Rectangular Parking Functions

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Joint work with F. Bergeron (UQAM, Montréal)

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Main result

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$$\sum_{d\geq 0} \mathrm{Frob}\left(\mathcal{P}_{ad,bd}\right) z^d = \exp\left(\sum_{k\geq 1} \frac{1}{ak}\, h_{bk}[ak\,\mathbf{x}]\, z^k\right)$$

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2. $h_{bk}[ak\,\mathbf{x}]$?
3. Frob ?

Outline

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Dyck paths

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(square $n \times n$)

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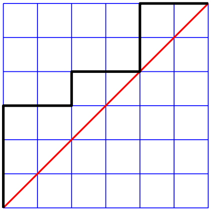

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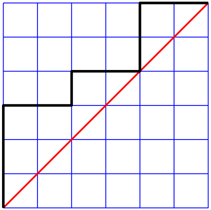
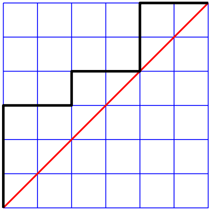


Catalan numbers
$\frac{1}{n+1}\binom{2n}{n}$

Dyck paths



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Dyck paths
(rational $a \times b$)
$(a \wedge b = 1)$

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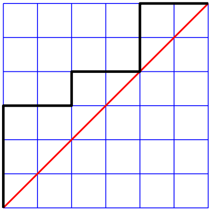

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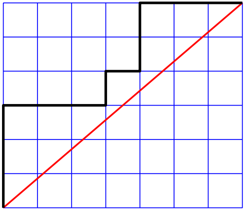

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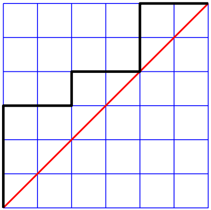

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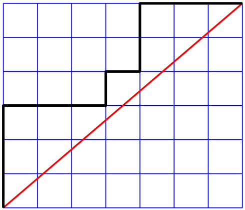

Rational
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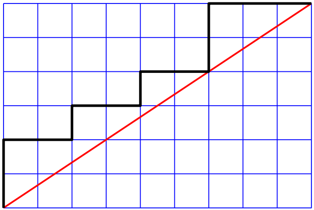

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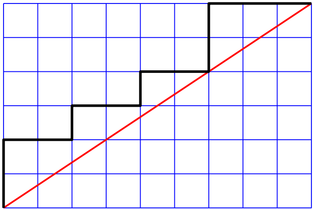

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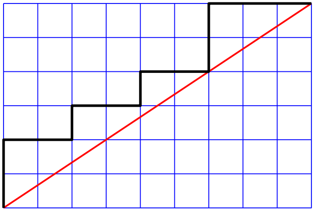

$(m, n) = (ad, bd)$, $a \wedge b = 1$
$\mathcal{D}_{m,n}$ set of (rectangular) Dyck paths $m \times n$.

Rectangular parking functions

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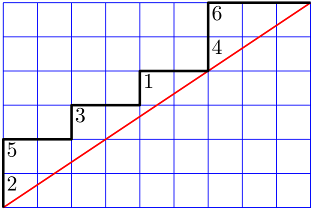

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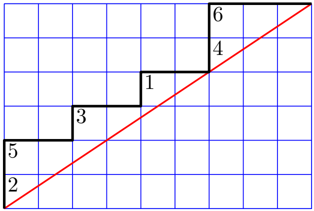

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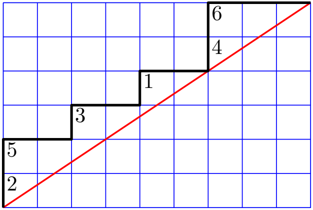

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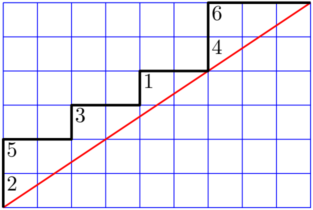
Rectangular parking functions



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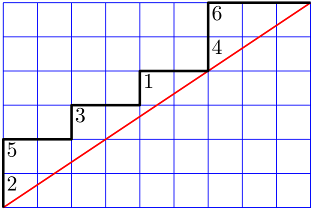

Rectangular parking functions $(m \times n)$

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$\mathcal{L}_\alpha$ set of parking functions with (Dyck) path α

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Outline

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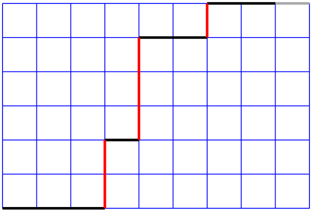

$$(0,0,0,2,3,0,1,0,0)$$

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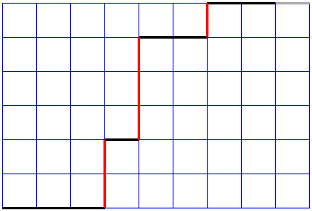

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$\mathcal{B}_{m,n}$ set of rectangular paths $m \times n$ (with last step = East)

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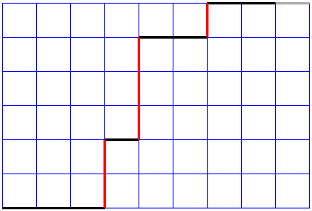

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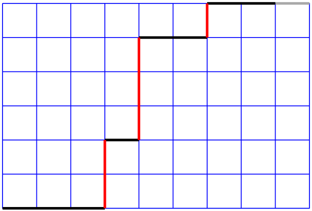

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$\rho(\gamma) = (3, 2, 1)$

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$$= \sum_{\gamma\in\mathcal{B}_{m,n}} h_{\rho(\gamma)}(\mathbf{x}) \qquad (h_0 = 1)$$

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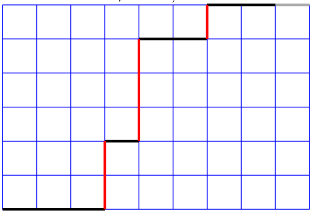

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group of permutations over n letters

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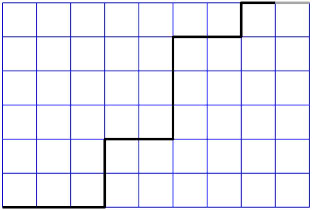

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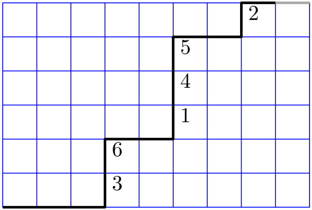

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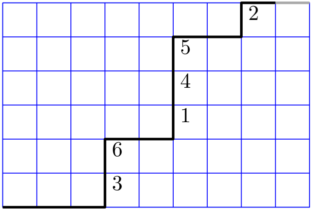

$\mathcal{L}_\gamma$ increasing labellings of γ

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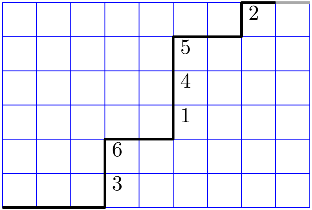

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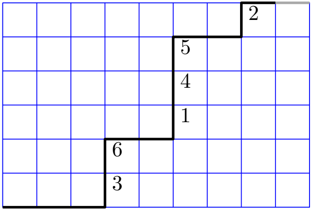

$\mathcal{L}_\gamma$ increasing labellings of γ

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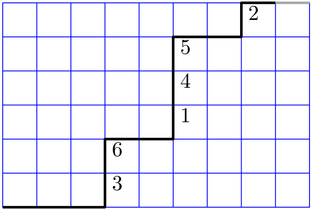

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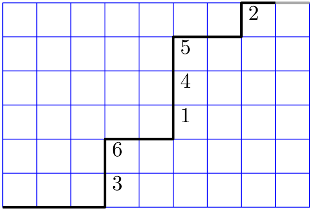

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$$\sum_{\gamma\in\mathcal{B}_{m,n}} |\mathcal{L}_\gamma| = m^n$$

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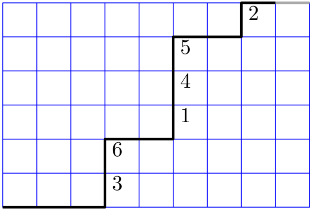

$(6, 8, 4, 6, 6, 4)$

$\mathcal{L}_\gamma$ increasing labellings of γ

$|\mathcal{L}_\gamma| = \binom{n}{\rho(\gamma)} = \frac{n!}{r_1! \, r_2! \cdots r_l!}$ for $\rho(\gamma) = (r_1, r_2, \dots, r_l)$

$$\sum_{\gamma \in \mathcal{B}_{m,n}} |\mathcal{L}_\gamma| = m^n$$

Symmetric group action

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$\mathcal{S}_n$ acts on $\mathcal{L}_\gamma$ by permuting the labels

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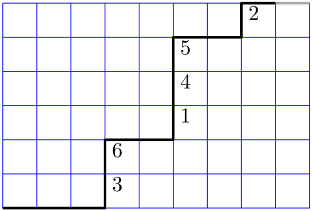

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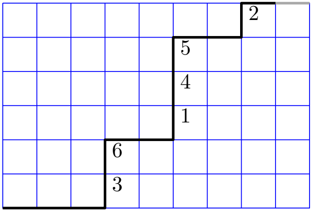

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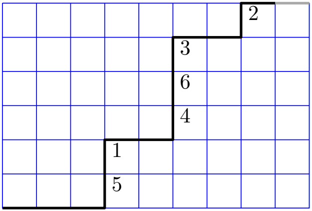

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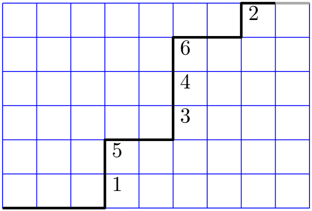

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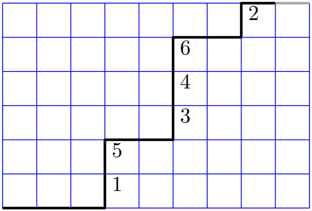

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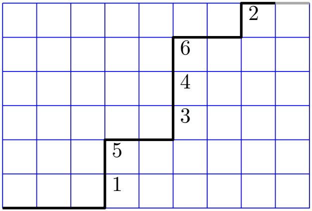

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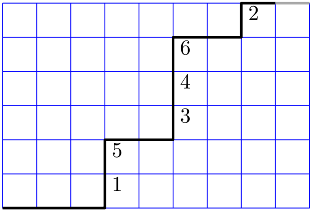

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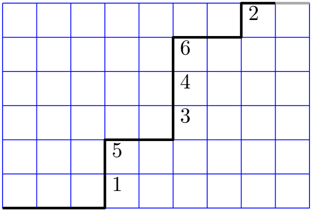

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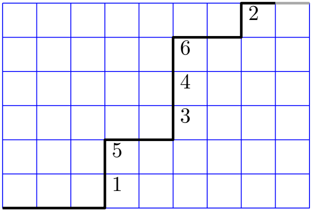

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$\longrightarrow$ Trivial action of $\mathcal{S}_{r_1} \times \mathcal{S}_{r_2} \times \cdots \times \mathcal{S}_{r_l}$ induced on $\mathcal{S}_n$

Frobenius characteristic

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Action of $\mathcal{S}_n$ with character χ

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Cyclic type

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Outline

The formula

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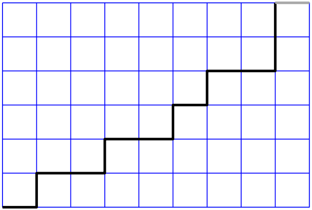
$$\sum_{d\geq 0} \mathrm{Frob}\left(\mathcal{P}_{ad,bd}\right) z^d = \exp\left(\sum_{k\geq 1} \frac{1}{ak}\, h_{bk}[ak\,\mathbf{x}]\, z^k\right)$$

$\gamma \in \mathcal{B}_{m,n}$



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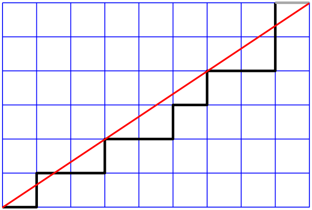

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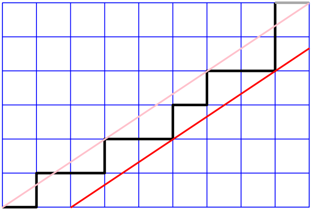

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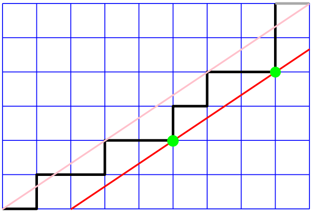

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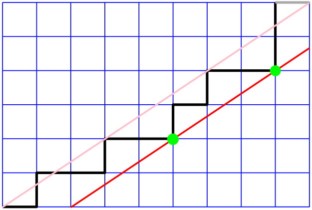
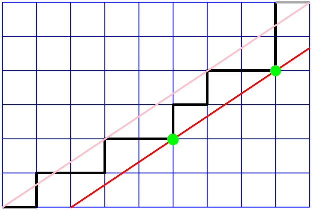


2 low points $(1 \leq t \leq d)$



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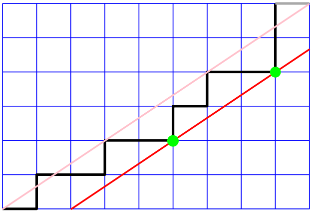

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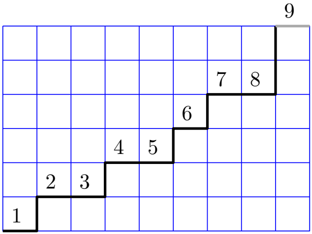

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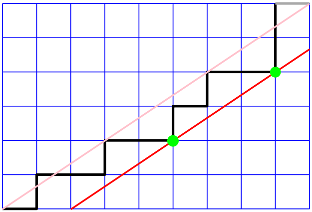

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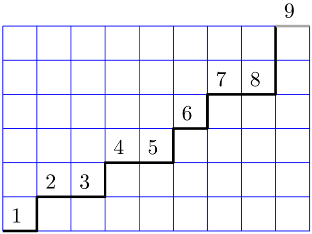

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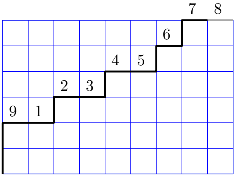

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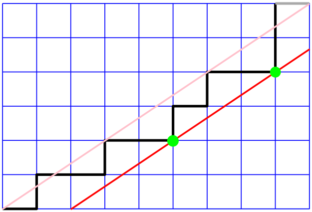

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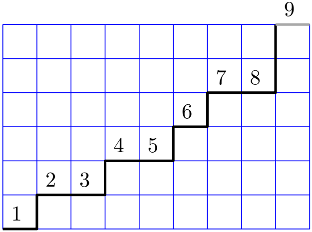

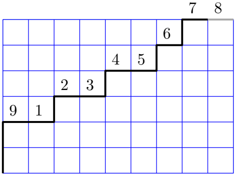

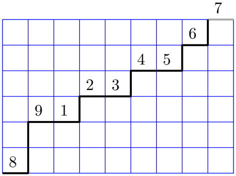

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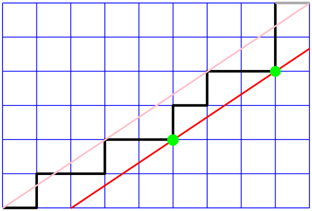

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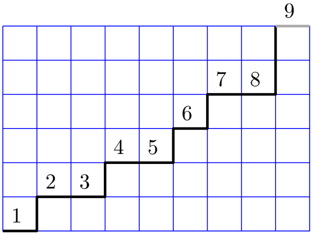

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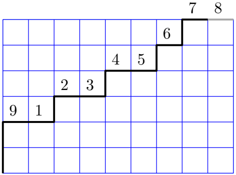

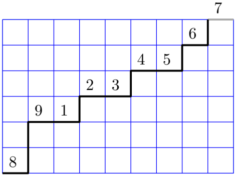

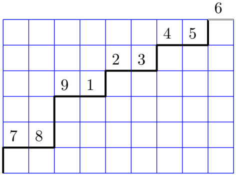

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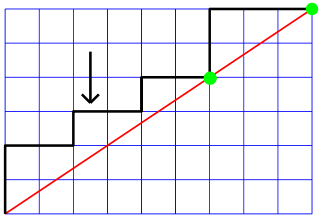
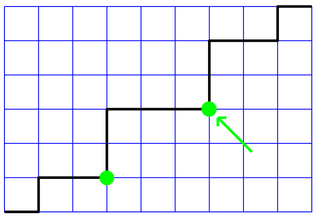


$\longleftrightarrow$



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$$\sum_{t>0}\frac{1}{t}\Phi^t_d(\mathbf{x}) = \frac{1}{ad}\sum_{t>0}\sum_{\gamma\in\mathcal{B}^t_{m,n}} h_{\rho(\gamma)} = \frac{1}{ad}h_{bd}[ad\,\mathbf{x}]$$

$$\Phi^t_d(\mathbf{x}) = \sum_{\substack{c_1,c_2,\dots,c_t>0\\ c_1+c_2+\dots+c_t=d}} \Phi^1_{c_1}(\mathbf{x})\,\Phi^1_{c_2}(\mathbf{x})\cdots\Phi^1_{c_t}(\mathbf{x})$$

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$$\Phi_d^t(\mathbf{x}) = \sum_{\substack{c_1, c_2, \ldots, c_t > 0 \\ c_1 + c_2 + \cdots + c_t = d}} \Phi_{c_1}^{1}(\mathbf{x})\, \Phi_{c_2}^{1}(\mathbf{x}) \cdots \Phi_{c_t}^{1}(\mathbf{x})$$

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$$\frac{1}{ad} h_{bd}[ad\,\mathbf{x}] = \big(-\log(1 - P(\mathbf{x}; z))\big)\Big|_{z^d}$$

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Outline

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$$\begin{aligned}\sum_{d\geq 0} \left|\mathcal{P}_{ad,bd}\right| z^d &= \exp\left(\sum_{k\geq 1} \frac{1}{ak}\, (ak)^{bk}\, z^k\right)\\ &= \exp\left(\sum_{k\geq 1} (ak)^{bk-1}\, z^k\right)\end{aligned}$$

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$m_\mu! = \prod_j m_j!$
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$$\sum_{d\geq 0} \text{Frob}\left(\mathcal{P}_{ad,bd}\right) z^d = \exp\left(\sum_{k\geq 1} \frac{1}{ak} h_{bk}[ak\,\mathbf{x}]\, z^k\right)$$

$$\text{Frob}\left(\mathcal{P}_{ad,bd}\right) = \sum_{\mu\vdash d} \frac{1}{m_\mu!} H_\mu^{a/b}(\mathbf{x})$$

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[Bizley 1954]

Outline

Generalization - Schröder parking functions

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Rectangular Schröder parking functions
$(m \times n)$

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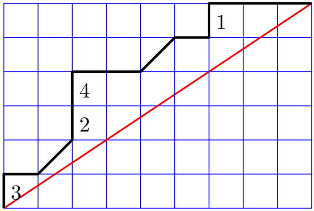
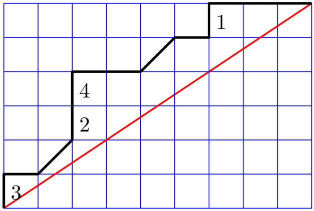
Rectangular Schröder parking functions ($m \times n$)



Generalization - Schröder parking functions

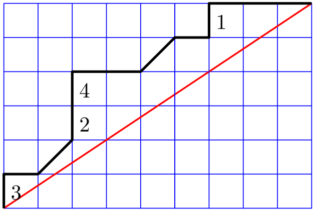
Rectangular Schröder parking functions $(m \times n)$



$\mathcal{S}_{m,n}{}^{(k)}$ set of (rectangular) Schröder parking functions $m \times n$ with k diagonal steps

Generalization - Schröder parking functions

Rectangular Schröder parking functions ($m \times n$)



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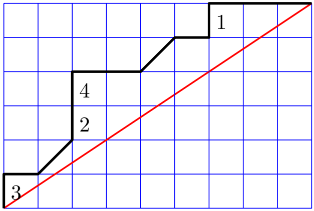
$$\sum_{k \geq 0} \mathrm{Frob}\, \mathcal{S}_{m,n}^{(k)}(\mathbf{x})\, y^k$$

Generalization - Schröder parking functions

Rectangular Schröder parking functions ($m \times n$)



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$$\sum_{k \geq 0} \mathrm{Frob}\, \mathcal{S}_{m,n}^{(k)}(\mathbf{x})\, y^k = \mathrm{Frob}\, \mathcal{P}_{m,n}(\mathbf{x} + y)$$

Références

J.-C. Aval, F. Bergeron, *Interlaced Rectangular Parking Functions*, preprint (2015) `arXiv:1503.03991`.

J.-C. Aval, F. Bergeron, *Rectangular Schröder Parking Functions Combinatorics*, preprint (2016) `arXiv:1603.09487`.