Regularity of the optimal sets for spectral functionals - Part I

Susanna Terracini

Dipartimento di Matematica "Giuseppe Peano"
Università di Torino
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a joint work with Dario Mazzoleni and Bozhidar Velichkov

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Outline

Outline

The Problem

We consider problem

$$\min\left\{\lambda_1(\Omega)+\cdots+\lambda_k(\Omega)\ :\ \Omega\subset\mathbb{R}^d,\ \text{open}\ ,\ |\Omega|=1\right\},\qquad (1)$$

which, as a simple scaling argument can show that this is equivalent to

$$\min\left\{\lambda_1(\Omega)+\cdots+\lambda_k(\Omega)+\Lambda|\Omega|\ :\ \Omega\subset\mathbb{R}^d\ \text{open}\right\},\qquad (2)$$

where $\Lambda>0$ is a Lagrange multiplier.

Theorem (G. Buttazzo and G. Dal Maso, 1993)

There exists an optimal set Ω^* for problem (2) in the class of quasi-open sets.

Optimal Shape Problems

The solvability of

$$\min\left\{\varphi(\lambda_1(\Omega),\dots,\lambda_k(\Omega))\ :\ \Omega\in\mathcal{A}\ ,\quad |\Omega|=1\right\},$$

for general φ monotone in all variable, strongly depends on the choice of the class $\mathcal{A}$.

In general, for reasonable classes, such as open sets, such a problem does not admit a solution $\Longrightarrow$ a relaxation is needed.
[Buttazzo and Dal Maso (1993)], [Buttazzo and Timofte (2002)]

Reference: The book *Variational methods in shape optimization problems* [Bucur and Buttazzo (2005)]

Energy

For a vector $W = (w_1, \dots, w_k) \in H_0^1(\mathbb{R}^d, \mathbb{R}^k)$, we consider the following energy functional

$$\begin{aligned}\mathcal{F}_0(W) &= \int_{\mathbb{R}^d} |\nabla W|^2\,dx + \Lambda \big|\overbrace{\{|W|>0\}}^{=\Omega_W}\big| \\ &= \sum_{i=1}^{k} \int_{\mathbb{R}^d} |\nabla w_i|^2\,dx + \Lambda \left|\left\{w_1^2+\dots+w_k^2>0\right\}\right|.\end{aligned} \tag{3}$$

Then the vector of normalized eigenfunctions $U = (u_1, \dots, u_k)$ on the optimal set for (2) is a solution to the problem

$$\min\left\{\mathcal{F}_0(W) \ : \ W = (w_1, \dots, w_k) \in H_0^1(\mathbb{R}^d, \mathbb{R}^k),\ \int_{\mathbb{R}^d} w_i w_j\,dx = \delta_{ij}\right\}. \tag{4}$$

Regularity

$$\min\left\{\lambda_1(\Omega)+\cdots+\lambda_k(\Omega)\ :\ \Omega\subset\mathbb{R}^d \text{ quasi-open},\ |\Omega|=1\right\}.$$

Goals:

- Regularity of the optimal set.
- Regularity of associated eigenfunctions.

Theorem

(i) *Bucur (2012); Mazzoleni-Pratelli (2013): every solution Ω^* is bounded.*

(ii) *Bucur (2012): Every solution Ω^* has finite perimeter.*

(iii) *Bucur-Mazzoleni-Pratelli-Velichkov (2015): Let Ω^* be a solution. Then the first k normalized eigenfunctions $u_1,\dots,u_k$ on Ω^*, extended by zero over $\mathbb{R}^d\setminus\Omega^*$, are Lipschitz continuous on $\mathbb{R}^d$ and $\|\nabla u_i\|_{L^\infty}\le C_{d,k}$, for every $i=1,\dots,k$, where $C_{d,k}$ is a constant depending only on k and d. In particular, every solution is an open set.*

d^*-regularity

Definition (d^*-regularity)

Let d^* be any integer. A set $\Omega \subset \mathbb{R}^d$ d^*-regular if $\partial\Omega$ is the disjoint union of a regular part $Reg(\partial\Omega)$ and a (possibly empty), singular part $Sing(\partial\Omega)$ such that:

- $Reg(\partial\Omega)$ is an open subset of $\partial\Omega$ and locally a $\mathcal{C}^\infty$ hypersurface of codimension one ;
- $Sing(\partial\Omega)$ is a closed subset of $\partial\Omega$ and has the following properties:
  - If $d < d^*$, then $Sing(\partial\Omega)$ is empty,
  - If $d = d^*$, then the singular set $Sing(\partial\Omega)$ contains at most a finite number of isolated points,
  - If $d > d^*$, then the Hausdorff dimension of $Sing(\partial\Omega)$ is at most $d - d^*$.

Main result

Our main result is the following.

Theorem (D. Mazzoleni, S. Terracini and B. Velichkov, 2016)

Let the open set $\Omega_k^* \subset \mathbb{R}^d$ be an optimal set for problem (4). Then Ω_k^* is connected and d^*-regular. Moreover the vector $U = (u_1, \dots, u_k)$ of the normalized eigenfunctions on Ω_k^* satisfies the optimality condition

$$\big|\nabla |U|\big| = \sqrt{\Lambda} \quad on \quad Reg(\Omega_k^*),$$

where the constant Λ is given by $\Lambda = \dfrac{2}{d} \displaystyle\sum_{i=1}^{k} \lambda_i(\Omega_k^*)$.

The natural number $d^* \geq 5$ comes from the classical theory of one phase free boundary problems and it is the smallest dimension admitting a nontrivial minimal cone.

Outline

Alt-Caffarelli problem

Define, for a real valued function u,

$$\mathcal{E}_\Lambda(u) = \int |\nabla u|^2 \, dx + \Lambda |\{u > 0\}|$$

Let D be a given open and regular set, together with $\phi \in \mathcal{C}^0(\partial D) \cap H^{1/2}(\partial D)$, $\phi \geq 0$. Consider the minimization problem

The Alt-Caffarelli one phase minimization problem

$$\min \left\{ \mathcal{E}_\Lambda(u) \; : \; u \in H^1(D), \; u = \phi \text{ on } \partial D \right\}.$$

Regularity of solutions to the one-phase problem

Theorem (Alt-Caffarelli, 1981, Weiss, 1999)

Let u^* be a minimizer of the Alt-Caffarelli one phase minimization problem and let $\Omega^* = \{x \in D \ : \ u^*(x) > 0\}$. There exists $d^* \geq 4$ such that Ω^* is (relatively) d^*-regular. Moreover the function u^* satisfy the optimality condition

$$\begin{cases} -\Delta u^* = 0 & \text{in } \Omega^* \\ \left|\nabla u^*\right| = \sqrt{\Lambda} & \text{on} \quad Reg(\Omega^*). \end{cases}$$

Definition (of d^*)

The natural number d^* it is the smallest dimension admitting a nontrivial minimal 1-homogeneous non negative function which is locally minimal for $\mathcal{E}$ with respect to variations having compact support (non containing the origin).

The critical dimension

The 1-homogeneous non negative function satisfies

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega = \{x \in \mathbb{R}^d \ : \ u(x) > 0\} \\ |\nabla u| = \text{const} & \text{on } \partial\Omega. \end{cases}$$

As soon as $d \geq 3$ there do exist nontrivial solutions. But are these solutions minimal?

Theorem

(i) *Alt-Caffarelli (1981): in two dimensions there are no minimal cones which are no hyperplanes;*

(ii) *Weiss (1999):* $d^* \geq 3$;

(iii) *Caffarelli-Jerison-Kenig (2004):* $d^* \geq 4$;

(iv) *De Silva-Jerison (2009):* $d^* \leq 7$;

(v) *Jerison-Savin (2015):* $d^* \geq 5$. *Local convexity with respect to domain variations.*

Outline

Weiss Theorem for vector valued functions

Our main theorem follows from the following statement, which better highlights the analogy with the one phase Alt-Caffarelli problem solved by Weiss.

Theorem (D. Mazzoleni, S. Terracini and B. Velichkov)

Let $D \subset \mathbb{R}^d$ be an open smooth set, $k \in \mathbb{N}$ and consider the problem

$$\min\left\{ \sum_{i=1}^{k} \int_{\mathbb{R}^d} |\nabla u_i|^2\,dx + \Lambda \Big| \overbrace{\cup_{i=1}^{k} \{u_i \neq 0\}}^{=:\Omega} \Big|,\ u_i \in H^1(D), \right.$$
$$\left. u_i = \phi_i \text{ on } \partial D,\ \phi_i \in C^0(\partial D),\ \phi_1 > 0 \right\}. \tag{5}$$

Then every solution $U = (u_1, \dots, u_k)$ has all the components which are locally Lipschitz continuous in D and moreover the free-boundary $\Omega = \partial \cup_{i=1}^{k} \{u_i \neq 0\}$ is (relatively) d^*-regular.

Some papers

- *A minimization problem with free boundary related to a cooperative system,*
  Luis A. Caffarelli, Henrik Shahgholian, Karen Yeressian
  posted on 27 Aug 2016
  arXiv:1608.07689

- *Regularity of the optimal sets for some spectral functionals,*
  Dario Mazzoleni, Susanna Terracini, Bozhidar Velichkov
  posted on 5 Sep 2016
  arXiv:1609.01231

- *Regularity for Shape Optimizers: The Nondegenerate Case,*
  Dennis Kriventsov, Fanghua Lin
  posted on 9 Sep 2016
  arXiv:1609.02624

Quasi-minimality of the eigenfunctions

Proposition (Minimality of U)

Indeed, suppose that the set $\Omega \subset \mathbb{R}^d$ is a solution to the shape optimization problem (2). Then the vector $U = (u_1, \dots, u_k) \in H^1_0(\Omega; \mathbb{R}^k)$ of normalized eigenfunctions on Ω satisfies the following quasi-minimality condition. There are constants $C > 0$ and $\varepsilon > 0$ such that:

$$\int_{\mathbb{R}^d} |\nabla U|^2 \, dx + \Lambda \big| \{ |U| > 0 \} \big| \leq$$

$$\leq \Big(1 + C \| U - \widetilde{U} \|_{L^1} \Big) \int_{\mathbb{R}^d} |\nabla \widetilde{U}|^2 \, dx + \Lambda \big| \{ |\widetilde{U}| > 0 \} \big|,$$

$$\text{for every} \quad \widetilde{U} \in H^1(\mathbb{R}^d; \mathbb{R}^k) \quad \text{such that}$$

$$\| \widetilde{U} \|_{L^\infty} \leq \varepsilon^{-1} \quad \text{and} \quad \| U - \widetilde{U} \|_{L^1} \leq \varepsilon. \tag{6}$$

Outline

Boundary behaviour of the eigenfunctions

Let us consider the normalized eigenfunctions $u_1, \dots, u_k$ on an optimal set Ω for problem (2).

Proposition (Boundary behavior of the eigenfunctions)

Let Ω be optimal for (2) and let $U = (u_1, \dots, u_k) \in H^1_0(\Omega; \mathbb{R}^k)$ be the vector of the first k normalized eigenfunctions on Ω.

1. $|U| = (u_1^2 + \cdots + u_k^2)^{1/2}$ *is non-degenerate, i.e. there are constants* $c_0 > 0$ *and* $r_0 > 0$ *such that for every* $x_0 \in \mathbb{R}^d$ *and* $r \in (0, r_0]$ *the following implication holds*

$$\Big(\fint_{B_r(x_0)} |U|\, dx < c_0 r \Big) \Rightarrow \Big(U \equiv 0 \quad in \quad B_{r/2}(x_0) \Big).$$

2. u_1 *is non-degenerate, i.e. there are constants* $c_0 > 0$ *and* $r_0 > 0$ *such that for every* $x_0 \in \mathbb{R}^d$ *and* $r \in (0, r_0]$ *the following implication holds*

$$\Big(\fint_{B_r(x_0)} u_1\, dx < c_0 r \Big) \Rightarrow \Big(u_1 \equiv 0 \quad in \quad B_{r/2}(x_0) \Big).$$

Density Estimates

Corollary (Density estimate)

Let Ω be optimal for (2). Then $\Omega = \{|U| > 0\}$ and there are constants ε_0, r_0 and δ such that:

1. *We have the density estimate*

$$\varepsilon_0 |B_r| \leq \big|\Omega \cap B_r(x_0)\big| \leq (1-\varepsilon_0)|B_r|,$$

   for every $x_0 \in \partial\Omega$ and $r \leq r_0$.

2. *For every $x_0 \in \partial\Omega$ and $r \leq r_0$ there is a point $x_1 \in \partial B_{r/2}(x_0)$ such that $B_{\delta r}(x_1) \subset \Omega$.*

Outline

Weiss inequality

The following Theorem and the idea of the radial extension, which is a key point in our proof of a monotonicity formula for the functional ϕ was brought to us by Dorin Bucur.

Theorem (Weiss inequality)

Suppose that the function U is a local minimizer of $\mathcal{F}_0$ in B_{r_0}. Then, for every $r \in (0, r_0)$, we have the estimate

$$\int_{B_r} (|\nabla U|^2 - \lambda \cdot U^2)\, dx + \left|B_r \cap \{|U|^2 > 0\}\right| \leq$$

$$\leq \frac{r}{d} \int_{\partial B_r} \left(|\nabla_\tau U|^2 + \frac{|U|^2}{r^2}\right) d\mathcal{H}^{d-1} - \frac{r}{d+2} \int_{\partial B_r} \lambda \cdot U^2 \, d\mathcal{H}^{d-1}$$

$$+ \frac{r}{d} \mathcal{H}^{d-1}\left(\partial B_r \cap \{|U|^2 > 0\}\right).$$

Weiss type monotonicity formula

Theorem (Weiss type monotonicity formula)

Suppose that the vector U is a local minimizer of $\mathcal{F}_0$ in B_{r_0}. Then the function

$$\phi(r) := \frac{1}{r^d}\int_{B_r}(|\nabla U|^2 - \lambda \cdot U^2)\,dx + \frac{1}{r^d}\|\{|U|^2>0\}\cap B_r|+ \\ - \frac{1}{r^{d+1}}\int_{\partial B_r}|U|^2\,d\mathcal{H}^{d-1},$$

is non-decreasing on the interval $(0, r_0)$ up to an integrable term. More precisely, there exists a constant $C_\phi > 0$ such that

$$\psi(r) := e^{C_\phi r^2}\phi(r),$$

is non-decreasing in $(0, r_0)$. Moreover, if $\phi'(r) = 0$ for some $r > 0$, then U is 1-homogeneous in B_r.

Outline

Blow-up sequences

Let U be a quasi-minimizer of $\mathcal{F}_C$ in the open set $D \subset \mathbb{R}^d$, let $r_n \to 0$ be a sequence of positive real numbers and $x_n \in \partial\{|U| > 0\}$ a sequence converging to some $x_0 \in \partial\{|U| > 0\}$.

Proposition

Then, there is a function $U_0 \in H^1_{loc}(\mathbb{R}^d;\mathbb{R}^k) \cap W^{1,\infty}_{loc}(\mathbb{R}^d;\mathbb{R}^k)$ and subsequences of $(r_n)_n$ and $(x_n)_n$ such that, for every ball $B_R \subset \mathbb{R}^d$, the following properties hold.

(a) *The sequence $U_{r_n}(x) = \dfrac{1}{r_n} U(x_n + r_n x)$ converges to U_0 uniformly in B_R and strongly in $H^1(B_R;\mathbb{R}^k)$.*

(b) *The characteristic functions $\mathbf{1}_{\Omega_n}$ converge in $L^1(B_R)$ to $\mathbf{1}_{\Omega_0}$,*

(c) *The sequences of closed sets $\overline{\Omega}_n$ and Ω_n^c converge Hausdorff in B_R respectively to $\overline{\Omega}_0$ and Ω_0^c, where we have set*

$$\Omega_n := \{|U_{r_n}| > 0\} \quad \text{and} \quad \Omega_0 := \{|U_0| > 0\}.$$

Blow-up limits

Moreover, the limit function U_0 is non-degenerate in zero, i.e. there is a dimensional constant $c_d > 0$ such that

$$\|U_0\|_{L^\infty(B_r)} \geq c_d\, r \quad \text{for every} \quad r > 0.$$

Definition (Tangent functions)

Let U be a local minimizer in the open set $D \subset \mathbb{R}^d$ and let $x_0 \in \Delta \cap \partial\{|U| > 0\}$. We denote by $\mathcal{BU}_U(x_0)$ the space of all blow-up limits of U, i.e. the set of all functions $U \in H^1_{loc}(\mathbb{R}^d;\mathbb{R}^k) \cap W^{1,\infty}_{loc}(\mathbb{R}^d;\mathbb{R}^k)$ such that $U = \lim_{n\to\infty} U_{r_n}$, where $(r_n)_n$ is an infinitesimal sequence.

Properties of the blow-ups

Proposition

Suppose that $U = (u_1, \dots, u_k) : \mathbb{R}^d \to \mathbb{R}^k$ *is a solution of* (4) *and* $x_0 \in \partial\Omega_U$. *Suppose that* $U_0 \in \mathcal{BU}_U(x_0)$. *Then* U_0 *is a* 1*-homogeneous function which is a local minimizer of the functional* $\mathcal{F}_0$. *In particular, all the components of* U_0 *are proportional to* $|U_0|$.

Definition (Critical dimension)

The critical dimension d^* is the minimal dimension carrying a positive 1-homogeneous function which is a local minimizer of the functional $\mathcal{F}_0$ and is not a hyperplane.

Outline

The optimality condition on the free boundary

We first prove an optimality condition on the free boundary for the local minimizers of $\mathcal{F}_0$ in the case when the free boundary $\partial\Omega_U$ is smooth.

Lemma

Suppose that U is a local minimizer for $\mathcal{F}_0$. Suppose that the set $\Omega_U = \{|U| > 0\}$ has C^1 regular boundary and that $U \in C^2(\overline{\Omega}_U; \mathbb{R}^k)$. Then $|\nabla U| = \sqrt{\Lambda}$ on $\partial\Omega_U$.

Lemma

Let U be a solution of (4). Then U is a viscosity solution to the problem

$$\begin{aligned} -\Delta U = \lambda U \quad & \text{in} \quad \Omega_U, \\ U = 0 \quad & \text{on} \quad \partial\Omega_U, \\ |\nabla |U|| = \sqrt{\Lambda} \quad & \text{on} \quad \partial\Omega_U. \end{aligned}$$

Viscosity solutions

Definition (Vector viscosity solution)

Let $\Omega \subset \mathbb{R}^d$ be an open set and $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$ a vector with positive coordinates. We say that the continuous function $U = (u_1, \ldots, u_k) : \overline{\Omega} \to \mathbb{R}^k$ is a viscosity solution of the problem

$$-\Delta U = \lambda \cdot U \quad \text{in} \quad \Omega, \qquad U = 0 \qquad |\nabla |U|| = \sqrt{\Lambda} \quad \text{on} \quad \partial\Omega,$$

if for every $i = 1, \ldots, k$ the component u_i is a solution of the PDE

$$-\Delta u_i = \lambda_i u_i \quad \text{in} \quad \Omega, \qquad u_i = 0 \quad \text{on} \quad \partial\Omega,$$

and

$$|\nabla |U|| = \sqrt{\Lambda} \quad \text{on} \quad \partial\Omega$$

holds in viscosity sense,

Viscosity solutions

Definition (Continued)

i.e.

- for every function $\varphi \in C_c^\infty(\mathbb{R}^d)$ such that "φ touches $|U|$ from below in x_0" (which means that $|U| - \varphi$ has a local minimum equal to zero in x_0), we have $|\nabla\varphi|(x_0) \le \sqrt{\Lambda}$.
- for every function $\varphi \in C_c^\infty(\mathbb{R}^d)$ such that "φ_+ touches $|U|$ from above in x_0" (which means that $|U| - \varphi_+$ has a local maximum equal to zero in x_0), we have $|\nabla\varphi|(x_0) \ge \sqrt{\Lambda}$.

Outline

Classification of boundary points

The regular part of the free boundary will be simply given by the points of density 1/2 of the set Ω_U, i.e. we define

$$Reg(\partial\Omega_U) := \Omega_U^{(1/2)},$$

where for every $\gamma \in [0,1]$ and every measurable set $E \subset \mathbb{R}^d$ we recall the classical notation

$$E^{(\gamma)} := \left\{ x_0 \in \mathbb{R}^d \ : \ \lim_{r\to 0} \frac{|E \cap B_r(x_0)|}{|B_r|} = \gamma \right\},$$

while the singular part is given by

$$Sing(\partial\Omega_U) = \partial\Omega_U \setminus Reg(\partial\Omega_U).$$

The Gap

We will prove that $Reg(\partial\Omega_U)$ is relatively open in $\partial\Omega_U$, while the estimate on the Hausdorff dimension of the singular part will follow simply from the Federer's Reduction Theorem.

Lemma (The Gap)

There exists a constant $\delta > 0$ such that for every 1-homogeneous local minimizer U of $\mathcal{F}_0$ we have that

$$0 \notin \Omega_U^{(\gamma)}, \quad \textit{for} \quad \gamma \in (1/2, 1/2 + \delta).$$

Lemma

Let U be a solution of (4). *Then,*

(i) *for every $x_0 \in \partial\Omega_U$ we have*

$$\liminf_{r\to 0} \frac{|B_r(x_0) \cap \Omega_U|}{|B_r|} \geq \frac{1}{2} \ ;$$

(ii) *for every $\gamma \geq 1/2$ we have*

$$\Omega_U^{(\gamma)} = \left\{ x_0 \in \partial\Omega_U \ : \ \lim_{r\to 0} \psi(x_0, r) = \omega_d \gamma \right\},$$

where $\omega_d = |B_1|$ and $\psi(r)$ is the monotone function defined in (13) *;*

(iii) $\partial\Omega_U = \cup_{\gamma \in \{1/2\} \cup [1/2+\delta, 1)} \Omega_U^{(\gamma)}$.

Outline

Reifenberg flatness

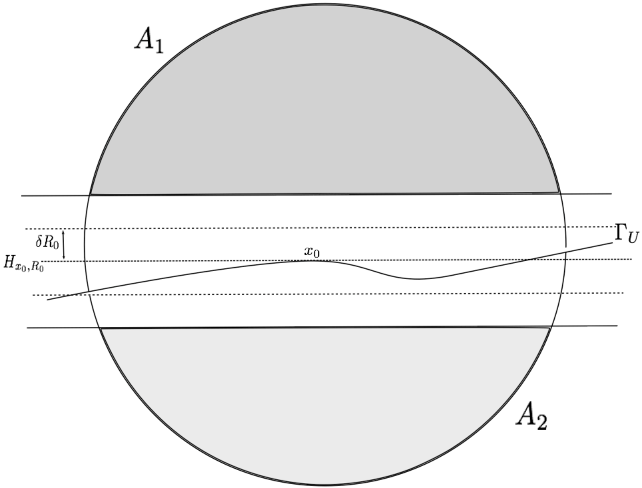
Definition (Reifenberg flatness)

Let $\Omega \subset \mathbb{R}^d$ be an open set and let $0 < \delta < 1/2$, $R > 0$. We say that Ω is a (δ, R)-Reifenberg flat domain if:

1. For every $x \in \partial\Omega$ and every $0 < r \le R$ there is a hyperplane $H = H_{x,r}$ containing x such that
$$\mathrm{dist}_{\mathcal{H}}(B_r(x) \cap H, B_r(x) \cap \partial\Omega) < r\delta.$$
2. For every $x \in \partial\Omega$, one of the connected components of the open set $B_R(x) \cap \{x \ : \ \mathrm{dist}(x, H_{x,R}) > 2\delta R\}$ is contained in Ω, while the other one is contained in $\mathbb{R}^d \setminus \overline{\Omega}$.

Reifenberg flat domains have a special property: they are NTA (non tangentially accessible): harmonic functions vanshing on the same portion of the boundary are nicely comparable.

Reifenberg flatness



Boundary Harnack Principle in NTA domains

Theorem (C. Kenig and T. Toro, 1999)

There exists a $\delta_0 > 0$ such that if $\Omega \subset \mathbb{R}^d$ is a (δ, R)-Reifenberg flat domain for $\delta < \delta_0$, then is NTA.

Theorem (D. Jerison and C. Keing, 1982)

Let Ω be an NTA domain, $x_0 = 0 \in \partial\Omega$. Then there exist R_0, C, depending only on the NTA constants, such that for every $r \in (0, R_0/2)$ and every u, v positive harmonic functions in $B_{2r} \cap \Omega$ vanishing on $\partial\Omega \cap B_{2r}$, Moreover, there exists $\alpha > 0$, depending only on the NTA constants, such that

$$\left|\frac{v(x)}{u(x)} - \frac{v(y)}{u(y)}\right| \leq C\frac{v(z)}{u(z)}\frac{|x-y|^\alpha}{r^\alpha}, \qquad \forall x, y \in \overline{\Omega \cap B_r},\ z \in \Omega \cap B_r.$$

In particular, for every $x_0 \in \partial\Omega \cap B_r$, the limit of $v(x)/u(x)$ as $x \to x_0$ exists.

Consequences

The Hölder regularity of the ratios u_i/u_1 and the existence of the limit till the boundary give the same Hölder regularity also to the normal derivatives of the u_i on the boundary of Ω_U, defined as the limit of the difference quotient.

Corollary

Let U be a solution of (4) and $x_0 = 0 \in \Omega_U^{(1/2)}$. Then there exist $R_0, \alpha > 0$ such that for all $r \in (0, R_0/2)$ there are $\gamma_i \in C^{0,\alpha}(\Omega_U^{(1/2)} \cap B_r)$ for $i = 1, \dots, k$ such that

$$\frac{\partial u_i}{\partial \nu} = \gamma_i \frac{\partial u_1}{\partial \nu}, \quad \text{on } \Omega_U^{(1/2)} \cap B_r,$$

for all $i = 2, \dots, k$ and where ν is the usual external unit normal.

It is now sufficient to join this with the optimality condition for $|U|$ already proved .

End of the proof for the regular part of the boundary

So far we have proved:

Lemma

Let U be a solution of (4). Then u_1 is a viscosity solution to the problem

$$-\Delta u_1 = \lambda_1 u_1 \quad \text{in } \Omega_U, \quad u_1 = 0 \quad \text{on } \partial\Omega_U, \quad |\nabla u_1| = g \quad \text{on } \Omega_U^{(1/2)},$$

where $g \in C^{0,\alpha}(\Omega^{(1/2)})$, for some positive α.

Now we can apply the a general Theorem by D. De Silva:

flatness implies $C^{1,\alpha}$

and we obtain local regularity of the regular part of the free boundary.

Flatness implies $\mathcal{C}^{1,\alpha}$

Let u be a viscosity solution of

$$\begin{cases} -\Delta u = f & \text{in } \Omega := \{u(x) > 0\} \\ |\nabla u| = g(x) & \text{in } \partial\Omega. \end{cases}$$

Theorem (De Silva 2011)

There exists a universal constant $\bar{\varepsilon} > 0$ such that, if the graph of u is $\bar{\varepsilon}$-flat in B_1, i.e.

$$(x_n - \bar{\varepsilon})^+ \leq u(x) \leq (x_n + \bar{\varepsilon})^+$$

and $\|f\|_{L^\infty} \leq \bar{\varepsilon}$, $\|g\|_{\mathcal{C}^{0,\alpha}} \leq \bar{\varepsilon}$. Then, $\partial\Omega$ is $\mathcal{C}^{1,\alpha}$ in $B_{1/2}$.

Applying De Silva-Savin (2015), the theorem can be boot-strapped to increase further the regularity up to $\mathcal{C}^\infty$

Outline

Functions of eigenvalues and extremality conditions

We wish to extend our result to the general case:

$$\min\left\{\varphi(\lambda_1(\Omega),\dots,\lambda_k(\Omega)) \ : \ \Omega\subset\mathbb{R}^d,\ \text{open}\ ,\ |\Omega|=1\right\},$$

where φ is increasing in each variable, as well as to the case of functions which depend only on one or few eigenvalues:

$$\min\left\{\lambda_k(\Omega) \ : \ \Omega\subset\mathbb{R}^d,\ \text{open}\ ,\ |\Omega|=1\right\}.$$

A main problem, in this case id to derive proper extremality conditions (even formally), in particular in the case of multiple eigenvalues.
The following picture is contained in *Numerical Optimization of Low Eigenvalues of the Dirichlet and Neumann Laplacians* by Pedro R.S. Antunes and Pedro Freitas.

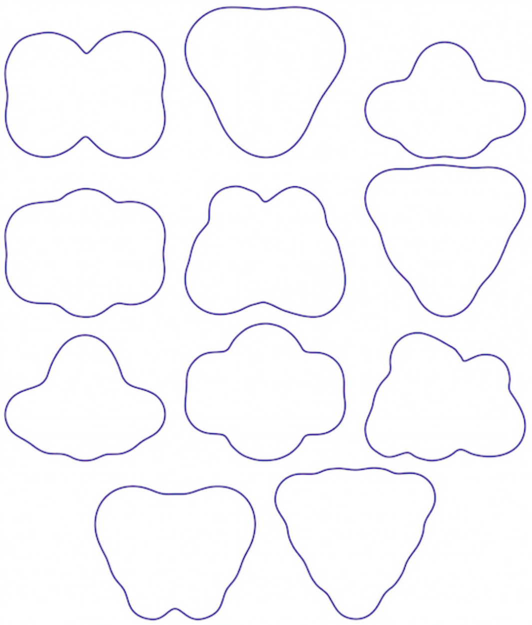


Fig. 3 The optimizers for the Dirichlet eigenvalue problem (5) with $i = 5, 6, \ldots, 15$