,r+1} + \delta\Big) \in H^R_{r,s}(q,\rho),$$

where

$$E_{j,k} := (S_{k-1}\cdots S_{r+1})(S_j^{-1}\cdots S_{r-1}^{-1})E_{r,r+1}(S_{r-1}^{-1}\cdots S_j^{-1})(S_{r+1}\cdots S_{k-1})$$
$$\text{for } j \le r < k.$$

Then we define

$$\widetilde{L}_i := \begin{cases} S_i^{-1}\widetilde{L}_{i-1}S_i^{-1} + S_i^{-1} & \text{if } 2 \le i \le r, \\ S_i\widetilde{L}_{i-1}S_i + S_i & \text{if } r+2 \le i \le r+s \end{cases}$$

We call these $\widetilde{L}_i$'s the *Jucys-Murphy elements of* $H^R_{r,s}(q,\rho)$.

Center of quantized walled Brauer algebra

Remark

If we take a limit $q \mapsto 1, \rho \mapsto 1$, then we have

$$\widetilde{L}_i \longmapsto L_i \quad (i = 1, \ldots, r+s).$$

Theorem

The center $Z(H_{r,s}^{\mathbb{C}(q,\rho)}(q,\rho))$ of $H_{r,s}^{\mathbb{C}(q,\rho)}(q,\rho)$ is generated by the supersymmetric polynomials in the Jucys-Murphy elements $\widetilde{L}_1, \ldots, \widetilde{L}_{r+s}$.

Set

$$T(j) := -(q - q^{-1})\widetilde{L}_j + 1 \quad (1 \le j \le r),$$
$$U(k) := \rho^{-2}(q - q^{-1})\widetilde{L}_{k+r} + \rho^{-2} \quad (1 \le k \le s).$$

Corollary

The elements

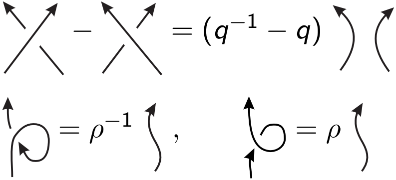
$$\sum_{j=1}^{r}(\rho^{-1}\, T(j))^m - \sum_{k=1}^{s}(\rho\, U(k))^m \quad (m \in \mathbb{Z}_{\ge 0})$$

generate the center $Z(H_{r,s}^{\mathbb{C}(q,\rho)}(q,\rho))$ *of* $H_{r,s}^{\mathbb{C}(q,\rho)}(q,\rho)$*.*

Proof. It is equivalent to saying that $p_m(\widetilde{L}_1, \dots, \widetilde{L}_{r+s})$ $(m \ge 1)$ generate the center. □

framed HOMFLYPT skein modules

Let R be the subring of $\mathbb{C}(q, \rho)$ generated by $q^{\pm 1}, \rho^{\pm 1}$ and $(q^k - q^{-k})^{-1} (k \in \mathbb{Z}_{>0})$.
Let F be a planar surface with some designated input and output boundary points.
The framed HOMFLYPT skein module $\mathcal{S}(F)$ of F is a R-linear combination of oriented tangle diagrams on F, modulo Reidemeister move II and III, and the two additional local relations:



(Switch and smooth)

(Framing change)

(∗These diagrams are taken from [Morton-Samuelson, 2015])

framed HOMFLYPT skein modules

We consider the following cases:

1. $F = \mathcal{A}$ is the annulus without designated boundary points, so that $\mathcal{S}(\mathcal{A})$ consists of oriented link diagrams on the annulus. Then $\mathcal{S}(\mathcal{A})$ has a product induced by placing two diagrams together on the annulus (the product is commutative).
2. $F = \mathcal{R}_{r,s}^{r,s}$ is the rectangle with r outputs and s inputs on the top, and r inputs and s outputs on the bottom. In $\mathcal{S}(\mathcal{R}_{r,s}^{r,s})$, a tangle diagram is either a link diagram or a matching two designated boundary points according to the orientation. The product on $\mathcal{S}(\mathcal{R}_{r,s}^{r,s})$ is defined just by stacking two diagrams.

   Then $\mathcal{S}(\mathcal{R}_{r,s}^{r,s})$ is isomorphic to $H_{r,s}^{R}(q,\rho)$ as an R-algebra.

framed HOMFLYPT skein modules and the center

In 2002, Hugh Morton introduced an interesting R-algebra homomorphism

$$\psi_{r,s} : \mathcal{S}(\mathcal{A}) \to Z(H^R_{r,s}(q,\rho)).$$

Theorem (H. Morton (2002))

$$\psi_{r,s}(P_m) = (q^{-m} - q^m)\Big(\sum_{j=1}^{r} (\rho^{-1}\, T(j))^m - \sum_{k=1}^{s} (\rho\, U(k))^m \Big) + \langle P_m \rangle \mathbf{1}$$

where $\langle P_m \rangle \in R$.

Corollary (Conjecture in Morton (2002))

The image of $\psi_{r,s}$ *generates the center of* $H^R_{r,s}(q,\rho)$.