Characters of symmetric groups, combinatorics and free probability

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I will investigate asymptotics of characters and representations of symmetric groups of large order.

In order to describe them one needs to find the right "macroscopic" parameters.

These parameters come from the theory of random matrices.

Once these parameters are found, they yield new *exact* combinatorial formulas for the characters.

I. Random matrices and free cumulants

Example:

Π_1 Π_2, $N \times N$ matrices
=orthogonal projections on subspaces of dimension $N/2$.

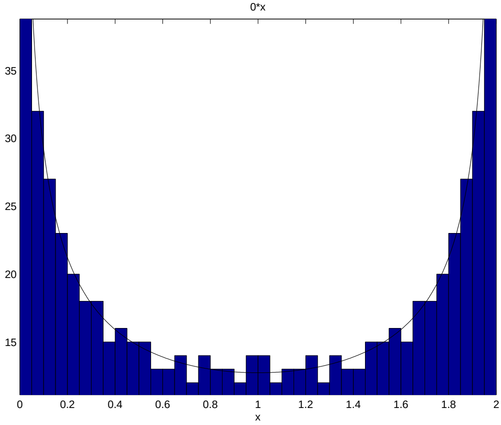
choose subspaces uniformly at random i.e.

$$\Pi_i = U_i \begin{pmatrix} I_{N/2} & 0 \\ 0 & 0 \end{pmatrix} U_i^*$$

U_i are independent, chosen with Haar measure on $U(N)$.

Compute the spectrum of $\Pi_1 + \Pi_2$.

Histogram of the spectrum of $\Pi_1 + \Pi_2$ $\quad (N = 800)$



$$y = \frac{1}{\pi\sqrt{x(2-x)}}$$

Prologue: high dimensional random vectors and concentration of measure

$v_1, v_2, \dots v_n \in \mathbf{R}^N$.

$a_i = \|v_i\|$ fixed.

v_i are chosen randomly and independently
on the sphere of radius a_i

If $N \to \infty$, with high probability v_i are almost orthogonal:

for any $\epsilon > 0$, $i \neq j$

$$P(|\langle v_i, v_j \rangle| > \epsilon) \to_{N \to \infty} 0$$

These angles encode the geometry of the set of vectors.
When N is large, this geometry becomes frozen with high probability.

Replace vectors by matrices considered up to unitary conjugation.

The geometry of matrices is more complicated than that of vectors.

In order to encode it we need more numbers than just inner products.

BASIC INVARIANT THEORY FOR MATRICES

Two complex self-adjoint $N \times N$ matrices L and M have the same spectrum if and only if there exists a unitary U such that

$$L = UMU^*$$

Moreover L and M have the same spectrum if and only if for all $r \geq 0$

$$\frac{1}{N} Tr(L^r) = \frac{1}{N} Tr(M^r)$$

i.e. the matrices have the same moments
In the sequel I will use

$$tr = \frac{1}{N} Tr$$

Let $M_1, \dots, M_n$ be $N \times N$ self-adjoint matrices.

Theorem (Procesi, 1978)

The "non-commutative moments"

$$tr(M_{i_1} \dots M_{i_r}), \qquad r \geq 1,\ i_1, \dots, i_r \in \{1, \dots, n\}$$

form a complete set of invariants of the matrices up to conjugation.

This means that

$$tr(L_{i_1} \dots L_{i_r}) = tr(M_{i_1} \dots M_{i_r}) \qquad \text{for all } r, i_1, \dots, i_r$$

if and only if there exists a unitary matrix U such that

$$L_i = UM_iU^* \qquad \text{for all } i$$

U **does not depend on** i**!**

Let $X_i = U_i D_i U_i^{-1}$
D_i=diagonal matrices
U_i= independent unitary random matrices taken with Haar measure.
The spectra of the X_i are fixed, their eigenvectors are chosen at random.

Theoreme

(Voiculescu, 1990) *As* $N \to \infty$ *the mixed moments*

$$tr(X_{i_1} \dots X_{i_k})$$

are given by explicit polynomials in the $tr(D_i^k) = tr(X_i^k)$ *(with high probability and with a small error)*

Exemples:

$$tr(X_1X_2) \sim tr(X_1)tr(X_2)$$

$$tr(X_1^kX_2^l) \sim tr(X_1^k)tr(X_2^l)$$

$$tr(X_1X_2X_1X_2) \sim tr(X_1^2)tr(X_2)^2 + tr(X_1)^2tr(X_2^2) - tr(X_1)^2tr(X_2)^2$$

Here $\sim$ means that the difference is small with high probability.

Corollary

If we know the spectra of $X_1, \dots, X_n$ we can compute, with a good approximation the spectra of any polynomial in the X_i.

Example:

$$tr((X_1 + X_2)^n) = \sum_{i_1 \dots i_n} tr(X_{i_1} \dots X_{i_n})$$

can be computed in terms of the numbers

$$tr(X_1^k), tr(X_2^k), \qquad k = 1, 2, ...$$

The explicit computation of these polynomials can be done using the theory of *free cumulants* due to R. Speicher.

This theory relies on some combinatorial objects: non-crossing partitions.
A set partition of $\{1, \ldots, n\}$ is *non-crossing* if there are no (i, j, k, l) with

$$i < j < k < l$$

and

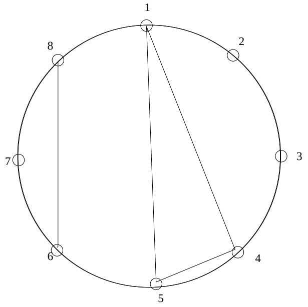
$$i \sim k, \qquad k \sim l$$

and i, j in the same part

Example

$$\{1, 4, 5\} \cup \{2\} \cup \{3\} \cup \{6, 8\} \cup \{7\}$$

is non-crossing



Example: the free convolution.
Suppose you know

$$tr(L^n), tr(M^n), n = 0, 1, 2, \ldots$$

how do you compute $tr((L + M)^n)$?
The answer is given by *free cumulants*.

Introduce the generating functions of the moments of a matrix X

$$G_X(z) = tr((z-X)^{-1}) = \frac{1}{z} + \sum_{n=1}^{\infty} z^{-n-1} tr(X^n)$$

then

$$K_X(G_X(z)) = G_X(K_X(z)) = z; \qquad K_X(z) = \frac{1}{z} + \sum_{n=1}^{\infty} R_n(X) z^{n-1}$$

the $R_n(X)$ are the **free cumulants** of X.

Free cumulants and moments determine each other by a triangular polynomial system:

$$m_1 = R_1$$
$$m_2 = R_2 + R_1^2$$
$$m_3 = R_3 + 3R_1R_2 + R_1^3$$

etc.

Free cumulants and random matrices

Recall the model of random matrices

$$L = UDU^* \qquad M = VEV^*$$

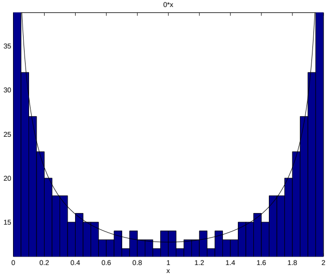
D, E diagonal with given spectrum;
U, V independent Haar unitaries.

for N large one has

$$R_n(L + M) \sim R_n(L) + R_n(M)$$

Random matrix model: compute the spectrum of $\Pi_1 + \Pi_2$ where $\Pi_1, \Pi_2 =$ orthogonal projections on random subspaces of dimensions $N/2$.
The free cumulants computation gives the moments of an arcsine distribution:

$$\frac{dx}{\pi\sqrt{x(2-x)}} \qquad \text{arcsine distribution}$$



Example: the free compression

$$X = UDU^*$$

$$0 < p < 1$$

$X^{(p)}$=$pN \times pN$ upper left corner of X.

$$X = \begin{pmatrix} X^{(p)} & Y \\ Z & W \end{pmatrix}$$

$$R_n(X^{(p)}) \sim_{N\to\infty} p^{n-1} R_n(X)$$

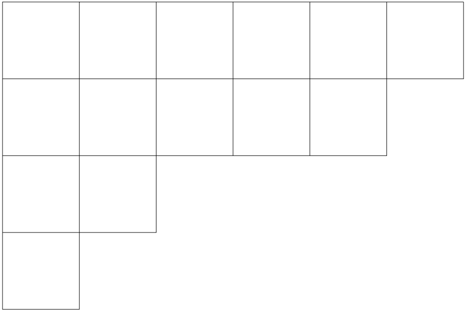
II. Characters of symmetric groups and free cumulants

I will give an asymptotic formula for *characters of symmetric groups*

in terms of *free cumulants* of Young diagrams

PARTITIONS

A partition is a nonincreasing finite sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$.
Partitions label *irreducible representations of symmetric group* on $\lambda_1 + \lambda_2 + \ldots + \lambda_n$ letters.



$$6 + 5 + 2 + 1 = 14$$

$$4 + 3 + 2 + 2 + 2 + 1 = 14$$

FRENCH CONVENTION

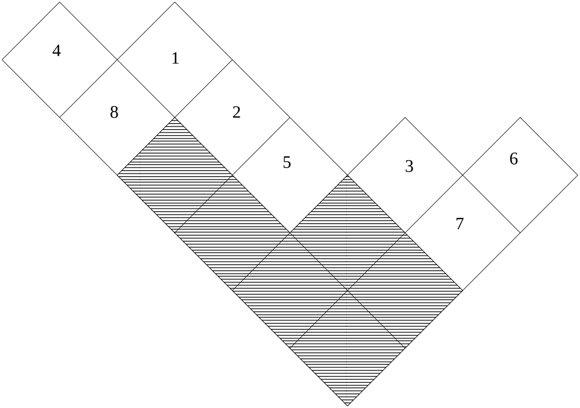
14			
12	13		
4	11		
3	8		
2	6	10	
1	5	7	9

1			
3	1		
4	2		
5	3		
7	5	1	
9	7	3	1

Dimension of a representation=number of Young tableaux.
Hook formula

$$\frac{n!}{\prod_{i,j} h_{ij}}$$

RUSSIAN CONVENTION



Restriction of a representation $S_{14} \downarrow S_6$.

The multiplicity is the number of ways to erase boxes.

LARGE SYMMETRIC GROUPS, KEROV-VERSHIK THEORY

Normalized characters $\chi_\lambda(\mu) = \frac{Tr(\rho_\lambda(\mu))}{\dim(\lambda)}$

$\mu =$ fixed conjugacy class of $S_\infty = \cup_n S_n$

If

$$N = \sum_i \lambda_i$$

$$\lambda_i / N \to \alpha_i$$

$$\lambda_i' / N \to \beta_i$$

$\chi_\lambda(\mu) \to \chi^\infty_{\alpha,\beta}(\mu)$ for some factor representation of S_∞.

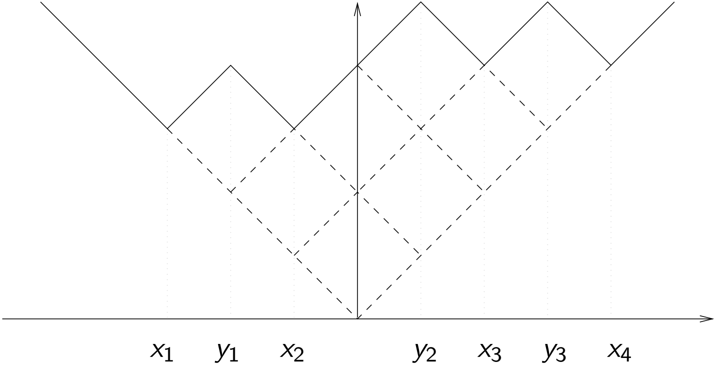
Representation theory of S_∞ is obtained as limit of representation theory of S_N for $N \to \infty$.

For "most" Young diagrams one has $\lambda_i = o(N)$ and

$$\chi_\lambda(\sigma) \sim \delta_e(\sigma)$$

corresponding to the character of the regular representation of S_∞.

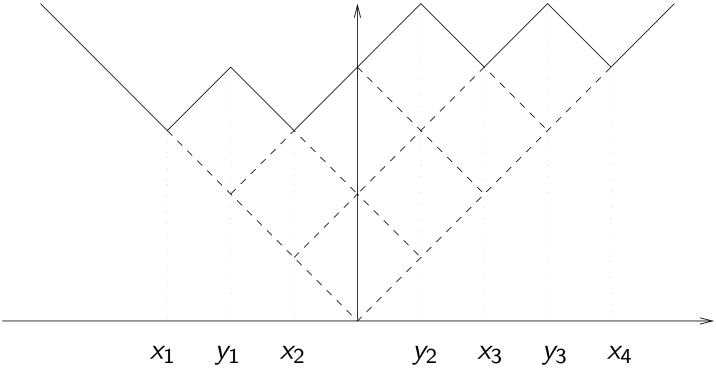
In this regime, $\lambda_i = o(N)$, we need to renormalize characters in order to get a nontrivial limit. Representation theory of symmetric groups is governed by *free probability*.



A diagram may be identified with a function $\omega(x)$ such that

$$|\omega(x)| = |x| \text{ for } x >> 1$$

$$|\omega(x) - \omega(y)| \leq |x - y|.$$



Introduce the generating function of a partition (or Young diagram) λ:

$$G_\lambda(z) = \frac{\prod_{i=1}^{n-1}(z - y_k)}{\prod_{i=1}^{n}(z - x_k)}$$

$$K_\lambda = G_\lambda^{\langle -1 \rangle}$$

$$K_\lambda(z) = \frac{1}{z} + \sum_{n=1}^{\infty} R_n(\lambda) z^{n-1}$$

$R_n(\lambda)$= the free cumulants of the diagram.

By scaling and approximation, this definition can be extended continuously to *continuous diagrams*
i.e. functions $\omega(x)$ such that

$$|\omega(x)| = |x| \text{ for } x >> 1$$

$$|\omega(x) - \omega(y)| \leq |x - y|.$$

ASYMPTOTIC EVALUATION OF CHARACTERS

$\lambda =$ Young diagram with q boxes, $\lambda \sim \sqrt{q}\omega$.

Number of rows and columns $= O(\sqrt{q})$.

$\chi_\lambda =$ normalized character of λ.

$$\chi_\lambda(\sigma) = q^{-|\sigma|/2}\Big(\prod_{c|\sigma} R_{|c|+2}(\omega) + O(q^{-1})\Big)$$

$|\sigma|=$ length of σ w.r.t generating set of all transpositions, the product is over cycles of σ.

Conversely, if a (non-irreducible) character factorizes aproximately over disjoint cycles

$$\chi = \sum_{\lambda} a_\lambda \chi_\lambda$$

$$\chi(\sigma) \sim \prod_{c|\sigma} \chi(c)$$

then a high proportion of Young diagrams occuring in its decomposition have a shape close to the shape ω with free cumulants

$$R_n(\omega) \sim \chi(c_{n-1})$$

ASYMPTOTIC OF RESTRICTION

$\omega=$ continuous diagram, $0 < t < 1$,
define a continuous diagram ω_t by

$$R_n(\omega_t) = t^{n-1} R_n(\omega)$$

The restriction of λ to $S_p \times S_{q-p} \subset S_q$ splits into irreducible

$$\bigoplus c^{\lambda}_{\mu\nu} [\mu] \otimes [\nu] \quad \text{(Littlewood-Richarson rule).}$$

Give a weight $c^{\lambda}_{\mu\nu}\, dim(\mu)\, dim(\nu)$ to the pair (μ, ν).
Then as $q \to \infty$ and $p/q \to t$, almost all pairs (μ, ν) (rescaled by $\sqrt{q}$),
become close to (ω_t, ω_{1-t}).

ASYMPTOTIC OF INDUCTION

For continuous diagrams ω, ω', define $\omega \boxplus \omega'$ by

$$R_n(\omega \boxplus \omega') = R_n(\omega) + R_n(\omega')$$

The induction of $[\mu] \otimes [\nu]$ from $S_p \times S_{q-p}$ to S_q splits into

irreducible

$$\bigoplus c_{\mu\nu}^{\lambda} \, [\lambda]$$

Frobenius duality: the coefficients are given by Littlewood-Richardson rule.

Give a weight $c_{\mu\nu}^{\lambda} \, dim(\lambda)$ to λ.

Rescaling by a common factor $\mu \to \omega$ and $\nu \to \omega'$, then almost all λ become close to the shape $\omega \boxplus \omega'$.

AN IDENTITY IN THE CENTER OF THE GROUP ALGEBRA OF S_n

$$J = (1*) + (2*) + \ldots + (n*) \in \mathbf{C}(S_{[n]\cup\{*\}})$$

$$\Sigma_k = \sum_{1 \le i_1, \ldots, i_k \le n;\, i_j \ \mathit{distinct}} (i_1 \ldots i_k)$$

M_k = projection onto $\mathbf{C}(S_n)$ of J^k
$R_k \in \mathbf{C}(S_n)$ cumulants associated with the moments M_k. Then

$$\chi_\lambda(R_k) = k^{th} \ \mathit{cumulant\ of}\, \lambda$$

There exist universal polynomials (independent of n) such that

$$\Sigma_k = P_k(R_{k+1}, \ldots R_2)$$

Corollary: Kerov's formula for characters.

$$
\begin{aligned}
\Sigma_1 &= R_2 \\
\Sigma_2 &= R_3 \\
\Sigma_3 &= R_4 + R_2 \\
\Sigma_4 &= R_5 + 5R_3 \\
\Sigma_5 &= R_6 + 15R_4 + 5R_2^2 + 8R_2 \\
\Sigma_6 &= R_7 + 35R_5 + 35R_3R_2 + 84R_3 \\
\Sigma_7 &= R_8 + 70R_6 + 84R_4R_2 + 56R_3^2 + 14R_2^3 + 469R_4 + 224R_2^2 + 180R_2
\end{aligned}
$$

Kerov's conjecture: the coefficients are nonnegative.
Proved by V. Féray in his PhD thesis.

A FORMULA FOR KEROV POLYNOMIALS

Consider the formal power series

$$H(z) = z - \sum_{j=2}^{\infty} B_j z^{1-j}.$$

Define

$$\Sigma_k = -\frac{1}{k}[z^{-1}]H(z)\dots H(z-k+1)$$

and

$$R_{k+1} = -\frac{1}{k}[z^{-1}]\,H(z)^k$$

then the expression of Σ_k in terms of the R_k's is given by Kerov's polynomials.

SOME COEFFICIENTS

$$\begin{aligned}\Sigma_8 \quad &= R_9 + 126R_7 + 169R_5R_2 + 252R_4R_3 + 30R_3R_2^2 \\ &\quad +1869R_5 + 3392R_3R_2 + 3044R_3\end{aligned}$$

The coefficient of R_{n+1} in Σ_n is 1.

The coefficient of R_{k+1-2l} in Σ_k is equal to the number of cycles $c \in S_k$,
of length k, such that $(1\,2\ldots k)\,c^{-1}$ has $k-2l$ cycles.

In general the coefficients count more complicated factorizations in the symmetric group (Dołega, Féray, Śniady, 2010)