PITMAN'S THEOREM, RANDOM MATRICES AND CRYSTALS

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Based on joint work with Philippe Bougerol and Neil O'Connell

I. Pitman's theorem

GUE

A GUE matrix is a hermitian $n \times n$ matrix M chosen with the probability distribution

$$\frac{e^{-Tr(M^2)/2}}{(2\pi)^{n/2}} dM$$

The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are random variables whose distribution occur in many fields:

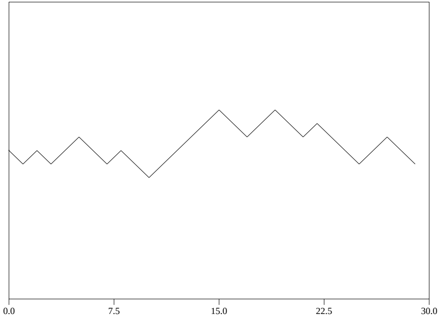
- Statistics
- Wireless communications
- Zeros of Riemann zeta function
- Atomic physics
- Integrable systems
- ...

A good way to obtain a gaussian variable is to use the *central limit theorem*.

Equivalently we will construct a random matrix by using a random walk on the space of matrices.

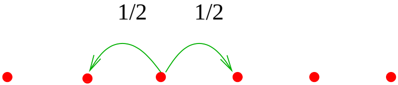
This will be closely related to representation theory.

Classical random walk



$$A_n = a_1 + \ldots + a_n$$

$$a_k = \pm 1$$



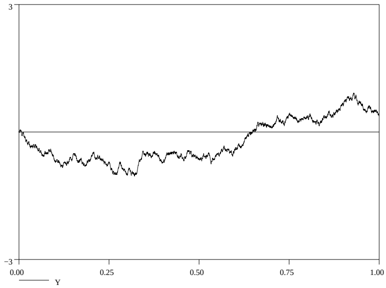
Brownian motion

Scale by ε in time and $\sqrt{\varepsilon}$ in space.

$$\sqrt{\varepsilon} \quad \uparrow$$

$$\rightarrow$$

$$\varepsilon$$



Matrix Brownian motion $M(t)$ is obtained by letting coordinates of the matrix evolve as independent Brownian motions (modulo relations $M_{ij} = \bar{M}_{ji}$).
The eigenvalues $(\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t))$ evolve according to a Brownian motion "conditioned to stay forever in the Weyl chamber"

$$x_1 > x_2 > \ldots > x_n$$

At time 1 the matrix is a GUE matrix.

Conditional probability:

P a probability measure, B an event, we condition by the formula:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

here B=the Brownian motion remains forever in the Weyl chamber.

One has $P(B) = 0$ so the definition of $P(A|B)$ is somewhat indirect.

It requires some theory of harmonic functions, Martin boundaries etc.

Brownian motion on 2×2 matrices:

$$\begin{pmatrix} U_t & V_t + iW_t \\ V_t - iW_t & -U_t \end{pmatrix}$$

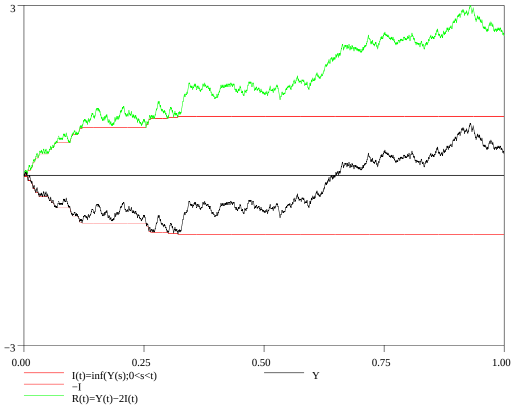
eigenvalues:

$$\pm\sqrt{U_t^2 + V_t^2 + W_t^2}$$

this is the norm of a three dimensional Brownian motion.
This is the same as "Brownian motion conditioned to remain positive"

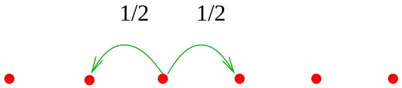
PITMAN THEOREM (1975)

$B_t; t \geq 0$ Brownian motion; $I_t = \inf_{0 \leq s \leq t} B_s$
$R_t = B_t - 2I_t; t \geq 0$ is distributed as the norm of a three dimensional Brownian motion(=Bessel 3 process)



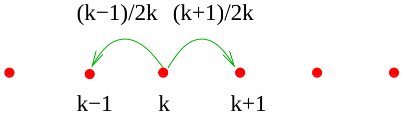
DISCRETE VERSION

$X_i = \pm 1; \quad S_n = X_1 + X_2 + \ldots + X_n$



$R_n = S_n - 2\min_{0 \le k \le n} S_k$ is a Markov chain(=discrete Bessel 3 process)

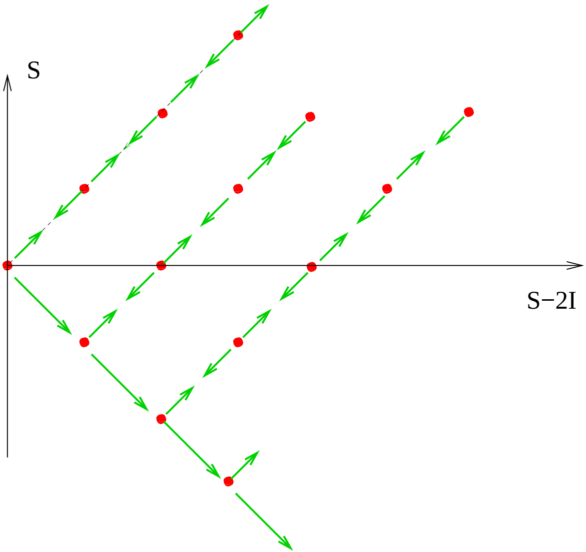
$$P(R_{n+1} = k+1 | R_n = k) = \frac{k+1}{2k}$$
$$P(R_{n+1} = k-1 | R_n = k) = \frac{k-1}{2k}$$



when $n \to \infty$ $S_{[nt]}/\sqrt{n} \to_{n\to\infty}$ Brownian motion

$R_{[nt]}/\sqrt{n} \to_{n\to\infty}$ norm of 3D-Brownian motion

PROOF OF PITMAN'S THEOREM



I will explain how Pitman's theorem is related to representation theory of $SU(2)$ and $SU_q(2)$.

$1=\text{rank}(SU(2))$

$3=\text{dimension}(SU(2))$

A crash course on quantum mechanics

$H =$ (complex) Hilbert space

Observables=self-adjoint operators on H

φ unit vector $+$ A observable
$\rightarrow$ probability measure

$$P(\lambda) = |\pi_\lambda \varphi|^2$$

$$\pi_\lambda = \text{orthogonal projection on eigenspace of } \lambda$$

P is supported on the spectrum of A.

Expectation of A is

$$\langle A\varphi, \varphi\rangle = Tr(A\pi_\varphi)$$

More generally: expectation of $f(A)$ is

$$\langle f(A)\varphi, \varphi\rangle = Tr(f(A)\pi_\varphi)$$

One can convexify: replace π_φ with a positive operator of trace 1.

$$E[f(A)] = Tr(\rho f(A))$$

Basic example

(Ω, F, P) probability space

$H = L^2(\Omega, F, P)$

x=real random variable

$$\begin{aligned} X_x : H &\rightarrow H \\ X_x(z) &= xz \end{aligned}$$

is a self-adjoint operator
Spectral theorem: any self-adjoint operator on a Hilbert space can be put in this form.

If $A_1, \ldots, A_n$ commute $\to$ diagonalized simultaneously

Their joint distribution makes sense:

$$Tr(\rho f(A_1, \ldots, A_n)) = \int f(x_1, \ldots, x_n) d\mu$$

for μ proba on $\mathbf{R}^n$

Spins

dim(H)=2

The space of observable has dimension 3

Pauli matrices give a basis

$$x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In the state e_1,
X and Y are symmetric Bernoulli
$Z = 1$ a.s.

In the central state $Tr(.\frac{1}{2}Id)$ all three are symmetric Bernoulli.

By choosing state appropriately on can realize any Bernoulli distribution.

Quantum Bernoulli random walks

We "quantize" the set of increments of the random walk $\{\pm 1\}$ to obtain $M_2(\mathbf{C})$.
The subset of hermitian operators in $M_2(\mathbf{C})$ is a four dimensional real subspace, generated by the identity matrix I as well as the three matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The matrices $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices. They satisfy the commutation relations

$$[\sigma_x, \sigma_y] = 2i\sigma_z; \quad [\sigma_y, \sigma_z] = 2i\sigma_x; \quad [\sigma_z, \sigma_x] = 2i\sigma_y \qquad (1)$$

and form (up to i) a basis for the Lie algebra $su(2)$

The random walk

$$x_n = I^{\otimes(n-1)} \otimes \sigma_x \otimes I^{\otimes(N-n-1)}, \; y_n = I^{\otimes(n-1)} \otimes \sigma_y \otimes I^{\otimes(N-n-1)}$$

$$z_n = I^{\otimes(n-1)} \otimes \sigma_z \otimes I^{\otimes(N-n-1)}$$

x_n is a commuting family of operators, a sequence of independent Bernoulli random variables.

$$X_n = \sum_{i=1}^{n} x_i; \quad Y_n = \sum_{i=1}^{n} y_i \quad Z_n = \sum_{i=1}^{n} z_i$$

are Bernoulli random walks.

$$2^{-N} Tr(f_1(X_1) \dots f_n(X_n)) = E[f_1(A_1) \dots f_n(A_n)]$$

They do not commute but obey

$$[X_n, Y_m] = 2iZ_{n \wedge m} \tag{2}$$

as well as the similar relations obtained by cyclic permutation of X, Y, Z.

$$j_n(X, Y, Z) = (X_n, Y_n, Z_n); n \geq 1$$

is a *quantum Bernoulli random walk*.
It is obtained by tensoring the basic representation of $su(2)$, n times.

The spin process

Let $S_n = \sqrt{I + X_n^2 + Y_n^2 + Z_n^2}$

Proposition For all n, m one has

$$[S_n, S_m] = 0$$

We can diagonalize simultaneously the S_n.
Then we can evaluate

$$2^{-N} Tr(f_1(S_1) \dots f_n(S_n))$$

S_n is the image of the Casimir operator generating the center of $U(su(2))$.

Theorem
S_n is distributed as a Markov chain on the positive integers, with probability transitions

$$p(k, k+1) = \frac{k+1}{2k}; \quad p(k, k-1) = \frac{k-1}{2k}.$$

i.e. the quantities

$$2^{-n} Tr(f_1(S_1) \dots f_n(S_n)) = E[f_1(R_1) \dots f_n(R_n)]$$

where R is a Markov chain as above

$E =$ a set (e.g. Z^d)
Ω a probability space

A random variable with values in E: $X : \Omega \to E$

this gives an algebra morphism:

$$F(E) \to F(\Omega)$$

$$f \to f \circ X$$

We could drop the condition that the algebras are commutative
An algebra morphism between two (non-commutative) algebras is a (non-commutative) random variable.

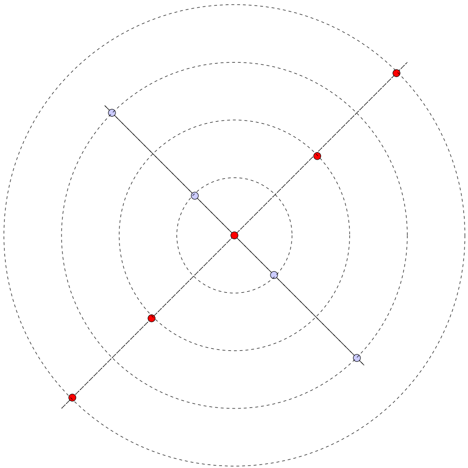
$A =$ group algebra of $SU(2)$
$j_n : A \to M_2(\mathbb{C})^{\otimes\infty}$=$n$-fold tensor product of 2-dimensional representations for $n = 1, 2, ...$ form a sequence of non-commutative random variables =stochastic process with values in a non-commutative space (the "dual" of $SU(2)$).

The dual of $SU(2)$ as a noncommutative space

$G = SU(2) =$ unitary 2×2 matrices with determinant 1.
Irreps are parametrized by $\{1, 2, 3, \dots\}$

$$\mathcal{A}(SU(2)) = \oplus_{n=1}^{\infty} M_n(\mathbb{C})$$

is the noncommutative space dual to $SU(2)$.
The Pauli matrices belong to the Lie algebra $su(2)$, they define unbounded operators X, Y, Z, on $L^2(SU(2))$.
They generate oneparameter sugbroups isomorphic to $U(1)$. This is true also of any linear combination $xX + yY + zZ$ with $x^2 + y^2 + z^2 = 1$.



Noncommutative space underlying $\mathcal{A}(SU(2))$

If you are in this space and measure your coordinate in some direction (x, y, z) using the operator $xX + yY + zZ$, and you will always find an integer.

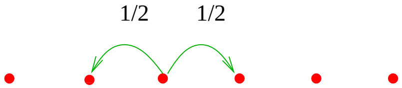
You cannot measure coordinates in two different directions at the same time.

The operator $D = \sqrt{I + X^2 + Y^2 + Z^2} - I$ is in the center of the algebra $\hat{\mathcal{A}}(SU(2))$, and therefore can be measured simultaneoulsy with any other operator.
Its eigenvalues are the nonnegative integers $0, 1, 2 \ldots,$ and its spectral projections are the identity elements of the algebras $M_n(\mathbb{C})$

$$D = \sum_{n=1}^{\infty} (n-1) I_{M_n(\mathbb{C})}$$

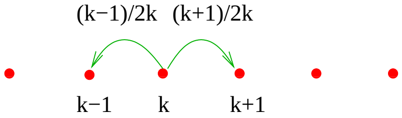
$M_n(\mathbb{C})$ is a kind of "noncommutative sphere of radius $n-1$".
Looking at the eigenvalues of the operators $xX + yY + zZ$ the coordinate on this "radius" can only take the $n+1$ values $n, n-2, n-4, \ldots, -n$.

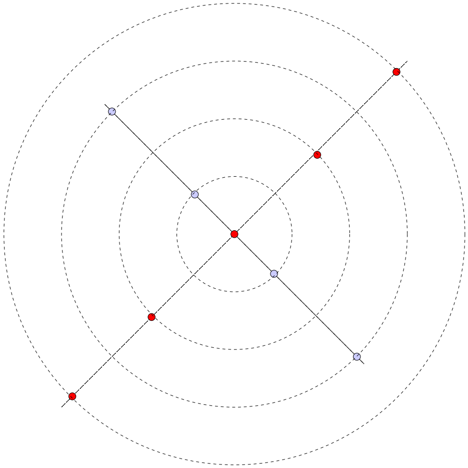
One can restrict the j_n to commutative subalgebras.
Restriction to a one parameter subgroup gives a Bernoulli random walk.



The spin process (radial part) is obtained by restriction of j_n to the center of the group algebra.
The restriction of the Quantum random walk to this center can be computed by the Clebsch Gordan formula

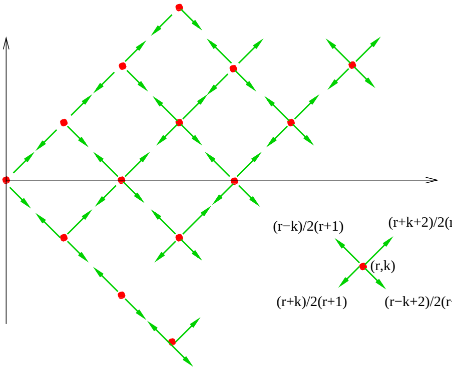
$$\rho_2 \otimes \rho_k = \rho_{k-1} \oplus \rho_{k+1}$$

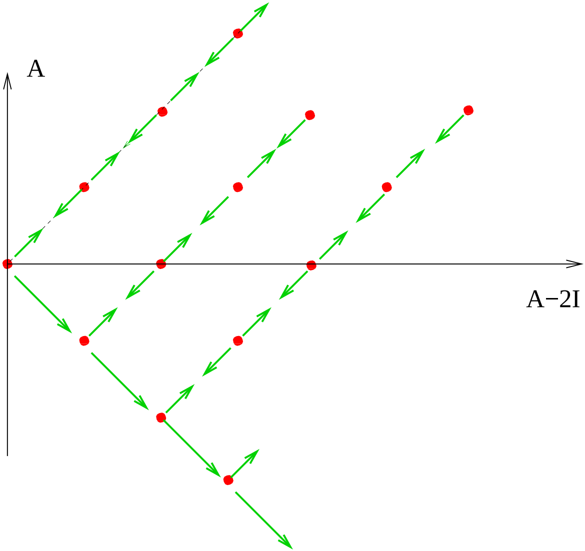




RESTRICTION TO A MAXIMAL ABELIAN ALGEBRA

Restrict the random walk to the maximal abelian subalgebra generated by the center and a one parameter subgroup.
In the decomposition $\mathcal{A}(SU(2)) = \oplus M_n(\mathbb{C})$ this is the algebra of diagonal operators.
One gets probability transitions

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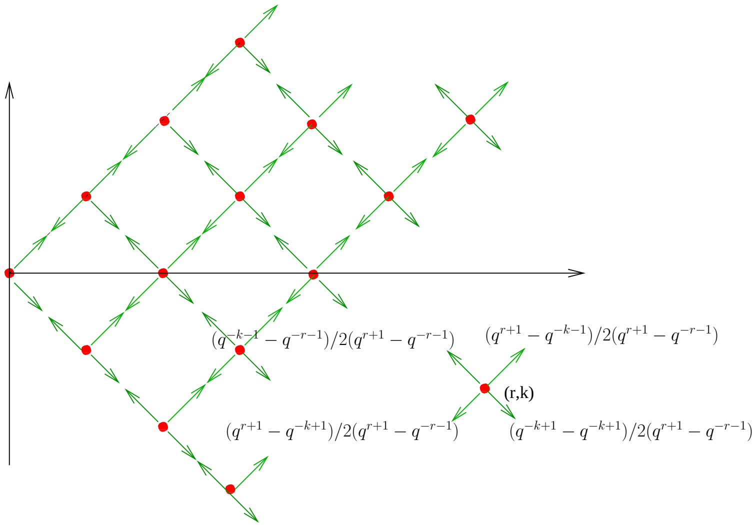
Kashiwara's crystallization

Replace $SU(2)$ by $SU_q(2)$ then



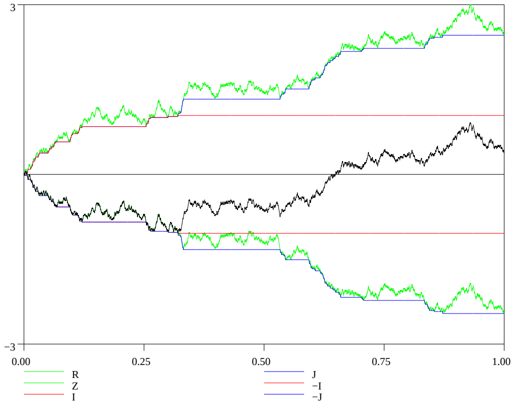
Let $q \to 0$ then one obtains Pitman's theorem.

We can generalize the preceding construction to the quantum groups $SU_q(n)$.

II. Generalized Pitman theorem and Littelmann paths

PITMAN OPERATORS

$$Y : [0, T] \to \mathbf{R}, \qquad Y(0) = 0$$



$$PY(t) = Y(t) - 2 \inf_{0 \leq s \leq t} Y(s)$$

For all t one has $PY(t) \geq 0$, in particular $PPY = PY$.

MULTIDIMENSIONAL PITMAN OPERATORS

V=real vector space, $\alpha \in V$, $\alpha^\vee \in V^*$ $\alpha^\vee(\alpha) = 2$.

$$P_\alpha Y(t) = Y(t) - \inf_{0 \leq s \leq t} \alpha^\vee(Y(s))\alpha$$

$$P_\alpha P_\alpha Y = P_\alpha Y$$

BRAID RELATIONS

Let $a, b \in V$, of norm 1, $\langle a, b \rangle = -\cos\theta$. If $n\theta \leq \pi$ then

$$(n \text{ terms}) \quad P_a P_b P_a \ldots f(t) = f(t) -$$

$$2 \inf_{t \geq s_1 \geq \ldots \geq s_n \geq 0} [\tfrac{\sin\theta}{\sin\theta} \langle f(s_1), a \rangle + \tfrac{\sin 2\theta}{\sin\theta} \langle f(s_2), b \rangle + \tfrac{\sin 3\theta}{\sin\theta} \langle f(s_3), a \rangle + \ldots + \tfrac{\sin n\theta}{\sin\theta} \langle f(s_n), c \rangle] \, a$$

$$-2 \inf_{t \geq s_1 \geq \ldots \geq s_{n-1} \geq 0} [\tfrac{\sin\theta}{\sin\theta} \langle f(s_1), b \rangle + \tfrac{\sin 2\theta}{\sin\theta} \langle f(s_2), a \rangle + \tfrac{\sin 3\theta}{\sin\theta} \langle f(s_3), b \rangle + \ldots + \tfrac{\sin(n-1)\theta}{\sin\theta} \langle f(s_{n-1}), d \rangle] \, b$$

If $\theta = \pi/n$ then

$$P_a P_b P_a \ldots = P_b P_a P_b \ldots \qquad (n \text{ termes})$$

Corollary: Let (W, S)=Coxeter system on V and $\alpha, \alpha^\vee$=simple roots and coroots,
C=Weyl chamber. To each $s_\alpha \in S$ associate P_{s_α}. For each $w \in W$ with reduced decomposition $w = s_{\alpha_1} \dots s_{\alpha_k}$ there exists

$$P_w = P_{s_{\alpha_1}} \dots P_{s_{\alpha_k}}$$

If w_0=longest element then $P_{w_0} X$ takes values in C.

Relation with Littelmann's path model

Let W be crystallographic, then each path starting from 0 belongs to some Littelmann module (if we take a lattice with infinitesimal spacing!). The Pitman operator P_{w_0} sends the path to the dominant path in the Littelmann module.
When W is not crystallographic there is no lattice any more, but P_{w_0} still makes sense and there is a "continuous Littelmann module (or "continuous crystal").

GENERALIZED PITMAN THEOREM

Let X be Brownian motion in V

then $P_{w_0}X$ is Brownian motion "conditioned to stay in C".

Random matrices and representation theory

Kirillov's formula for characters:

$a = \sum_{ij} a_{ij} e_{ij} \in gl_n$

$$tr((e^{\sum_{ij} a_{ij}\rho_\lambda(e_{ij})})) = \frac{\int_{O_\lambda} e^{\sum_{ij} a_{ij} u_{ji}} du}{\varphi(a)}$$

O_λ is the coadjoint orbit of λ=matrices with spectrum $\lambda_1, \dots, \lambda_n$.
This formula shows that the "non-commutative random variables" $\rho_\lambda(e_{ij})$ behave almost like the coordinates u_{ji} of a random matrix u taken with uniform measure on O_λ.
The matrix

$$\sum_{ij} \rho_\lambda(e_{ij}) \otimes e_{ji}$$

commutes with the action of gl_n.

The u_{ij} commute and the e_{ij} "almost" commute:

$$[e_{ij}, e_{kl}] = \text{term of degree } 1$$

Connexions with symmetric groups:
The operators

$$x_{ij} = \sum_{k=1}^{N} I \otimes \ldots \otimes I \otimes e_{ij} \otimes I \otimes \ldots \otimes I$$

behave almost like the coordinates of of a GUE matrix. Let us put them in a matrix:

$$M = \sum x_{ij} \otimes e_{ji}$$

This matrix commutes with the action of gl_n on $(\mathbf{C}^n)^{\otimes(N+1)}$ hence can be expressed using S_{N+1}.

$$M = \sum e_{ij} \otimes e_{ji}$$

is the transposition on $(\mathbf{C}^n) \otimes \mathbf{C}^n)$

$$M = (1\ N+1) + (2\ N+1) + \ldots + (N\ N+1)$$

is the Jucys Murphy element

UNIFORM DISTRIBUTION ON AN ORBIT

Let $(\lambda_1, \ldots, \lambda_N) \in \mathbf{R}^N$, the orbit

$$\mathcal{O}_\lambda = \{UDU^* \mid U \in U(N)\}$$

of

$$D = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_N \end{pmatrix}$$

by conjugation has a unique probability distribution invariant under $U(N)$.

HIZ FORMULA

The Fourier transform is given by Harish Chandra formula

$$\int_{U(N)} \exp(iTr(UDU^* A))dU = Z_N \frac{\text{dét}[(e^{i\lambda_j \mu_k})_{j,k}]}{V(\lambda)V(\mu)}$$

where μ_j are the eigenvalues of A and $V(\lambda)$ is the Vandermonde

$$V(\lambda) = \prod_{j<k} (\lambda_k - \lambda_j)$$

The Fourier transform is determined by its values on diagonal matrices A.
Remark: the formula is given by the stationary phase method.

DUISTERMAAT-HECKMAN MEASURE

For $A = diag(a_1, \dots, a_N)$

$$\begin{aligned} F(a_1, \dots, a_N) &= \int_{U(N)} \exp(iTr(UDU^* A))dU \\ &= \int_{U(N)} \exp(i \sum_j a_j (UDU^*)_{jj})dU \end{aligned}$$

is the Fourier transform of the distribution of

$$((UDU^*)_{11}, \dots, (UDU^*)_{NN})$$

This measure on $\mathbf{R}^N$ is supported by the hyperplane

$$\sum_i x_i = Tr(D)$$

It is the Duistermaat-Heckman measure.

SOME PROPERTIES OF DH MEASURE

The support of the Duistermaat-Heckman measure is the convex hull of the points $(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(N)})$, where $\sigma \in S_N$.
It has a piecewise polynomial density on this set.

It is the image by an affine map of Lebesgue measure on a convex polytope of dimension $\frac{N(N-1)}{2}$.

EXAMPLE; N=2

$D = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$ $\lambda > 0$, the orbit is

$$\begin{pmatrix} x & z \\ \bar{z} & -x \end{pmatrix}$$

such that $x^2 + |z|^2 = \lambda^2$. A sphere S^2 of radius λ.
D-H measure is the projection of uniform measure on S^2 onto a diameter.
It is Lebesgue measure on $[-\lambda, \lambda]$, for $A = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$

$$\int_{U(2)} \exp(\frac{i}{2} Tr(UDU^* A) dU = Z_2 \frac{\det \begin{pmatrix} e^{\frac{i}{2}\lambda a} & e^{-\frac{i}{2}\lambda a} \\ e^{-\frac{i}{2}\lambda a} & e^{\frac{i}{2}\lambda a} \end{pmatrix}}{i\lambda a}$$

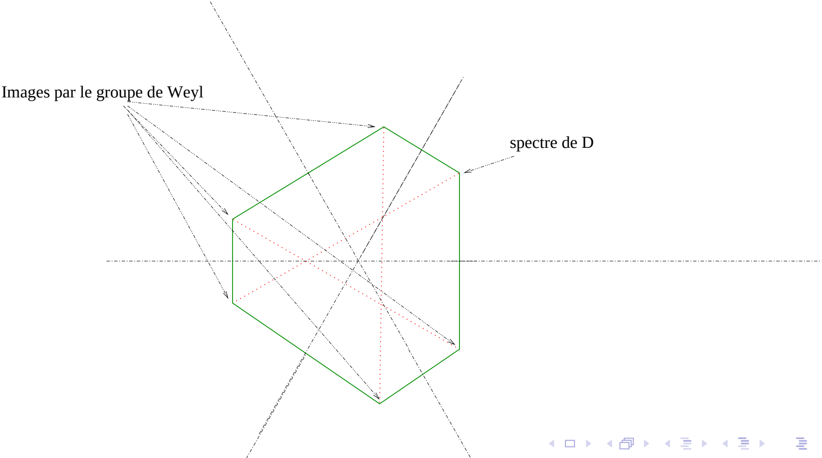
$$= \frac{\sin(\lambda a)}{\lambda a} = \frac{1}{2\lambda} \int_{-\lambda}^{\lambda} e^{iax} dx$$

N=3

Take

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \qquad \lambda_1 + \lambda_2 + \lambda_3 = 0$$

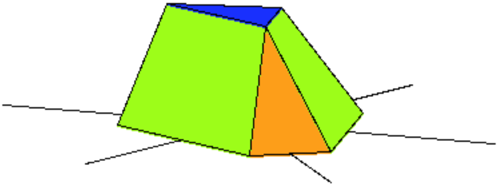
the measure is supported by a convex set.



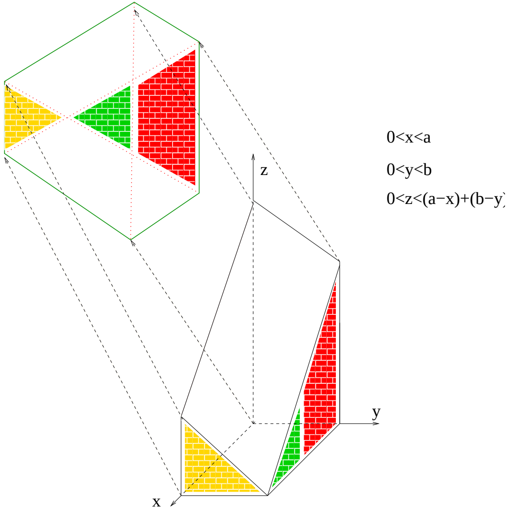
Density is piecewise polynomial with degree

$$\frac{(N-1)(N-2)}{2}$$

degree= 1 for N=3



This measure is the image by an affine map affine of Lebesgue measure on a convex polytope



BROWNIAN INTERPRETATION

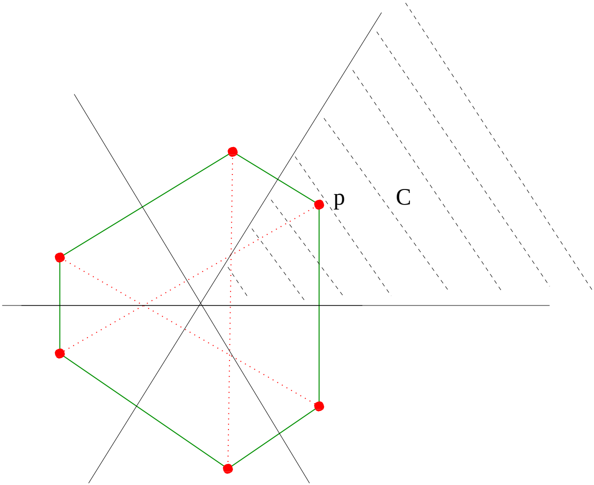
OF DUISTERMAAT-HECKMAN MEASURE

$B = (B_{ij}(t))_{1 \leq i,j \leq N}$ $U(N)$-invariant brownian motion on $N \times N$ hermitian matrices

Duistermaat-Heckmann measure is the conditional distribution of $(B_{11}(t), \ldots, B_{NN}(t))$ knowing that $B(t)$ has spectrum $(\lambda_1, \ldots, \lambda_N)$.

CONVERSE THEOREM

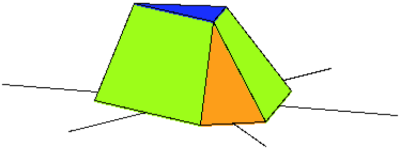
The conditional distribution of $X(t)$ knowing $P_{w_0}X(t) = p$ is the Duistermaat-Heckmann measure on the convex polytope with vertices $w(p); w \in W$.



Its Fourier transform is

$$\frac{1}{\prod_{\beta \in R} \beta(y)} \sum_{w \in W} \varepsilon(w) e^{i\langle p,y \rangle}$$

density is piecewise polynomial

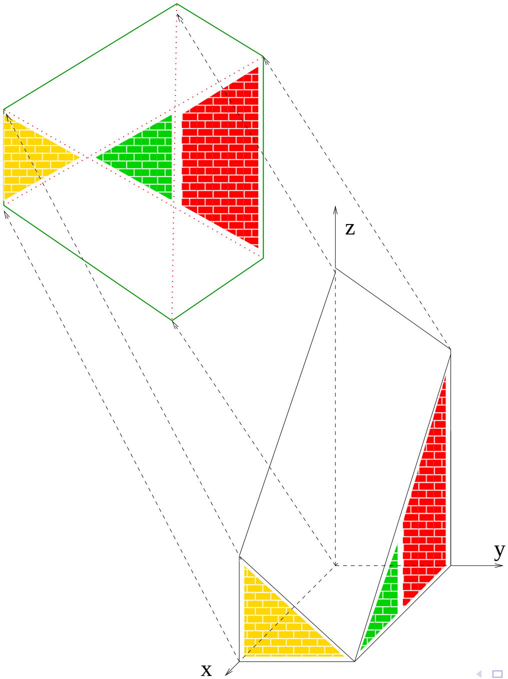


In order to recover X from $P_{w_0}X$ we need a positive real number x_i for each s_i in $P_{w_0} = P_{s_1} \dots P_{s_q}$.

Lemma Given $P_{w_0}X(t)$ the numbers $(x_1, \dots, x_q)$ belong to a certain convex polytope. Their distribution is the normalized Lebesgue measure on this polytope.

Cristallographic case: string polytopes

The Duistermaat-Heckman measure is the image of this measure by an affine map.



$0<x<a$

$0<y<b$

$0<z<(a-x)+(b-y$

THANK YOU