Triple crystal for Fock spaces

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Algebraic Combinatorics in Representation Theory
CIRM, Luminy

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Motivation: classical objects in representation theory

1. **(Some) finite classical groups**
   $G_n(q) \in \{GL_n(q), GU_n(q), Sp_{2n}(q), SO_{2n+1}(q)\}$ where $q =$ power of a prime p.

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   Representation theory over $\mathbb{C}$: ✓(Lusztig 1970's)
   Classification of the irreducible representations of $G_n(q)$ via *Harish-Chandra theory*. Ingredients:
   - Harish-Chandra induction functor,
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   Representation theory over $k = \bar{k}$ with $\mathrm{char}(k) = m > 0$: **?**
   Important questions:
   - modular Harish-Chandra theory ?
     - classic approach (Geck-Hiss-Malle 1990's)
     - *weak* approach (G.-Hiss-Jacon 2015)
   - branching rules?

   First step: restriction to the category of *unipotent* representations.

Motivation: classical objects in representation theory

2. **Cyclotomic rational Cherednik algebras**
   $W_n = (\mathbb{Z}/\ell\mathbb{Z}) \wr \mathfrak{S}_n = G(\ell, 1, n)$. V=reflection representation of W_n.
   Parameter $\mathbf{c} : \{\text{reflections of } W_n\} \longrightarrow \mathbb{C}$, invariant on conjugacy classes.
   The cyclotomic rational Cherednik algebra is the vector space

$$\mathcal{C}_{\mathbf{c}}(W_n) = \mathrm{Sym}(V^*) \otimes_{\mathbb{C}} \mathbb{C}W_n \otimes_{\mathbb{C}} \mathrm{Sym}(V)$$

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   First step: restriction to the *category* $\mathcal{O}$ for Cherednik algebras.

   - $\mathrm{Irr}_{\mathbb{C}}(\mathcal{C}_{\mathbf{c}}(W_n)) \overset{1:1}{\longleftrightarrow} \mathrm{Irr}_{\mathbb{C}}(W_n) \overset{1:1}{\longleftrightarrow} \{\ell\text{-partitions of } n\}$.
   - There exist induction/restriction functors $\mathcal{C}_{\mathbf{c}}(W_n) \leftrightsquigarrow \mathcal{C}_{\mathbf{c}}(W_{n+1})$ (Bezrukavnikov-Etingof 2008).

   Important questions:

   - branching rules?
   - finite-dimensional representations?

Fock spaces

Fix $\ell \geq 1$, v indeterminate.

$$\mathcal{F} \;=\; \bigoplus_{\substack{\boldsymbol{\lambda}_\ell \\ \ell\text{-partition}}} \mathbb{Q}(v)\,\boldsymbol{\lambda}_\ell \quad .$$

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Example: $\ell = 3$, $\boldsymbol{\lambda}_\ell = (5.3.1, 3.1, 1)$, $\mathbf{s}_\ell = (0, -1, 1)$.

$$|\boldsymbol{\lambda}_\ell, \mathbf{s}_\ell\rangle = \begin{array}{|c|c|c|c|c|}\hline 0 & 1 & 2 & 3 & 4 \\ \hline -1 & 0 & 1 \\ \cline{1-3} -2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|}\hline -1 & 0 & 1 \\ \hline -2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|}\hline 1 \\ \hline \end{array}$$

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Theorem (Jimbo-Misra-Miwa-Okado 1991)

$\mathcal{F}_{\mathbf{s}_\ell}$ is an integrable $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$-module (action depends on $\mathbf{s}_\ell$ and e).

NB: $\mathcal{U}_v(\widehat{\mathfrak{sl}_e}) = \langle e_i, f_i, t_i^{\pm 1} \, ; \, i \in \{0, \dots, e-1\}\rangle_{\mathbb{Q}(v)\text{-alg.}}$ + relations.

$\rightsquigarrow$ $\mathcal{F}_{\mathbf{s}_\ell}$ has a *Kashiwara crystal*.

Fock spaces

Example: Crystal for $\ell = 2$, $e = 3$, $\mathbf{s}_\ell = (0, 1)$.

$$\emptyset\,\emptyset$$

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$$\begin{tikzcd}
 & \emptyset\,\emptyset \arrow[dl] \arrow[dr] & \\
\emptyset\,\boxed{1} & & \boxed{0}\,\emptyset
\end{tikzcd}$$

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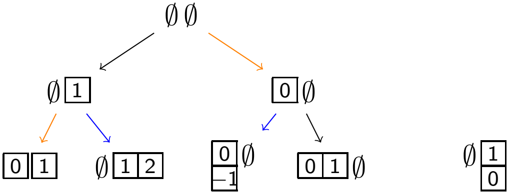

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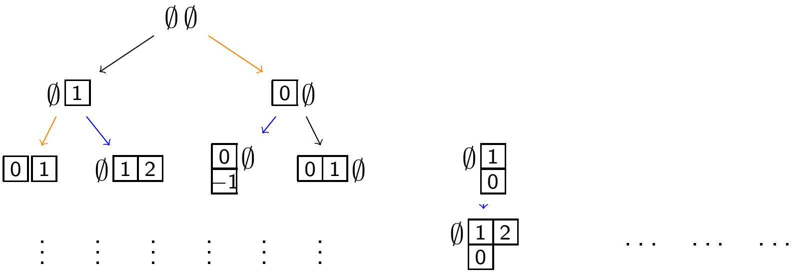

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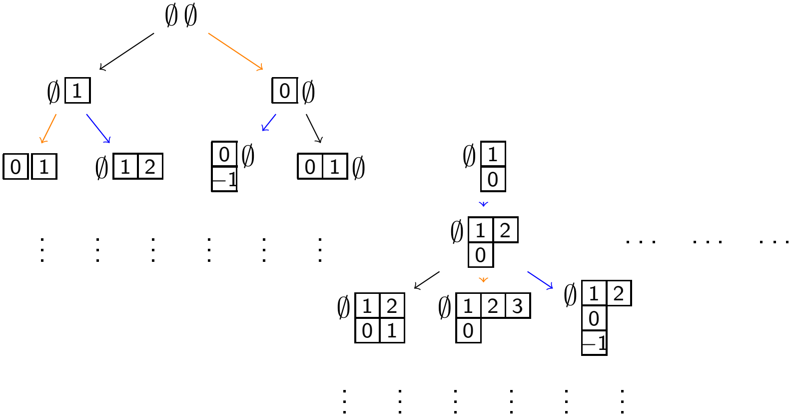

Categorifications

1. **Cherednik algebras**

Theorem (Shan 2011)

The crystal of $\mathcal{F}_{\mathbf{s}_\ell}$ gives the branching rule for $\mathcal{C}_{\mathbf{c}}(W_n)$, $n \in \mathbb{N}$, where $\mathbf{s}_\ell$ and e are determined by $\mathbf{c}$.

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2. **Classical groups**

Conjecture (G.-Hiss-Jacon 2015)
Theorem (Dudas-Varagnolo-Vasserot 2015-2016)

Let $\ell = 2$. The crystals of $\mathcal{F}_{\mathbf{s}_2}$, $|\mathbf{s}_2| = s$, give the weak branching rule for $G_n(q)$, $n \in \mathbb{N}$, where $\mathbf{s}_2$ and e are determined by q and m.

Triple module structure on the Fock space

Idea: consider $\Lambda^s = \bigoplus_{|\mathbf{s}_\ell|=s} \mathcal{F}_{\mathbf{s}_\ell}$. Set $u = -1/v$.

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$\rightsquigarrow$ Λ^s is an H-module, where H is the Heisenberg algebra:
$H = \langle \mathbf{1}, b_n \,;\, n \in \mathbb{Z}^\times \rangle_{\mathbb{Q}(v)-\mathrm{alg.}}$ + relations $b_n b_m = \delta_{n,-m}\gamma_n$ with $\gamma_n \in \mathbb{Q}(v)$.

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Theorem (Uglov 1999)

1. The three actions pairwise commute.
2. $$\Lambda^s = \bigoplus_{\mathbf{r}_\ell \in D(s)} \left(\mathcal{U}_u(\widehat{\mathfrak{sl}_\ell}) \times H \times \mathcal{U}_v(\widehat{\mathfrak{sl}_e})\right) |\emptyset_\ell, \mathbf{r}_\ell\rangle$$ where
$D(s) = \{\mathbf{r}_\ell \in \mathbb{Z}^\ell \mid |\mathbf{r}_\ell| = s \text{ and } r_1 \leq \cdots \leq r_\ell < r_1 + e\}$.

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Theorem (G. 2016)

Via the identification $*$, the $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$-crystal and the $\mathcal{U}_u(\widehat{\mathfrak{sl}_\ell})$-crystal commute.

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For $|\boldsymbol{\lambda}_\ell, \mathbf{s}_\ell\rangle$ source in both Kashiwara crystals, set $\tilde{b}^c|\boldsymbol{\lambda}_\ell, \mathbf{s}_\ell\rangle = \ell$-partition obtained by adding e boxes of consecutive contents $c, c-1, ..., c-e+1$ and of same column.

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Example: $|\boldsymbol{\lambda}_\ell, \mathbf{s}_\ell\rangle = \begin{array}{|c|c|} \hline 0 & 1 \\ \hline -1 & 0 \\ \hline -2 \\ \cline{1-1} -3 \\ \cline{1-1} -4 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 0 \\ \cline{1-1} -1 \\ \cline{1-1} -2 \\ \cline{1-1} \end{array}$ and $e = 3$

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Example: $|\boldsymbol{\lambda}_\ell, \mathbf{s}_\ell\rangle = \begin{ytableau} 0 & 1 \\ -1 & 0 \\ -2 \\ -3 \\ -4 \end{ytableau} \; \begin{ytableau} 1 & 2 \\ 0 \\ -1 \\ -2 \end{ytableau}$ and $e = 3$ $\qquad \tilde{b}^1|\boldsymbol{\lambda}_\ell, \mathbf{s}_\ell\rangle = \begin{ytableau} 0 & 1 \\ -1 & 0 \\ -2 & -1 \\ -3 \\ -4 \end{ytableau} \; \begin{ytableau} 1 & 2 \\ 0 & 1 \\ -1 & 0 \\ -2 \end{ytableau}$

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Theorem (G. 2016)

1. The H-crystal commutes with the $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$-crystal and the $\mathcal{U}_u(\widehat{\mathfrak{sl}_\ell})$-crystal.
2. Every charged ℓ-partition is obtained from $|\boldsymbol{\emptyset}_\ell, \mathbf{r}_\ell\rangle$ with $\mathbf{r}_\ell \in D(s)$ by moving down the three crystals.

Applications

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Definition (Foda-Leclerc-Okado-Thibon-Welsh 1999)

A charged ℓ-partition $|\boldsymbol{\lambda}_\ell, \mathbf{s}_\ell\rangle$ is *e-regular* if:

- $\forall 1 \leq d \leq \ell - 1$, $\lambda^d_k \geq \lambda^{d+1}_{k+s_{d+1}-s_d}$ $\forall k \geq 1$
  and $\lambda^d_k \geq \lambda^1_{k+e+s_1-s_\ell}$ $\forall k \geq 1$.
- $\forall \alpha > 0, \left\{ (\lambda^d_k - k + s_d) \mod e \,\middle|\, 1 \leq d \leq \ell - 1, \lambda^d_k = \alpha \right\}$
  $\neq \{0, \dots, e-1\}$.

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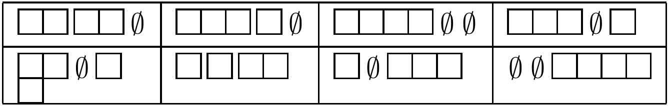
Example: The 2-regular 3-partitions of rank 4 associated to the 3-charge $(0, 0, 1)$ are:



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Theorem (Shan-Vasserot 2012 + Losev 2015 + G. 2016)

$|\boldsymbol{\lambda}_\ell, \mathbf{s}_\ell\rangle$ labels a **finite-dimensional** irreducible representation of the corresponding Cherednik algebra iff $|\boldsymbol{\lambda}_e, \mathbf{s}_e\rangle$ is ℓ-regular.

Applications

2. **Finite unitary groups**

   Recall: Representation theory over k, field of characteristic $m > 0$.
   Set $e =$ order of $-q$ modulo m, **odd**.

   Remark: The case e even is understood (Gruber-Hiss 1997).

Theorem (G. 2016 + Dudas-Varagnolo-Vasserot 2016)

$|\boldsymbol{\lambda}_2, \mathbf{s}_2\rangle$ labels a **cuspidal** irreducible representation of $GU_n(q)$ iff $|\boldsymbol{\lambda}_e, \mathbf{s}_e\rangle$ is 2-regular.