Optimal shape of a domain which minimizes the buckling load of a clamped plate

Kathrin Stollenwerk
RWTH Aachen University

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Introduction

- $\Omega \subset \mathbb{R}^n$ open, buckling load of Ω

$$\Lambda(\Omega) := \min \left\{ \frac{\int_\Omega |\Delta v|^2 \, dx}{\int_\Omega |\nabla v|^2 \, dx} : v \in H_0^{2,2}(\Omega) \right\}$$

- minimizer $u \in H_0^{2,2}(\Omega)$ solves

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Polya-Szegö Conjecture (1951)

Among all clamped plates of the same area subjected to a lateral compression, the disk has minimal buckling load.

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- S. (2016) [Existence, $n = 2, 3$, bounded and connected optimal domain]

Existence of an optimal domain

Let $B \subset \mathbb{R}^n$ a ball with $0 < \omega_0 << |B|$. For $v \in H_0^{2,2}(B)$ define

$$\mathcal{J}(v) := \frac{\int_B |\Delta v|^2 \, dx}{\int_B |\nabla v|^2 \, dx} \quad \text{and} \quad \mathcal{O}(v) := \{x \in B : v(x) \neq 0\}.$$

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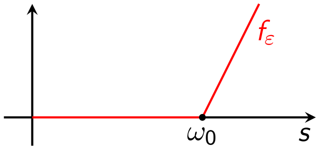
Minimizing problem (P)

Find a function $u \in H_0^{2,2}(B)$ with $|\mathcal{O}(u)| = \omega_0$ such that

$$\mathcal{J}(u) = \min\{\mathcal{J}(v) : v \in H_0^{2,2}(B), |\mathcal{O}(v)| = \omega_0\}.$$

For $\varepsilon > 0$ define the penalty term

$$f_\varepsilon(s) := \begin{cases} 0, & s \leq \omega_0 \\ \frac{1}{\varepsilon}(s - \omega_0), & s \geq \omega_0 \end{cases}.$$



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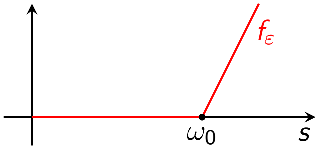

Penalized functional $\mathcal{J}_\varepsilon$ on $H_0^{2,2}(B)$

$$\mathcal{J}_\varepsilon(v) := \mathcal{J}(v) + f_\varepsilon(|\mathcal{O}(v)|).$$

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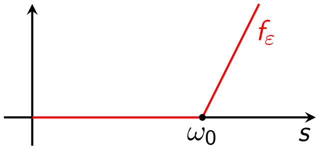

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Penalized minimizing problem (P_ε)

Find a function $u_\varepsilon \in H_0^{2,2}(B)$ such that

$$\mathcal{J}_\varepsilon(u_\varepsilon) = \min\{\mathcal{J}_\varepsilon(v) : v \in H_0^{2,2}(B)\}.$$

Existence of solutions

For every $\varepsilon > 0$ there exists a minimizer $u_\varepsilon \in H_0^{2,2}(B)$ of $\mathcal{J}_\varepsilon$.

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Regularity I

Let u_ε *be a solution of the penalized problem. Then* $u_\varepsilon \in C^{1,\alpha}(\overline{B})$ *for each* $\alpha \in [0, 1)$.

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Set $\Omega(u_\varepsilon) := \mathcal{O}(u_\varepsilon) \cup \{x \in \partial\mathcal{O}(u_\varepsilon) : |\nabla u_\varepsilon| > 0\}$. Then

$$\begin{cases} \Delta^2 u_\varepsilon + \Lambda_\varepsilon \Delta u_\varepsilon = 0, & \text{in } \Omega(u_\varepsilon) \\ u_\varepsilon = |\nabla u_\varepsilon| = 0, & \text{in } \partial\Omega(u_\varepsilon). \end{cases}$$

Note: $\Omega(u_\varepsilon)$ open and $|\mathcal{O}(u_\varepsilon)| = |\Omega(u_\varepsilon)|$.

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There exists an $\varepsilon_0 > 0$ such that for every $\varepsilon < \varepsilon_0$ there holds $|\Omega(u_\varepsilon)| = \omega_0$.

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From now on: choose $\varepsilon < \varepsilon_0$ and omit the index ε

- u is a minimizer for the problem (P)
- $\Omega(u)$ is an optimal domain for minimizing Λ
- $|\Omega(u)| = \omega_0$, $\Omega(u)$ is connected

Nondegeneracy

There exists a constant $c_0 > 0$ *such that for each* $B_r(x_0)$ *with* $x_0 \in \partial\Omega(u)$ *there holds*

$$c_0\, r \leq \sup_{B_r(x_0)} |\nabla u| \,.$$

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- $\partial\Omega(u)$ does not touch ∂B

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Missing: any information about the regularity of $\partial\Omega(u)$!

Uniqueness of the optimal domain

Joint work with A. Wagner.

Assumption

There exists a smooth optimal domain $\Omega \subset \mathbb{R}^n$ *and* $\partial\Omega$ *is connected.*

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Volume preserving perturbations of Ω:

- $\Omega_t := \left\{ x + t\, v(x) + \frac{t^2}{2}\, w(x) + o(t^2) : x \in \Omega \right\}$
- $|\Omega_t| = |\Omega| + o(t^2) \qquad \left(\Rightarrow \int_\Omega \operatorname{div} v\, dx = 0 \right)$
- $\Lambda(t) := \Lambda(\Omega_t) = \min \left\{ \frac{\int_B |\Delta v|^2 dx}{\int_B |\nabla v|^2 dx} : v \in H_0^{2,2}(\Omega_t) \right\}$

First domain variation

Let $u \in H_0^{2,2}(\Omega)$ *be a minimizer. Then*

$$\dot{\Lambda}(0) = -\int_{\partial\Omega} |\Delta u|^2 \, v(x) \cdot \nu(x) dS(x) = 0.$$

First domain variation

Let $u \in H_0^{2,2}(\Omega)$ be a minimizer. Then

$$\dot{\Lambda}(0) = -\int_{\partial\Omega} |\Delta u|^2 \, v(x) \cdot \nu(x) dS(x) = 0.$$

Consequently, u solves the overdetermined boundary value problem

$$\begin{cases} \Delta^2 u + \Lambda \Delta u = 0, & \text{in } \Omega \\ u = |\nabla u| = 0, & \text{in } \partial\Omega \\ \Delta u = const., & \text{in } \partial\Omega \end{cases}$$

Thus, $\Delta u + \Lambda u$ is constant in $\overline{\Omega}$.
For $n = 2$: Ω is a ball (Weinberger/Willms).

Second domain variation

Let u' be a shape derivative of u resulting from a volume preserving perturbation of Ω. Then

$$\ddot{\Lambda}(0) = 2 \int_{\Omega} \left|\Delta u'\right|^2 - \Lambda \left|\nabla u'\right|^2 dx \geq 0.$$

Second domain variation

Let u' be a shape derivative of u resulting from a volume preserving perturbation of Ω. Then

$$\ddot{\Lambda}(0) = 2 \underbrace{\int_\Omega \left|\Delta u'\right|^2 - \Lambda \left|\nabla u'\right|^2 dx}_{=:\mathcal{F}(u')} \geq 0.$$

Define

$$\mathcal{Z} := \left\{ \varphi \in H_0^{1,2} \cap H^{2,2}(\Omega) : \partial_\nu \varphi \not\equiv 0, \int_{\partial\Omega} \partial_\nu \varphi dS = 0, \int_\Omega \nabla u . \nabla \varphi dx = 0 \right\}.$$

- {shape derivatives u'} $\subsetneq \mathcal{Z}$
- $\mathcal{F}(\varphi) \geq 0$ for every $\varphi \in \mathcal{Z}$

Payne's inequality (1955)

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$$\lambda_2(G) \leq \Lambda(G).$$

Equality holds if and only if G is a ball.

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- $\psi(x) := (1-t)\,u_1(x) + t\,u_2(x) + c\,u(x)$, $x \in \overline{\Omega}$
- $\psi \in \mathcal{Z}$ for suitable c and suitable $t \in (0,1]$

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$\Rightarrow$ $\mathcal{F}(\psi) = (1-t)^2 \lambda_1 \left(\lambda_1 - \Lambda\right) + t^2 \lambda_2 \left(\lambda_2 - \Lambda\right) \geq 0$

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$\Rightarrow$ $t = 1$ and $\lambda_2(\Omega) = \Lambda(\Omega)$

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$\Rightarrow$ $t = 1$ and $\lambda_2(\Omega) = \Lambda(\Omega)$

$\Rightarrow$ Ω is a ball!

Thank you!

Constructing ψ: fix $t \in (0,1]$ such that

$$\int_\Omega (1-t)\,\lambda_1\, u_1(x) + t\,\lambda_2\, u_2(x)\, dx = 0.$$

and set

$$c := -\frac{1}{\Lambda}\int_\Omega (1-t)\,\lambda_1\, \nabla u_1.\nabla u + t\,\lambda_2\, \nabla u_2.\nabla u\, dx$$