(1)

Linear degenerations of flag varieties.

$$\left\{ (U_1 \subset U_2 \subset \dots \subset U_n) \;\middle|\; \begin{array}{l} \dim U_i = i \\ U_i \subset U_{i+1} \subset \mathbb{C}^n \end{array} \right\}$$

$\mathbb{C}[P_I \mid 1 \le |I| \le n-1]$

$\cup$

$V(\text{Plücker relations})$

Flag variety $Fl_n = SL_n / B$

$SL_n . [v_\lambda] \subset \mathbb{P}(V(\lambda))$

irred. SL_n-representation of λ regular, dominant

$\overline{B^+ . t_{w_0}} \subset SL_n/B$

$X_{w_0} = \overline{B^+ [v_{w_0(\lambda)}]} \subseteq \mathbb{P}$

Schubert variety.

The degenerate flag variety:

Consider $sl_n = n^+ \oplus h \oplus n^- \rightsquigarrow U(sl_n) = U(n^+)\, U(h)\, U(n^-)$

set $\deg x = 1 \;\forall x \in n^-$

studied by F-Feigin-Littelmann

$$\Rightarrow \quad \begin{array}{ccccc} (4) & (3) & (1) & & (2) \\ U^- & n^- & U(n^-) & & V(\lambda) \\ \downarrow & \downarrow & \downarrow \text{gr} & & \downarrow \\ \mathbb{G}_a^{m} & n^{-,a} & S(n^-) & & \text{gr}\, V^a(\lambda) \end{array}$$

$\dim U^-$

E. Feigin:

$$Fl_a(\lambda) := \overline{\mathbb{G}_a^{m} . [v_\lambda]} \subset \mathbb{P}(V^a(\lambda))$$

the degenerate flag variety.

$$\left\{ (U_1 \subset \dots \subset U_n) \;\middle|\; \begin{array}{l} \dim U_i = i \\ pr_i\, U_i \subset U_{i+1} \end{array} \right\}$$ 3

V(degenerate Plücker relations) 2

$\mathcal{F}l_a(\lambda)$

$\overline{G_a^n \cdot [v_\lambda]} \subset \mathbb{P}(V^a(\lambda))$ 1

$X_{\tilde{w}}(\tilde{\lambda}) \subset SL_{2n}/B$ 4

~~Theorem (Feigin)~~ Let P_I be a Plücker coordinate

introduce the PBW degree of P_I

as $\#\{ j \mid 1 \le j \le |I|,\ i_j > |I| \}$

$I = \{i_1, \dots, i_{|I|}\}$

$\rightsquigarrow \quad X_1 X_{23} + X_2 X_{13} - X_3 X_{12}$

$\downarrow$

$t X_1 X_{23} + t^2 X_2 X_{13} - t X_3 X_{12} \overset{\text{degenerates to}}{\rightsquigarrow} X_1 X_{23} - X_3 X_{12}$

2 Theorem (Feigin) Let λ be regular

$\mathcal{F}l_a(\lambda) \cong V$(degenerate Plücker relations)

3 Theorem (Feigin - ~~Finkelberg~~)

$$\mathcal{F}l_a(\lambda) = \left\{ (U_1 \subset \dots \subset U_n) \;\middle|\; \begin{array}{l} \dim U_i = i \\ pr_i\, U_i \subset U_{i+1} \end{array} \right\}$$

$$pr_i\left(\sum_{j=1}^{n} a_j e_j\right) = \sum_{\substack{j=1 \\ j \neq i}}^{n} a_j e_j .$$

How about the last piece?

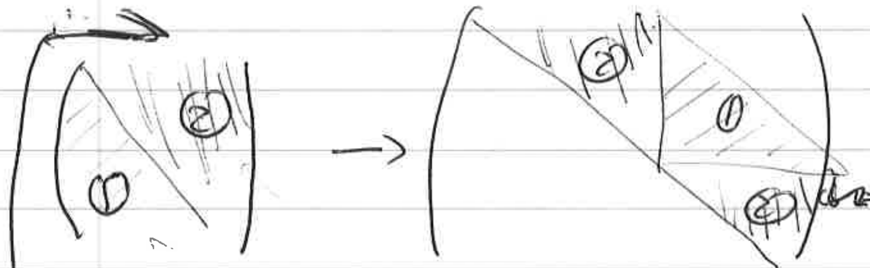
$$Fl_{\mathfrak{a}}(\lambda) \simeq X_{w_0}(\lambda) \quad , w_0 \text{ longest Weyl group element}$$

Now use a hammer to destroy the Lie theory $\quad U(\mathfrak{n}^-) \rightsquigarrow S(\mathfrak{n}^-)$

kind of magic:

(4) Theorem (Cerulli Irelli–Lanini)

$$Fl_a(\lambda) \simeq X_{\tilde{w}}(\tilde{\lambda})$$ inside a partial flag variety for $\mathfrak{sl}_{2n}$.



Xin will explain more on this.

Moreover (w. Littelmann): Compatible with global sections in the sense that

$$V^a(\lambda) \simeq V_{\tilde{w}}(\tilde{\lambda})$$

the Demazure module.

PBW filtration compatible with $\mathfrak{b}^+$

$\rightsquigarrow SL_n \rightsquigarrow SL_n^a = B \ltimes \mathbb{G}_a^{N}$

Then the PBW filtration is compatible with the adjoint action of $\mathfrak{b}^+$

$$\Rightarrow \mathfrak{b}^+ \circlearrowright V^a(\lambda)$$

$$s_4 s_5 s_6 s_7 s_5 s_4 s_5 s_2 s_3 s_1$$

How to generalize this? (jt. (I - F - FT - R)

Let us focus on the linear algebra part

$$\begin{array}{ccccccccc}
\mathbb{C}^n & \xrightarrow{pr_1} & \mathbb{C}^n & \xrightarrow{pr_2} & \mathbb{C}^n & \xrightarrow{pr_3} & \cdots & \xrightarrow{pr_{n-1}} & \mathbb{C}^n \\
\cup & & \cup & & \cup & & & & \| \\
U_1 & \xrightarrow[pr_1|U_1]{} & U_2 & \xrightarrow[pr_2|U_2]{} & U_3 & \longrightarrow & & \longrightarrow & U_n
\end{array}$$

Let us replace pr_i by arbitrary linear maps.

$$\begin{array}{ccccccccc}
\mathbb{C}^n & \xrightarrow{f_1} & \mathbb{C}^n & \xrightarrow{f_2} & \mathbb{C}^n & \xrightarrow{f_3} & & \xrightarrow{f_{n-1}} & \mathbb{C}^n \\
\cup & & & & & & & & \\
U_1 & \xrightarrow{f_1|U_1} & U_2 & \xrightarrow{f_2|U_2} & U_3 & \xrightarrow{f_3|U_3} & & \xrightarrow{f_{n-1}} & U_n
\end{array}$$

$$\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^n)\times\prod^{n-1} Gr(i,n)\supset Y = \left\{ (f_1,\dots,f_{n-1},U_1,\dots,U_{n-1}) \;\middle|\; \begin{array}{l}\dim U_i = i\\ f_i\, U_i \subset U_{i+1}\end{array}\right\}$$

Remark: Y is smooth and irreducible (since vector bundle / homogeneous bundle over $\prod Gr(U_i)$)

Definition. $\pi : Y \longrightarrow \left(\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^n)\right)^{n-1} = R$

is called the universal linear degenerate flag variety.

Do we understand the fibres?

Example: 1) all f_i invertible $\Rightarrow \pi^{-1}(f_1,\dots,f_{n-1})$

$$\simeq \pi^{-1}(id,\dots,id)$$
$$\simeq Fl$$

2) all $f_i = 0$ $\Rightarrow$ $\pi^{-1}(0, \dots, 0) = \prod_{i=1}^{n} Gr(i, n)$

3) $f_i = pr_i$ $\Rightarrow$ $\pi^{-1}(\quad) = Fl_n$

$G = (GL_n)^{n+1}$ acts on Y and R.

$(g_1, \dots, g_{n+1}) (f_1, \dots, f_{n-1}, U_1, \dots, U_n)$

$:= (g_2 f_1 g_1^{-1}, g_3 f_2 g_2^{-1}, \dots, g_n f_{n-1} g_{n-1}^{-1}, g_1 U_1, \dots, g_n U_n)$

$\left(\Leftarrow \quad g_{i+1} f_i g_i^{-1} (g_i U_i) = g_{i+1} f_i U_i = g_{i+1} U_{i+1} \right)$

Remark: 1) ~~Each~~ G-orbit on R can be represented by

$\underline{r} : (r_{ij}) = \operatorname{rank}(f_{j-1} \cdots f_i)$, denoted $O^{\underline{r}}$

Bongartz: 2) The orbit $O^{\underline{r}}$ degenerates to $O^{\underline{r}'}$ iff $O^{\underline{r}'}$ is in the closure of $O^{\underline{r}}$ iff $r_{ij} \geq r_{ij}'$ $\forall\, ij$

3) The fibre for a fixed $O^{\underline{r}}$ are isomorphic

~~Examples~~ Examples:

$$\begin{array}{ccccc} \mathbb{C}^3 & \xrightarrow{A} & \mathbb{C}^3 & \xrightarrow{B} & \mathbb{C}^3 \\ \cup & & \cup & & \cup \\ U_1 & & U_2 & & U_0 \end{array}$$

$B \leftarrow$ irrelevant

$Y = \{ (A, v, \varphi) \in (M_{3,3}(\mathbb{C}), \mathbb{P}^2, (\mathbb{P}^2)^*)$

s.t. $\varphi \cdot A \cdot v = 0 \}$.

$\Rightarrow$ rk A	fibre	
3	Fl	
2	Fl_a	$\mathbb{P}_1 \times \mathbb{P}_2 \vee_{\mathbb{P}_1\mathbb{P}_1} \mathbb{P}_2 \times \mathbb{P}_1$
1	two irred. components, not normal	
0	$Gr(1,3) \times Gr(2,3)$	

Question: ~~Which fibre is irreducible~~
For which I is the fibre of π ~~irred~~ $\swarrow Fl^{I}$

a) irreducible?

b) a flat degeneration of Fl / reduced, complete intersection, cell decomposition)

c) normal

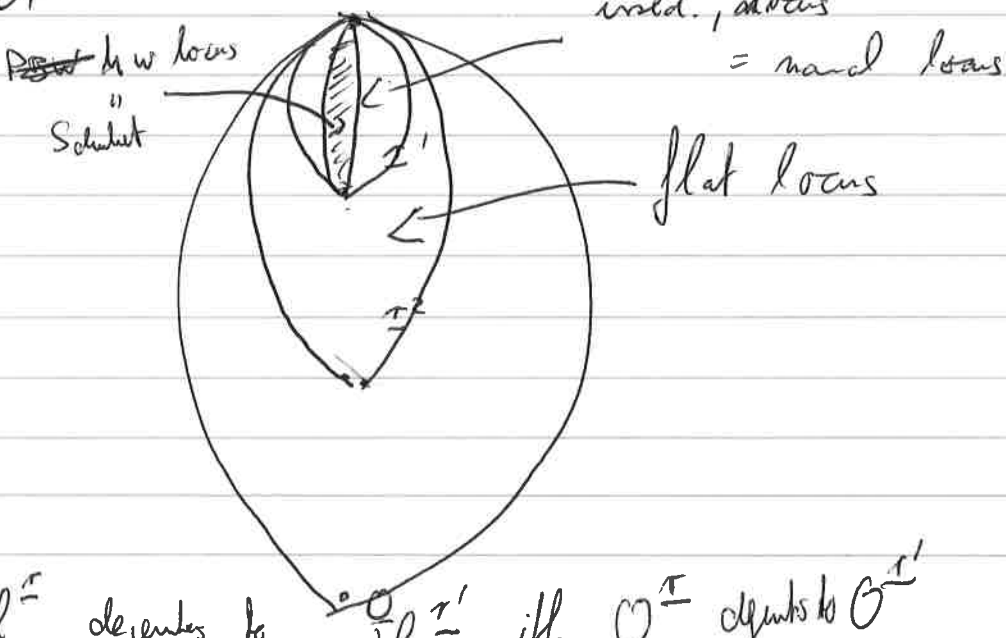
d) h.w. orbit / Schubert / Cohen-Macaulay /

$$\boxed{\begin{aligned} &\text{recall: } I^0 = (r_{ij} = n) \\ &I^1 := (r_{ij} = n-(j-i)) \\ &I^2 := (r_{ij} = n-(j-i)-1) \end{aligned}}$$

Theorem (CI-F-FF-Reineke): a) Fl^{I} is irreducible iff O^{I} degenerates to O^{I^1}
normal

b) Fl^{I} is a flat degen. of Fl iff O^{I} degenerates to O^{I^2}

Complete description!! c) ~~Fl^{I}~~



$d_i = id$

$d_i = pr_i$

$d_i = pr_i \circ pr_{i+1}$

c) Fl^{I} degenerates to $Fl^{I'}$ iff O^{I} degenerates to $O^{I'}$

fibre of $\mathcal{O}^{1^2}$:

C_n number of irred. components (equi-dimensional)
(parametrized by non-crossing arcs)

~~h.w. locus~~ irred. locus: ~~each~~ broken rhyme schemes
(choose for each f_i one projection pr_j
starting with $j=1,2,\dots$

$\cup$

h.w. orbit: the kernels of the projection maps are linearly independent.

$\underline{000}$ $\underline{001}$ $\underline{010}$ 011 $\underline{012}$ (id, pr_1, pr_1)

$\underline{100}$ 101 $\underline{102}$ 110 111

112 $\underline{120}$ 121 $\underline{122}$ $\underline{123}$

(pr_1, pr_2, pr_1)

$(b_1, \dots, b_{n-1})$

s.t. $b_1 \in \{0,1\}$
$b_{j+1} \le \max(b_1, \dots, b_j) + 1$