Spectral minimal partitions for a family of tori

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Minimal partitions

Notation

- $\mathrm{T}(a,b)$: two-dimensional flat torus $(\mathbb{R}/a\mathbb{Z}) \times (\mathbb{R}/b\mathbb{Z})$ with $0 < b \leq a$

- $\mathfrak{P}_k$: set of all k-partitions $\mathcal{D} = (D_i)_{i=1,\ldots,k}$ of $\mathrm{T}(a,b)$

  D_i open, connected, and mutually disjoint subsets of $\mathrm{T}(a,b)$

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  D_i open, connected, and mutually disjoint subsets of $\mathrm{T}(a,b)$
- $\lambda_1(D)<\lambda_2(D)\leqslant\cdots$ eigenvalues of the Dirichlet-Laplacian on D

$$\Lambda_k(\mathcal{D}) = \max_{i=1,\dots,k}\lambda_1(D_i)$$

$$\mathfrak{L}_k(\mathrm{T}(a,b)) = \inf\left\{\Lambda_k(\mathcal{D})\ ;\ \mathcal{D}\text{ is a }k\text{-partition}\right\}$$

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$$\begin{aligned} \Lambda_k(\mathcal{D}) &= \max_{i=1,\dots,k} \lambda_1(D_i) \\ \mathfrak{L}_k(\mathrm{T}(a,b)) &= \inf\left\{\Lambda_k(\mathcal{D})\ ;\ \mathcal{D} \text{ is a } k\text{-partition}\right\} \end{aligned}$$

- If $\Lambda_k(\mathcal{D}^*) = \mathfrak{L}_k(\mathrm{T}(a,b))$, then $\mathcal{D}^*$ is called a minimal k-partition

Minimal partitions

Some theoretical results

Theorem

For any k, there exists a minimal k-partition $\mathcal{D}$

[*Conti–Terracini–Verzini, Helffer–Hoffmann-Ostenhof–Terracini, Bucur–Buttazzo–Henrot, Caffarelli–Lin*]

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For any k, there exists a minimal k-partition $\mathcal{D}$

Up to zero capacity sets, $\mathcal{D}$ is strong

- $\mathrm{Int}\,\overline{D_i} = D_i$ and $\cup_{1\leq j\leq k}\overline{D_i} = \mathsf{T}(a,b)$

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- $N(\mathcal{D}) = \cup_{1\leq j\leq k}\partial D_i$ is smooth curve except at finitely many points and $N(\mathcal{D})$ satisfies the Equal Angle Property

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- The D_i are connected and $\Lambda_k(\mathcal{D}) = \lambda_1(D_j)$, for any $1 \leq j \leq k$

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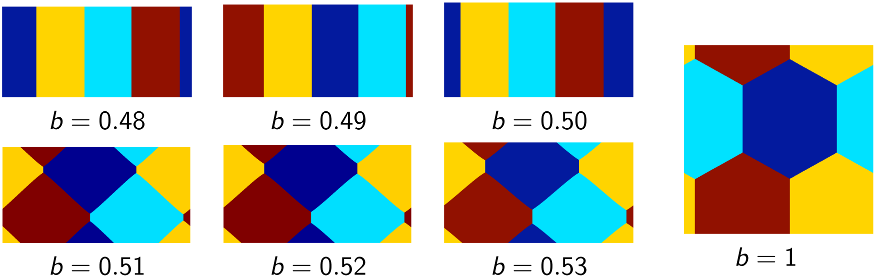
Aim:

determine $\mathfrak{L}_k(\mathsf{T}(1,b))$ and minimal k-partitions according to b $(a=1)$

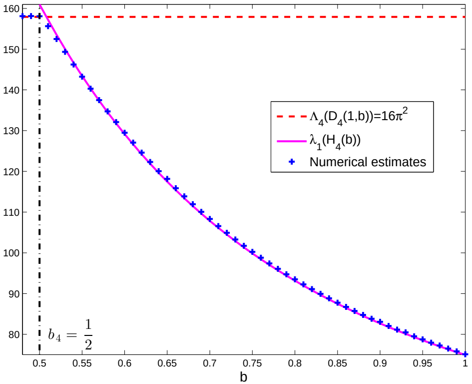
[Conti–Terracini–Verzini, Helffer–Hoffmann-Ostenhof–Terracini, Bucur–Buttazzo–Henrot, Caffarelli–Lin]

Numerical simulations for the 4-partitions of $\mathrm{T}(1, b)$



[*Bourdin–Bucur–Oudet, Bonnaillie-Noël-Léna*]

Numerical simulations for the 4-partitions of $T(1, b)$



Upper bounds of $\mathfrak{L}_4(T(1, b))$ for $b \in \{j/100\,;\, j = 48\,, \ldots\,, 100\}$

[*Bourdin–Bucur–Oudet, Bonnaillie-Noël-Léna*]

Nodal partitions

Definition

Let u be an eigenfunction of $-\Delta$ on $\mathrm{T}(a,b)$

- The nodal sets of u are the components of $\mathrm{T}(a,b) \setminus N(u)$

$$N(u) = \overline{\{x \in \mathrm{T}(a,b) |\ u(x) = 0\}}$$

$\mu(u) =$ number of nodal sets of u

- The partition composed by the nodal sets is called nodal partition

Regularity

$N(u)$ is a $\mathcal{C}^\infty$ curve except on some critical points $\{x\}$
$N(u)$ is locally the union of an **even** number of half-curves ending at x with equal angle

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Any eigenfunction u associated with λ_k has at most k nodal domains

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An eigenfunction u associated with λ, is said to be *Courant-sharp* if $\mu(u) = \min\{\ell\,;\ \lambda_\ell(\mathrm{T}(a,b)) = \lambda\}$

Nodal partition

Minimal partition

Theorem

[*Helffer–Hoffman-Ostenhof–Terracini*]

$$\lambda_k(\mathsf{T}(a,b)) \leqslant \mathfrak{L}_k(\mathsf{T}(a,b))$$

$\Rightarrow$ *the nodal partition of a Courant-sharp eigenfunction is minimal*

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Finally, if $\mathfrak{L}_k(\mathsf{T}(a,b)) = \lambda_k(\mathsf{T}(a,b))$*, all minimal k-partitions are nodal*

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Remarks

- No minimal k-partitions are nodal for large k [*Pleijel*]
- For $k=2$, $\mathfrak{L}_2(\mathsf{T}(a,b)) = \lambda_2(\mathsf{T}(a,b))$

Eigenvalues on the torus

The eigenvalues of the Laplacian on $\mathrm{T}(a, b)$ are

$$\lambda_{m,n}(a, b) = 4\pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) \quad \text{with} \quad m,\ n \in \mathbb{N}_0$$

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Proposition

The only non-constant Courant-sharp eigenfunctions for the torus $\mathrm{T}(1,1)$ *are associated with* $\lambda_2(\mathrm{T}(1,1)) = 4\pi^2$ [*Léna 15*]

$\Rightarrow$ as soon as $k \geq 3$, a minimal k-partition of $\mathrm{T}(1,1)$ is not nodal

Strip partitions

Let $\mathcal{D}_k(a,b)$ be the k-partition of $\mathrm{T}(a,b)$ with domains

$$D_i = \left(\frac{i-1}{k}a, \frac{i}{k}a\right) \times (0,b)\ , \qquad \text{for } i = 1, \dots, k$$

We have

$$\Lambda_k(\mathcal{D}_k(a,b)) = \frac{k^2\pi^2}{a^2}$$

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Let

$$b_k = \sup\{b > 0\ ;\ \mathcal{D}_k(1,b) \text{ is a minimal } k\text{-partition of } \mathrm{T}(1,b)\}$$

k even

Theorem

If k is even, then $b_k = \frac{2}{k}$

- $\mathfrak{L}_k(\mathsf{T}(1,b)) = k^2\pi^2 \quad \text{if} \quad b \leq \frac{1}{k}$

  and $\mathcal{D}_k(1,b)$ is a minimal k-partition of $\mathsf{T}(1,b)$

  (unique, up to a translation, if $b < 1/k$)

- $\mathfrak{L}_k(\mathsf{T}(1,b)) < k^2\pi^2 \quad \text{if} \quad b > \frac{1}{k}$

k odd

For $b \in (0,1]$, we denote

- $\mathsf{S}_b = \mathbb{R} \times (0, b)$: infinite strip
- $j(b) = \inf_{\Omega \subset \mathsf{S}_b, |\Omega| \leqslant b} \lambda_1(\Omega)$

$$b_k^{\mathsf{S}} = \sup \left\{ b \in (0,1]\,;\, j(b) > k^2\pi^2 \right\}$$

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Theorem

If k is odd, then $b_k \geq b_k^{\mathsf{S}} > \frac{1}{k}$ [*Bonnaillie-Noël–Léna 16*]

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We have

$$\frac{1}{k} < b_k^{\mathsf{S}} < \frac{1}{\sqrt{k^2-1}}$$

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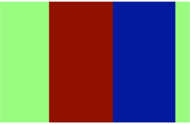
We have

$$\frac{1}{k} < \frac{1}{\sqrt{k^2 - \frac{1}{8}}} \leq b_k^S < \frac{1}{\sqrt{k^2-1}}$$

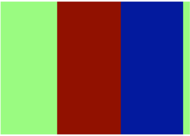
[*Bonnaillie-Noël-Léna 16, Léna 16*]

Numerical simulations

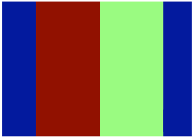
3-partitions



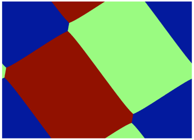
$b = 0.64$



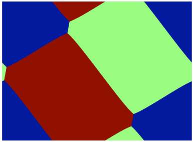
$b = 0.7$



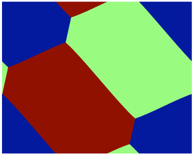
$b = 0.71$



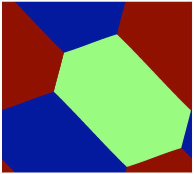
$b = 0.72$



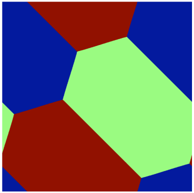
$b = 0.73$



$b = 0.8$



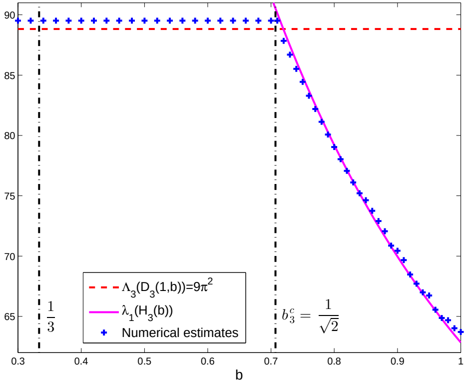
$b = 0.9$



$b = 1$

Numerical simulations

3-partitions



Upper bounds of $\mathfrak{L}_3(\mathsf{T}(1, b))$ for $b \in \{j/100\,;\, j = 30\,,\,\ldots\,,\,100\}$

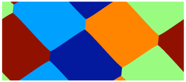
Numerical simulations

5-partitions

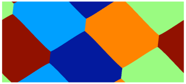
$b = 0.40$

$b = 0.41$

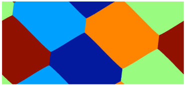
$b = 0.42$



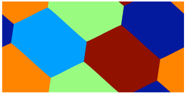
$b = 0.43$



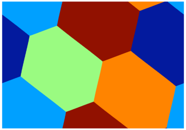
$b = 0.44$



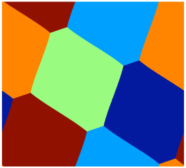
$b = 0.45$



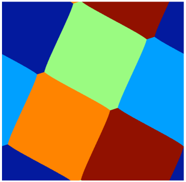
$b = 0.5$



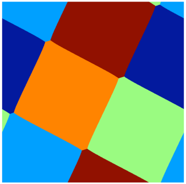
$b = 0.7$



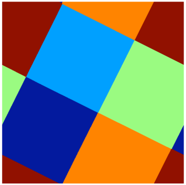
$b = 0.9$



$b = 0.98$



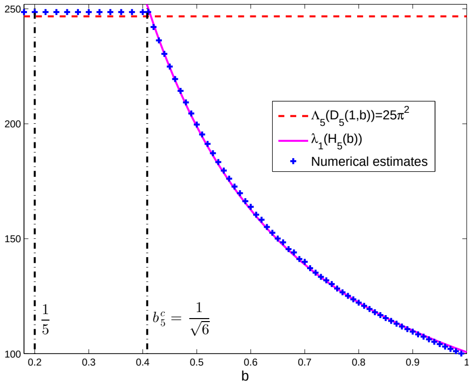
$b = 0.99$



$b = 1$

Numerical simulations

5-partitions



Upper bounds of $\mathfrak{L}_5(\mathsf{T}(1,b))$ for $b \in \{j/100\,;\, j = 18\,,\,\ldots\,,\,100\}$

Hexagonal partitions

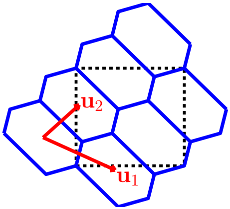
Construction

Theorem

For $k \in \{3, 4, 5\}$, *there exists* $b_k^{\mathsf{H}} \in (0, 1)$ *such that, for any* $b \in (b_k^{\mathsf{H}}, 1]$, *there exists a tiling* $\mathsf{H}_k(b)$ *of* $\mathsf{T}(1, b)$ *by* k *hexagons, that satisfies the equal angle meeting property*

We have

$$\mathfrak{L}_k(\mathsf{T}(1, b)) \leqslant \min\left(k^2\pi^2, \lambda_1(\mathsf{H}_k(b))\right), \quad \forall b \in (b_k^{\mathsf{H}}, 1]$$



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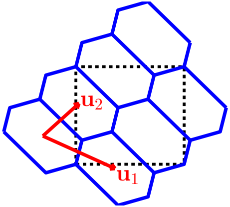
We have

$$\mathfrak{L}_k(\mathsf{T}(1,b)) \leqslant \min\left(k^2\pi^2, \lambda_1(\mathsf{H}_k(b))\right), \quad \forall b \in (b_k^{\mathsf{H}}, 1]$$



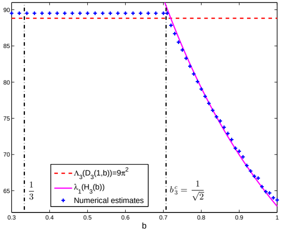
More explicitly, we can choose

$$b_4^{\mathsf{H}} = \frac{1}{2\sqrt{3}} \simeq 0.289 < b_4 = \frac{1}{2}$$

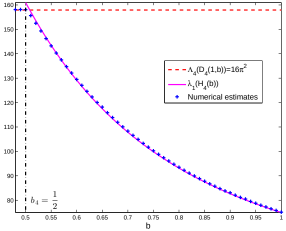
$$b_3^{\mathsf{H}} = \frac{\sqrt{11}-\sqrt{3}}{4} \simeq 0.396, \quad b_5^{\mathsf{H}} = \frac{\sqrt{291}-5\sqrt{3}}{36} \simeq 0.233$$

Hexagonal partitions

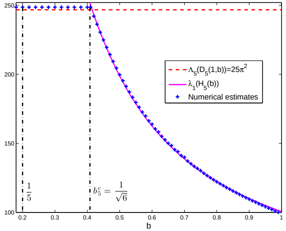
Numerical simulations



$k = 3$



$k = 4$



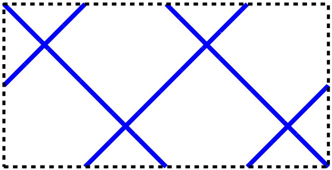
$k = 5$

Transition

k even

$$\mathfrak{L}_4(\mathrm{T}(1,b)) = 16\pi^2, \qquad \forall 0 < b \leq b_4 = \frac{1}{2}$$

- the minimal 4-partitions of $\mathrm{T}(1,1/2)$ are nodal
- $\lambda_4(\mathrm{T}(1,b))$ has multiplicity 4



A nodal 4-partition of $\mathrm{T}(1,1/2)$ (associated with $\sin(4\pi x) + \sin(4\pi y)$)

Transition

k even

$$\mathfrak{L}_4(\mathrm{T}(1,b)) = 16\pi^2, \qquad \forall 0 < b \leq b_4 = \frac{1}{2}$$

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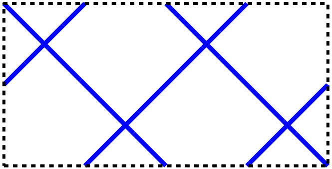

A nodal 4-partition of $\mathrm{T}(1, 1/2)$ (associated with $\sin(4\pi x) + \sin(4\pi y)$)

Conjecture.

- starting point for the apparition of non-nodal 4-partitions of $\mathrm{T}(1,b)$ when $b = 1/2 + \varepsilon$, $0 < \varepsilon \ll 1$
- each singular point of order four splits into two singular points of order three

Conjecture

k odd

If $k \geq 3$ is odd, we conjecture that $b_k = \dfrac{2}{\sqrt{k^2 - 1}}$

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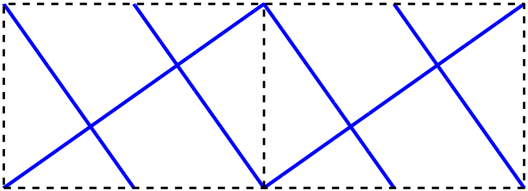
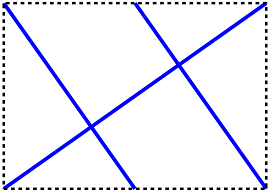


(a) A nodal 6-partition of $T(2, 1/\sqrt{2})$



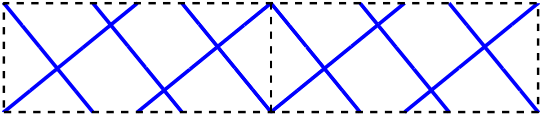
(b) The 3-partition of $T(1, 1/\sqrt{2})$ after projection

Construction of a 3-partition of $T(1, 1/\sqrt{2})$

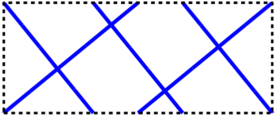
Conjecture

k odd

If $k \geq 3$ is odd, we conjecture that $b_k = \dfrac{2}{\sqrt{k^2 - 1}}$



(c) A nodal 10-partition of $T(2, 1/\sqrt{6})$



(d) The 5-partition of $T(1, 1/\sqrt{6})$ after projection

Construction of a 5-partition of $T(1, 1/\sqrt{6})$