# On the Stability of the Bossel Daners Inequality

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Shape Optimization and Isoperimetric
and Functional Inequalities

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# Faber-Krahn inequality

Let $\Omega \subset \mathbb{R}^n$ be an open set with finite measure.

Eigenvalue problem for the Laplacian with **Dirichlet boundary conditions**:

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

If $\lambda_1(\Omega)$ denotes the first eigenvalue of the Dirichlet Laplacian $\Omega^\star$ is a ball such that $|\Omega^\star| = |\Omega|$, we have:

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^\star) \qquad (\textit{Faber-Krahn inequality})$$

or equivalently, if $B$ denotes a ball in $\mathbb{R}^n$

$$\lambda_1(\Omega)|\Omega|^{\frac{2}{n}} \geq \lambda_1(B)|B|^{\frac{2}{n}}$$

[Faber, 1923], [Krahn, 1925] (Lord Rayleigh)

# Szegö - Weinberger inequality

Eigenvalue problem for the Laplacian with **Neumann boundary conditions**:

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

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If $\mu_2(\Omega)$ denotes the first nontrivial eigenvalue of the Neumann Laplacian, we have:

$$\mu_2(\Omega) \leq \mu_2(\Omega^\star)$$

or equivalently, if $B$ denotes a ball in $\mathbb{R}^n$

$$\mu_2(\Omega)|\Omega|^{\frac{2}{n}} \leq \mu_2(B)|B|^{\frac{2}{n}}$$

[Szegö, 1954], [Weinberger, 1956]

# Brock-Weinstock inequality

Eigenvalue problem for the laplacian with **Steklov boundary conditions**:

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = \sigma u & \text{on } \partial\Omega. \end{cases}$$

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If $\sigma_2(\Omega)$ denotes the first nontrivial eigenvalue of the Steklov Laplacian, we have:

$$\sigma_2(\Omega) \leq \sigma_2(\Omega^\star)$$

or equivalently, if $B$ denotes a ball in $\mathbb{R}^n$

$$\sigma_2(\Omega)|\Omega|^{\frac{1}{n}} \leq \sigma_2(B)|B|^{\frac{1}{n}}$$

[Weinstock, 1954], [Brock, 2001]

# Faber-Krahn inequality

If $\Omega$ is bounded and smooth the eigenvalue problem for the Laplacian with **Robin boundary conditions** ($\beta > 0$):

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ \\ \dfrac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial\Omega. \end{cases}$$

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If $\lambda_{1,\beta}(\Omega)$ denotes the first eigenvalue of the above problem, we have:

$$\lambda_{1,\beta}(\Omega) \geq \lambda_{1,\beta}(\Omega^\star)$$

[Bossel, 1988], [Daners, 2006],[Daners, Kennedy 2007] [Dai, Fu 2008], [Bucur, Daners 2010], [Bandle, Wagner 2015]

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In all the above inequalities equality holds (only) when $\Omega$ is a to a ball. This naturally leads to consider the question of the stability of such inequalities.

# Isoperimetric inequalities in sharp quantitative form

A well known stability result concerns the isoperimetric inequality

$$|\Omega|^{\frac{1-n}{n}} Per(\Omega) \geq |B|^{\frac{1-n}{n}} Per(B)$$

For the above inequality it has been widely investigated the following question:

> *Is it possible to add on the right hand side a remainder term which measures the deviation of $\Omega$ from spherical symmetry?*

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The answer to such a question is the quantitative isoperimetric inequality:

$$|\Omega|^{\frac{1-n}{n}} Per(\Omega) - |B|^{\frac{1-n}{n}} Per(B) \geq C(n)\, \mathcal{A}(\Omega)^2$$

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where $\mathcal{A}(\Omega)$ is the *Fraenkel asymmetry* of $\Omega$, defined as

$$\mathcal{A}(\Omega) = \min\left\{ \frac{|\Omega \triangle B|}{|B|} : B \text{ ball such that} |B| = |\Omega| \right\}$$

($\triangle$ is the symmetric difference). Observe that $0 \leq \mathcal{A}(\Omega) < 2$.

# Isoperimetric inequalities in quantitative form

In the inequality

$$|\Omega|^{\frac{1-n}{n}} Per(\Omega) - |B|^{\frac{1-n}{n}} Per(B) \geq C(n)\,\mathcal{A}(\Omega)^2$$

the exponent 2 is sharp.

[Fuglede, 1989], [Hall, 1992], [Fusco - Maggi - Pratelli, 2008], [Figalli - Maggi - Pratelli, 2010], [Cicalese - Leonardi, 2013].....

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## Remark

*The functional $\mathcal{A}(\Omega)$ is a "scaling invariant" asymmetry functional, i.e. a functional defined over sets such that*

$$\mathcal{A}(t\Omega) = \mathcal{A}(\Omega), \qquad \textit{for every } t > 0$$

*and*

$$\mathcal{A}(\Omega) = 0, \qquad \textit{if and only if } \Omega \textit{ is a ball}$$

# Quantitative inequalities for eigenvalues

The stability question arises also for the inequalities concerning the first (or first nontrivial) Laplacian-eigenvalues with different boundary conditions. For example, in the case of the first Dirichlet Laplacian eigenvalue the following quantitative Faber-Krahn inequality holds true:

$$|\Omega|^{2/n}\lambda_1(\Omega) - |B|^{2/n}\lambda_1(B) \geq C(n)\,\mathcal{A}(\Omega)^2$$

also in this case the exponent 2 is sharp.

[Melas, 1992], [Hansen - Nadirashvili, 1994], [Fusco - Maggi - Pratelli, 2009], [Brasco - De Philippis - Velichkov, 2015]

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Similar results have been obtained for the first non trivial Neumann Laplacian eigenvalue [Brasco - Pratelli, 2012]

and for the first non trivial Steklov Laplacian eigenvalue [Brasco - De Philippis - Ruffini, 2012]

# Robin problem

Then a natural question is:

*Prove a quantitative Faber-Krahn inequality for the first eigenvalue of the Robin Laplacian* $\lambda_{1,\beta}(\Omega)$

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\[2ex] \dfrac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial\Omega. \end{cases}$$

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Observe that for $\beta = +\infty$ the above problem becomes the Dirichlet problem, while for $\beta = 0$ it becomes the Neumann problem.

# The Robin eigenvalue problem

For smooth domains the variational characterization of $\lambda_{1,\beta}$ is given by

$$\lambda_{1,\beta}(\Omega) = \min_{\substack{v \in H^1(\Omega) \\ v \neq 0}} \frac{\displaystyle\int_\Omega |\nabla v|^2 \, dx + \beta \int_{\partial\Omega} v^2 \, d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega v^2 \, dx}.$$

As in the Dirichlet case, the ball minimizes this eigenvalue under the volume constraint, but, contrary to the Dirichlet case, scale invariance is not preserved because of the presence of the parameter $\beta$.

# A remark about $\Omega$

The Faber-Krahn inequality for the Robin Laplacian was extended in [Bucur - Giacomini, 2010], [Bucur - Giacomini, 2015] to arbitrary open sets with finite measure.

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If $A \subset \mathbb{R}^n$ is an open set with finite measure, without any smoothness assumption, the trace of a Sobolev function from $H^1(A)$ is not well defined on $\partial A$, then the Robin eigenvalue problem on $A$ can not be defined.

# A remark about $\Omega$

The Faber-Krahn inequality for the Robin Laplacian was extended in [Bucur - Giacomini, 2010], [Bucur - Giacomini, 2015] to arbitrary open sets with finite measure.

If $u \in BV(\mathbb{R}^n)$ we denote by $u^+, u^-$ the approximate upper and lower limits, and by $J_u = \{x \in \mathbb{R}^n : u^+(x) > u^-(x)\}$ the jump set.

The functions in $SBV(\mathbb{R}^n)$ are such that $D^s u$ (the singular part with respect to the Lebesgue measure of the finite Radon measure $Du$) is concentrated on $J_u$. For a function $u \in SBV(\mathbb{R}^n)$, $\nabla u$ denotes the approximate gradient.

[De Giorgi - Ambrosio, 1989], [Ambrosio, 1990]......

# A remark about $\Omega$

Let $SBV^{\frac{1}{2}}(\mathbb{R}^n)$ [Bucur - Giacomini, 2010, 2015] the set of nonnegative measurable functions $u$ such that $u^2 \in SBV(\mathbb{R}^n)$,
the fundamental Robin eigenvalue of the set $A$ is defined as

$$\lambda_{1,\beta}(A) = \min_{\substack{v \in SBV^{\frac{1}{2}}(\mathbb{R}^n) \\ v=0 \text{ a.e. in } \mathbb{R}^n \setminus A; J_v \subset \partial A \text{ a.e.} \mathcal{H}^{n-1}}} \frac{\displaystyle\int_{\mathbb{R}^n} |\nabla v|^2 \, dx + \beta \int_{J_v} (|v^+|^2 + |v^-|^2) \, d\mathcal{H}^{n-1}}{\displaystyle\int_{\mathbb{R}^n} v^2 \, dx}$$

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- The minimum is attained.
- All minimizers $u$ are called eigenfunctions and they satisfy an eigenvalue equation in a suitably weak sense.
- The Faber - Krahn inequality holds true.

**Theorem ([Bucur - Ferone - Nitsch - T., 2016])**

*Let $\Omega$ be an open set with finite measure and let $B$ a ball with the same measure as $\Omega$, then there exists a positive constant $c$ depending only on $n, \beta, |\Omega|$, such that*

$$\lambda_{1,\beta}(\Omega) - \lambda_{1,\beta}(B) \geq c\,\mathcal{A}(\Omega)^2.$$

# A refinement of the proof of Bossel and Daners

## Lemma ([Bucur - Ferone - Nitsch - T., 2016])

*Let $A$ be a connected, open set with finite measure, and let $B$ a ball with the same measure as $A$. Then if $u$ is a nonnegative, $L^2$-normalized eigenfunction we have*

$$\lambda_{1,\beta}(A) - \lambda_{1,\beta}(B) \geq \frac{\beta}{2}\left(\operatorname*{ess\,inf}_{x\in A} u(x)\right)^2 \left(\mathcal{H}^{n-1}(\partial^* A) - \mathcal{H}^{n-1}(\partial B)\right).$$

$\partial^* A$ stands for the reduced boundary of $A$ and the convention $0 \cdot \infty = 0$ is used.

# A refinement of the proof of Bossel and Daners

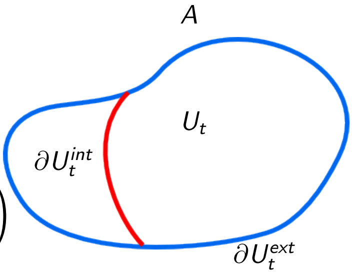
For simplicity we sketch the proof for $A$ smooth.

Let $u$ be a positive eigenfunction.

Let $U_t = \{u > t\}$, $t \geq 0$.

For $\varphi \geq 0$ define

$$H(U_t, \varphi) = \frac{1}{|U_t|}\left(\int_{\partial U_t^{int}} \varphi + \beta \int_{\partial U_t^{ext}} 1 - \int_{U_t} \varphi^2\right)$$



# A refinement of the proof of Bossel and Daners

For almost every $t \in (0, ||u||_{L^\infty})$ it results

$$\lambda_{1,\beta}(A) = H(U_t, \frac{|\nabla u|}{u})$$

$$\lambda_{1,\beta}(A) \geq H(U_t, \varphi) + \frac{1}{t|U_t|}\frac{d}{dt}(t^2 F(t)), \quad \forall\, \varphi$$

where $t^2 F(t)$ is an absolutely continuous function vanishing at 0 and $\|u\|_\infty$.

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where $t^2 F(t)$ is an absolutely continuous function vanishing at 0 and $\|u\|_\infty$. Denoting by $u_B$ a positive (radially symmetric) eigenfunction on $B$ we choose $\varphi$ as the derearrangement of $\dfrac{|\nabla u_B|}{u_B}$ on $U_t$, that is, we define $r(t)$ such that

- $|B_{r(t)}| = |U_t|$;
- $\varphi(x) = \dfrac{|\nabla u_B|}{u_B}(r(t)) \qquad \text{if } u(x) = t,\ x \in A$

moreover $\dfrac{|\nabla u_B|}{u_B}(r(t)) \leq \beta$

# A refinement of the proof of Bossel and Daners

$$\lambda_{1,\beta}(A) - \lambda_{1,\beta}(B) \geq \frac{1}{|U_t|} \left( \int_{\partial U_t^{int}} \varphi + \beta \int_{\partial U_t^{ext}} 1 - \int_{U_t} \varphi^2 \right)$$

$$- \frac{1}{|B(r(t))|} \left( \int_{\partial B(r(t))} \frac{|\nabla u_B|}{u_B} - \int_{B(r(t))} \frac{|\nabla u_B|^2}{u_B^2} \right) + \frac{1}{t|U_t|} \frac{d}{dt}(t^2 F(t))$$

# A refinement of the proof of Bossel and Daners

Using the properties of derearrangement one obtains

$$t|U_t|\left[\lambda_{1,\beta}(A)-\lambda_{1,\beta}(B)\right] \geq t\frac{|\nabla u_B|}{u_B}(r(t))\left[\mathcal{H}^{n-1}(\partial U_t)-\mathcal{H}^{n-1}(\partial B_{r(t)})\right]$$

$$+\frac{d}{dt}(t^2F(t)).$$

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$$+\frac{d}{dt}(t^2F(t)).$$

Integrating we have

$$\int_0^{||u||_{L^\infty}} t|U_t|\left[\lambda_{1,\beta}(A)-\lambda_{1,\beta}(B)\right]\,dt\geq$$

$$\geq\int_0^{||u||_{L^\infty}} t\frac{|\nabla u_B|}{u_B}(r(t))\left[\mathcal{H}^{n-1}(\partial U_t)-\mathcal{H}^{n-1}(\partial B_{r(t)})\right]\,dt.$$

Using the fact that $\dfrac{|\nabla u_B|}{u_B}(r(t))=\beta$ for $0\leq t\leq u_{min}$ we have:

$$\frac{||u||_{L^2}^2}{2}\left[\lambda_{1,\beta}(A)-\lambda_{1,\beta}(B)\right]\geq\frac{u_{min}^2}{2}\beta\left[\mathcal{H}^{n-1}(\partial A)-\mathcal{H}^{n-1}(\partial B)\right]$$

# The quantitative estimate: an idea of the proof

**Step 1**

Let $\Omega \subset \mathbb{R}^n$ be an open set of finite volume. For $k > 0$ the following free discontinuity problem

$$\min\{\lambda_{1,\beta}(A) + k|A| : A \subset \Omega, A \text{ open}\}.$$

admits a solution a solution exists and that for a nonnegative, $L^2$-normalized eigenfunction $u$ on $A$, we have

$$\operatorname*{ess\,inf}_{x\in A} u(x) \geq \alpha > 0,$$

where the value $\alpha$ depends only on $n, \beta, k, \lambda_{1,\beta}(\Omega), |\Omega|$. [Bucur - Giacomini, 2015], [Caffarelli - Kriventsov, 2016]

# The quantitative estimate: an idea of the proof

**Step 1**

Introducing the following relaxed problem

$$\min \left\{ \begin{array}{l} \displaystyle\int_{\mathbb{R}^n} |\nabla u|^2\,dx + \beta \int_{J_u} (|u^+|^2 + |u^-|^2)\,d\mathcal{H}^{n-1} + k|\{u>0\}| : u \in SBV^{\frac{1}{2}}(\mathbb{R}^n) \\ \\ u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega, \displaystyle\int_{\mathbb{R}^n} u^2\,dx = 1. \end{array} \right.$$

a solution $u$ of this problem exists [Bucur - Giacomini, 2010, 2015], is a function in $SBV^{\frac{1}{2}}(\mathbb{R}^n)$ and satisfies:

- $\operatorname{ess\,inf}_{x\in\{u>0\}} u(x) \geq \alpha > 0$,
- the jump set $J_u$ of the function $u$ is closed in $\Omega$,
- we dentify $A = \{u>0\} \setminus J_u$,
- the set $A$ is connected, otherwise a connected component of the set $\{u>0\} \setminus J_u$ would give strictly lower energy.

# The quantitative estimate: an idea of the proof

**Step 2: the refinement of the proof of Bossel and Daners**

$$\lambda_{1,\beta}(A) - \lambda_{1,\beta}(B_A) \geq \frac{\beta}{2}(\operatorname*{ess\,inf}_{x \in A} u(x))^2 \big(\mathcal{H}^{n-1}(\partial^* A) - \mathcal{H}^{n-1}(\partial B_A)\big).$$

# The quantitative estimate: an idea of the proof

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**Step 3**

We apply Step 2 to the set $A$ issued from Step 1. We use the quantitative isoperimetric inequality for the set $A$ and we prove that the Fraenkel asymmetry of $A$ and the loss of measure $|\Omega \setminus A|$ are comparable with the Fraenkel asymmetry of $\Omega$. Consequently we get the stated inequality.

# The quantitative estimate: an idea of the proof

Before giving some details about the proof we point out that the uniform convexity of the map

$$(0, +\infty) \ni r \mapsto \lambda_{1,\beta}(B_r) + k|B_r|$$

is needed.

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Before giving some details about the proof we point out that the uniform convexity of the map

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is needed.
For Dirichlet laplacian this property comes directly from the scaling properties of the eigenvalue

$$\lambda_1(B_r) = \frac{j^2_{n/2-1,1}}{r^2}.$$

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For Dirichlet laplacian this property comes directly from the scaling properties of the eigenvalue

$$\lambda_1(B_r) = \frac{j^2_{n/2-1,1}}{r^2}.$$

For the Robin laplacian the first eigenvalue $\lambda_1(B_r)$ is implicitely defined by the equation $G(r, \lambda) = 0$ where

$$G(r, \lambda) = \sqrt{\lambda} J_{n/2}(\sqrt{\lambda} r) - \beta J_{n/2-1}(\sqrt{\lambda} r)$$

# The quantitative estimate: an idea of the proof

A sketch of Step 3 goes as follows.

Assume that $\lambda_{1,\beta}(\Omega) \leq 2\lambda_{1,\beta}(B)$, otherwise the quantitative inequality holds with a suitable constant.

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A sketch of Step 3 goes as follows.

Assume that $\lambda_{1,\beta}(\Omega) \leq 2\lambda_{1,\beta}(B)$, otherwise the quantitative inequality holds with a suitable constant. It is possible to prove that there exists a constant $\bar{k}$

(depending on $|\Omega|$, $n$, $\beta$) such that

$$\lambda_{1,\beta}(A) + \bar{k}\,|A| \geq \lambda_{1,\beta}(B) + \bar{k}\,|B|, \qquad |B| = |\Omega|$$

for all $A \subset \mathbb{R}^n$ open sets with finite measure.

# The quantitative estimate: an idea of the proof

We denote by $\tilde{A}$ the set which minimizes

$$\min\{\lambda_{1,\beta}(A) + \bar{k}\,|A| : A \subset \Omega, A \text{ open}\}$$

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We denote by $\tilde{A}$ the set which minimizes

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and by $\tilde{u}$ a nonnegative eigenfunction associated to $\lambda_{1,\beta}(\tilde{A})$ normalized in such a way that $||\tilde{u}||_{L^2} = 1$. We denote as well $\tilde{B}$ a ball of the same volume as $\tilde{A}$ attaining the minimum in $\mathcal{A}(\tilde{A})$.
We use the refinement of the Bossel - Daners applied to $\tilde{A}$ and the quantitative isoperimetric inequality

# The quantitative estimate: an idea of the proof

We denote by $\tilde{A}$ the set which minimizes

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$$\begin{aligned}
\lambda_{1,\beta}(\tilde{A}) - \lambda_{1,\beta}(\tilde{B}) &\geq \frac{\beta}{2}(\operatorname*{ess\,inf}_{x\in\tilde{A}} \tilde{u}(x))^2 \big(\mathcal{H}^{n-1}(\partial^*\tilde{A}) - \mathcal{H}^{n-1}(\partial\tilde{B})\big), \\
&\geq C(n)|\tilde{A}|^{\frac{n-1}{n}} \frac{\tilde{u}^2_{min}}{2} \beta \mathcal{A}(\tilde{A})^2.
\end{aligned}$$

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&\geq C(n)|\tilde{A}|^{\frac{n-1}{n}}\frac{\tilde{u}_{min}^2}{2}\beta\mathcal{A}(\tilde{A})^2.
\end{aligned}$$

By the minimality of $\tilde{A}$ and the Step1 we have that there exists a positive constant $C(n,\beta,|\Omega|)$ such that

$$\begin{aligned}
\lambda_{1,\beta}(\Omega) + \bar{k}\,|\Omega| &\geq \lambda_{1,\beta}(\tilde{A}) + \bar{k}\,|\tilde{A}| \geq \\
&\geq \lambda_{1,\beta}(\tilde{B}) + \bar{k}\,|\tilde{B}| + C(n,\beta,|\Omega|)(|\tilde{A}\,\triangle\,\tilde{B}|)^2
\end{aligned}$$

# The quantitative estimate: an idea of the proof

By the convexity result we have that there exists a positive constant $C(n, |\Omega|)$ such that

$$\lambda_{1,\beta}(\tilde{B}) + \bar{k}\,|\tilde{B}| \geq \lambda_{1,\beta}(B) + \bar{k}\,|B| + C(n, |\Omega|)(|B| - |\tilde{B}|)^2$$

# The quantitative estimate: an idea of the proof

By the convexity result we have that there exists a positive constant $C(n, |\Omega|)$ such that

$$\lambda_{1,\beta}(\tilde{B}) + \bar{k}\,|\tilde{B}| \geq \lambda_{1,\beta}(B) + \bar{k}\,|B| + C(n, |\Omega|)(|B| - |\tilde{B}|)^2$$

and then

$$\begin{aligned}
\lambda_{1,\beta}(\Omega) + \bar{k}\,|\Omega| \quad &\geq \lambda_{1,\beta}(\tilde{A}) + \bar{k}\,|\tilde{A}| \geq \\
&\geq \lambda_{1,\beta}(\tilde{B}) + \bar{k}\,|\tilde{B}| + C(n, \beta, |\Omega|)(|\tilde{A} \triangle \tilde{B}|)^2 \\
\lambda_{1,\beta}(\Omega) \quad &\geq \lambda_{1,\beta}(B) + C(n, \beta, |\Omega|)(|B| - |\tilde{B}| + |\tilde{A} \triangle \tilde{B}|)^2 \\
&\geq \lambda_{1,\beta}(B) + C(n, \beta, |\Omega|)\mathcal{A}(\Omega)^2
\end{aligned}$$

where in the last inequality we used the very definition of Fraenkel asymmetry and the fact that $\tilde{A} \subseteq \Omega$.

# The quantitative estimate: an idea of the proof

By the convexity result we have that there exists a positive constant $C(n, |\Omega|)$ such that

$$\lambda_{1,\beta}(\tilde{B}) + \bar{k}\,|\tilde{B}| \geq \lambda_{1,\beta}(B) + \bar{k}\,|B| + C(n, |\Omega|)(|B| - |\tilde{B}|)^2$$

and then

$$\begin{aligned} \lambda_{1,\beta}(\Omega) + \bar{k}\,|\Omega| \quad &\geq \lambda_{1,\beta}(\tilde{A}) + \bar{k}\,|\tilde{A}| \geq \\ &\geq \lambda_{1,\beta}(\tilde{B}) + \bar{k}\,|\tilde{B}| + C(n, \beta, |\Omega|)(|\tilde{A} \triangle \tilde{B}|)^2 \\ \lambda_{1,\beta}(\Omega) \quad &\geq \lambda_{1,\beta}(B) + C(n, \beta, |\Omega|)(|B| - |\tilde{B}| + |\tilde{A} \triangle \tilde{B}|)^2 \\ &\geq \lambda_{1,\beta}(B) + C(n, \beta, |\Omega|)\mathcal{A}(\Omega)^2 \end{aligned}$$

where in the last inequality we used the very definition of Fraenkel asymmetry and the fact that $\tilde{A} \subseteq \Omega$.

The constants $C(n, \beta, |\Omega|)$ above do not depend explicitly on $\bar{k}$ and $\lambda_{1,\beta}(\Omega)$ since $\bar{k}$ depends only on $n, \beta$ and $|\Omega|$ and we have chosen to work with sets $\Omega$ such that $\lambda_{1,\beta}(\Omega) \leq 2\lambda_{1,\beta}(B)$, so that the upper bound of $\lambda_{1,\beta}(\Omega)$ depends only on $n, \beta$ and $|\Omega|$.

# The quantitative estimate: an idea of the proof

In order to compare $|B| - |\tilde{B}| + |\tilde{A} \triangle \tilde{B}|$ with $\mathcal{A}(\Omega)$ we observe that

$$\begin{aligned} |B| - |\tilde{B}| + |\tilde{A} \triangle \tilde{B}| &= |B| - |\tilde{B}| + 2(|\tilde{A}| - |\tilde{A} \cap \tilde{B}|) \\ &\geq |\Omega| + |\tilde{A}| - |\tilde{A} \cap \tilde{B}| - |\Omega \cap B| \\ &\geq |\Omega| - |\Omega \cap B| = \frac{1}{2}|\Omega \triangle B| \end{aligned}$$

It follows

$$|B| - |\tilde{B}| + |\tilde{A} \triangle \tilde{B}| \geq \frac{|\Omega|}{2}\mathcal{A}(\Omega).$$

Thank you for your attention