About the stability of Borell-Brascamp-Lieb inequalities

SALANI PAOLO
Università di Firenze

Shape Optimization and Isoperimetric and Functional Inequalities

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Advertising: papers and collaborators

D. Ghilli, P. S., *Stability of isoperimetric type inequalities for some Monge-Ampère functionals*, Ann. Mat. Pura Appl. (4) 193 (2014).

D. Ghilli, P. S. *Quantitative Borell-Brascamp-Lieb inequalities for power concave functions* (2015), to appear in *J. Convex Analysis*.

A. Rossi, P.S. *Stability for Borell-Brascamp-Lieb inequalities* (2016), to appear in the next volume of *GAFA Seminar Notes*.

Daria Ghilli: Postdoc at Karl-Franzens-Universität Graz, Austria.

Andrea Rossi: PhD student in Firenze.

Notations: p-means of non-negative numbers

Let $p \in [-\infty, +\infty]$ and $\mu \in (0, 1)$. Given two real numbers $a > 0$ and $b > 0$, the quantity

$$M_p(a, b; \mu) = \begin{cases} \max\{a, b\} & p = +\infty \\ ((1-\mu)a^p + \mu b^p)^{\frac{1}{p}} & \text{for } p \neq -\infty, 0, +\infty \\ a^{1-\mu} b^{\mu} & p = 0 \\ \min\{a, b\} & p = -\infty. \end{cases}$$

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Monotonicity w.r.t. p: $M_p(a,b;\mu) \leq M_q(a,b;\mu)$ if $p < q$
("$=$" iff $a = b$ or $ab = 0$).

The Borell-Brascamp-Lieb inequality

Let $0 < \lambda < 1$, $-\frac{1}{n} \leq p \leq \infty$. Let u_0, u_1, h be nonnegative integrable functions defined on $\mathbb{R}^n$, satisfying

$$h((1-\lambda)x + \lambda y) \geq M_p(u_0(x), u_1(y), \lambda) \quad x, y \in \mathbb{R}^n$$

Then

$$\int_{\mathbb{R}^n} h(x)\,dx \geq M_q\left(\int_{\mathbb{R}^n} u_0(x)\,dx, \int_{\mathbb{R}^n} u_1(x)\,dx, \lambda\right)$$

where

$$q = \begin{cases} 1/n & p = +\infty \\ \dfrac{p}{pn+1} & p \in (-1/n, +\infty) \\ -\infty & p = -1/n. \end{cases}$$

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Henstock-Macbeath (1953), Dinghas (1957)

Borell (1975), Brascamp-Lieb (1976)

The case $p = 0$

Prékopa-Leindler inequality

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Note: $$\log(h((1-\lambda)x + \lambda y)) \geq (1-\lambda)\log(u_0(x)) + \lambda \log(u_1(y))$$

$$h((1-\lambda)x + \lambda y)^p \geq (1-\lambda)u_0(x)^p + \lambda u_1(y)^p \quad (p > 0)$$

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NOOO! We have a very strong assumption!

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Let us define the function $u^*_{p,\lambda}$ as follows:

$$u^*_{p,\mu}(x) = \sup\{M_p(u_0(x_0), u_1(x_1); \lambda) \;:\; x = (1-\lambda)x_0 + \lambda x_1\}$$

and call it *p-concave convolution* of u_0 and u_1 (with weight λ).

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Remark: for $p > 0$, only the case of compactly supported functions is interesting...

Analysis versus Geometry

BBL inequalities are functional versions of the Brunn-Minkowski inequality.

The Brunn-Minkowski inequality

K_0, K_1 measurable sets, $\lambda \in [0,1]$, $K_\lambda = (1-\lambda)K_0 + \lambda K_1$ and $+$ is the Minkowski addition, then

$$|K_\lambda| \geq M_{1/n}(|K_0|, |K_1|; \lambda) = \left[(1-\lambda)\,|K_0|^{\frac{1}{n}} + \lambda\,|K_1|^{\frac{1}{n}}\right]^n \qquad (0.1)$$

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BM is equivalent to BBL (for any p).
Indeed, BM can be written in many different equivalent ways, for instance

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Rigidity

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Equality conditions in BBL - Dubuc (1977)

Equality holds in BBL for some $p \in [-1/n, \infty)$ if and only if

h is **p-concave**

and there exist suitable $A, B, m, n > 0$ and $x_1, x_\lambda \in \mathbb{R}^n$ such that

$$u_0(x) = A\, h(mx + x_1)\,, \qquad u_1(x) = B\, h(nx + x_\lambda)\,.$$

Power concave functions

Let Ω be a convex set in $\mathbb{R}^n$ and $p \in [-\infty, \infty]$. A nonnegative function u defined in Ω is said p ***-concave*** if

$$u((1-\lambda)x + \lambda y) \geq \mathcal{M}_p(u(x), u(y); \lambda)$$

for all x, $y \in \Omega$ and $\lambda \in (0,1)$. In the cases $p = 0$ and $p = -\infty$, u is also said log-concave and quasi-concave in Ω, respectively.

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In other words, a non-negative function u, with convex support Ω, is p-concave if:
- it is a non-negative constant in Ω, for $p = +\infty$;
- u^p is concave in Ω, for $p > 0$;
- $\log u$ is concave in Ω, for $p = 0$;
- u^p is convex in Ω, for $p < 0$;
- it is quasi-concave, i.e. all of its superlevel sets are convex, for $p = -\infty$.

For $p = 1$ corresponds to usual concavity.
From Jensen's inequality it follows that if u is p-concave, then u is q-concave for every $q \leq p$.

Stability/Improvements

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(All for log-concave functions or for some other special class of functions)

Stability/Improvements of PL for log-concave functions

Ball-Böröczky [1]

The first Ball-Böröczky result is for log-concave functions in dimension 1 and it is written as an L_1-stability result: if $\int_{\mathbb{R}} h dx \leq (1+\epsilon)\sqrt{\int_{\mathbb{R}} u_0 dx \int_{\mathbb{R}} u_1 dx}$, then there exist $a > 0$ and $b \in \mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} |a^{(-1)^i} u_i(x + (-1)^i b) - h(x)|\, dx \leq \gamma\, \omega(\epsilon) \int_{\mathbb{R}^n} h\, dx\,,$$

where $\omega(\epsilon) = \epsilon^{1/3} |\log \epsilon|^{4/3}$.

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Ball-Böröczky [2]

They extended it to dimension $n > 1$ in [2], but only for log-concave even functions (and with $\sqrt{\omega(\epsilon)}$ in place of $\omega(\epsilon)$).

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Bucur-Fragalà [3]

Bucur-Fragalà [3] use the 1-dimensional result by Ball-Böröczky to write a quantitative version of the PL for log-concave functions (not necessarily even) in terms of some suitable distance between u_0 and u_1, that is

$$\int_{\mathbb{R}^n} h\,dx \geq \left[1 + \Psi_{\lambda,n}(d_n(u_0,u_1))\right] \left(\int_{\mathbb{R}^n} u_0 dx\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} u_1\,dx\right)^{\lambda}$$

where d_n measure the distance of u_0 and u_1 from coinciding up to an homothety and $\Psi_{\lambda,n} \in C(\mathbb{R}^+)$ is a suitable increasing continuous function such that $\Psi_{\lambda,n}(0) = 0$.

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$$\int_{\mathbb{R}^n} h\,dx \geq \left[1+\Psi_{\lambda,n}(d_n(u_0,u_1))\right]\left(\int_{\mathbb{R}^n} u_0 dx\right)^{1-\lambda}\left(\int_{\mathbb{R}^n} u_1 dx\right)^{\lambda}$$

where d_n measure the distance of u_0 and u_1 from coinciding up to an homothety and $\Psi_{\lambda,n}\in C(\mathbb{R}^+)$ is a suitable increasing continuous function such that $\Psi_{\lambda,n}(0)=0$. The distance d_n is however weaker then the L^1 distance.

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They also use a 1-dimensional transportation to write more general results, not restricted to log-concave function, but for functions in suitable classes $A \subset L^1(\mathbb{R}^n, \mathbb{R}_+)$ and for a different distance d_n.

An application: stability for BM inequalities for variational functionals

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BM inequality for τ [Borell, 1985]

$$\tau(\Omega_\mu) \geq M_{1/(n+2)}(\tau(\Omega_0), \tau(\Omega_1); \mu) = \left[(1-\mu)\tau(\Omega_0)^{1/(n+2)} + \mu\,\tau(\Omega_1)^{1/(n+2)}\right]^{n+2}$$

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Equality holds if and only if Ω_0 and Ω_1 are homothetic [Colesanti 2005].

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Set

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It is possible to prove that $u^*_{1/2,\mu}$ is a subsolution to the torsion problem in Ω_μ.Then

$$u_\mu \geq u^*_{1/2,\mu}$$

and we can use the BBL with $p = 1/2$ inequality to get the desired result.

Quantitative BM inequalities for τ

In fact, also PL inequality is sufficient (in place of BBL with $p = 1/2$) and now it's easy to understand that it is possible to use the quantitative versions of PL or BBL to get corresponding quantitative versions of the BM inequality for τ.

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A disadvantage: the refinement is written in term of a distance between u_0 and u_1...

A desirable improvement: it would be more natural to write a quantitative version involving only some distance between Ω_0 and Ω_1.

Possible improvements for BBL stability

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Of course, most desirable: 1 + 2 ....

Quantitative BBL for p-concave functions with $p > 0$

Joint work with **D. Ghilli** - preprint 2015, to appear J. Convex Analysis.

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Let H denotes the Hausdorff distance between sets in $\mathbb{R}^n$, we set

$$H_0(K, L) = H(\tau_0 K, \tau_1 L), \tag{0.3}$$

where τ_1, τ_0 are two homotheties (i.e. translation plus dilation) such that $|\tau_0 K| = |\tau_1 L| = 1$ and such that the centroids of $\tau_0 K$ and $\tau_1 L$ coincide.

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Theorem 1 (Ghilli-S. 2015)

Let $p > 0$ and assume that u_0 and u_1 are L^1 p-concave functions, with convex compact supports Ω_0 and Ω_1 respectively. Then, if $H_0(\Omega_0, \Omega_1)$ is small enough, it holds

$$\int_{\Omega_\lambda} h(x)\, dx \geq \mathcal{M}_{\frac{p}{np+1}}(I_0, I_1, \lambda) \left[1 + \beta\, H_0(\Omega_0, \Omega_1)^{\frac{(n+1)(p+1)}{p}}\right] \tag{0.4}$$

where β is a constant depending on n, λ, p, $\mathcal{M}_{\frac{p}{np+1}}(I_0, I_1, \lambda)$ and the diameters and the measures of Ω_0 and Ω_1.

Quantitative BBL for p-concave functions with $p > 0$

Let A denote the *relative asymmetry* of two sets, that is

$$A(K, L) := \inf_{x \in \mathbb{R}^n} \left\{ \frac{|K \,\Delta (x + \lambda F)|}{|K|},\ \lambda = \left(\frac{|K|}{|L|} \right)^{\frac{1}{n}} \right\}, \qquad (0.5)$$

where Δ denotes the operation of symmetric difference, i.e.
$\Omega \,\Delta\, B = (\Omega \setminus B) \cup (B \setminus \Omega)$.

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Theorem 2 (Ghilli-S. 2015)

In the same assumptions and notation of Theorem 1, if $A(\Omega_0, \Omega_1)$ is small enough it holds

$$\int_{\Omega_\lambda} h(x)\,dx \geq \mathcal{M}_{\frac{p}{np+1}}(I_0, I_1, \lambda)\left[1 + \delta\, A(\Omega_0, \Omega_1)^{\frac{2(p+1)}{p}}\right], \qquad (0.6)$$

where δ is a constant depending only on n, λ, p, $\mathcal{M}_{\frac{p}{np+1}}(I_0, I_1, \lambda)$ and on the measures of Ω_0 and Ω_1.

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3. We in fact prove more than what stated above and the support sets Ω_0 and Ω_1 could be replaced by any couple of level sets of u_0 and u_1, suitably related. However, for the application we have in mind (quantitative BM inequalities for variational functionals), we are mainly interested in the support sets.

Quantitative BBL for p-concave functions with $p > 0$

The proof of both theorems essentially amounts to proving the following and then applying existing quantitative results for the classical BM inequality.

Main theorem (Ghilli-S. 2015)

If for some (small enough) $\epsilon > 0$ it holds

$$\int_{\Omega_\lambda} h(x)\,dx \leq \mathcal{M}_{\frac{p}{np+1}}\left(\int_{\Omega_0} u_0(x)\,dx, \int_{\Omega_1} u_1(x)\,dx\,;\,\lambda\right) + \epsilon, \tag{0.7}$$

then

$$|\Omega_\lambda| \leq \mathcal{M}_{\frac{1}{n}}(|\Omega_0|, |\Omega_1|, \lambda)\left[1 + \eta\epsilon^{\frac{p}{p+1}}\right]. \tag{0.8}$$

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THEOREM 2 = MAIN THEOREM + STABILIYT for BM by FIGALLI-MAGGI-PRATELLI (2009).

Sketch of the proof

Let

$$I_i = \int_{\Omega_i} u_i \, dx \quad i = 0, 1 \,,$$

$$I_\lambda = \int_{\Omega_\lambda} h \, dx$$

and

$$L_i = \max_{\Omega_i} u_i \quad i = 0, 1, \qquad L_\lambda = \max_{\Omega_\lambda} h$$

Consider the distribution functions

$$\mu_i(s) = |\{u_i \geq s\}| \quad i = 0, 1 \,, \qquad \mu_\lambda(s) = |\{u_{p,\lambda} \geq s\}|$$

Then

$$I_i = \int_0^{L_i} \mu_i(s) \, ds \quad i = 0, 1, \lambda.$$

Sketch of the proof

Notice that the assumption of BBL is equivalent to

$$\{h \geq \mathcal{M}_p(s_0, s_1; \lambda)\} \supseteq (1-\lambda)\{u_0 \geq s_0\} + \lambda\{u_1 \geq s_1\} \tag{0.9}$$

for $s_0 \in [0, L_0]$, $s_1 \in [0, L_1]$.

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for $s_0 \in [0, L_0]$, $s_1 \in [0, L_1]$. Then, using the Brunn-Minkowski inequality, we get

$$\mu_\lambda(\mathcal{M}_p(s_0, s_1; \lambda)) \geq \mathcal{M}_{\frac{1}{n}}(\mu_0(s_0), \mu_1(s_1), \lambda). \tag{0.10}$$

Now define the functions $s_i : [0,1] \to [0, L_i]$ for $i = 0, 1$ such that

$$s_i(t) : \frac{1}{I_i}\int_0^{s_i(t)} \mu_i(s)\, ds = t \quad \text{for } t \in [0,1]\,, \tag{0.11}$$

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and set

$$s_\lambda(t) = \mathcal{M}_p(s_0(t), s_1(t), \lambda) \quad t \in [0,1]\,.$$

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Thanks to (0.10), we get

$$\mu_\lambda(s_\lambda(t)) \geq \mathcal{M}_{\frac{1}{n}}(\mu_0(s_0(t)), \mu_1(s_1(t)), \lambda) \quad t \in [0,1] \tag{0.12}$$

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Now, given any $\alpha \in (0,1)$, set

$$F_\epsilon = \{t \in [0,1] \,:\, \mu_\lambda(s_\lambda(t)) > \mathcal{M}_{\frac{1}{n}}(\mu_0(s_0(t)), \mu_1(s_1(t)), \lambda) + \epsilon^{1-\alpha}\} \tag{0.13}$$

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$$\Gamma_\epsilon = \{s_\lambda(t) \,:\, t \in F_\epsilon\}\,. \tag{0.14}$$

We want to find a bound of $|\Gamma_\epsilon|$ in terms of ϵ and, playing with the integrals and using the assumption, it is actually possible and we find

$$|\Gamma_\epsilon| \leq \epsilon^\alpha. \tag{0.15}$$

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Now choosing the right power $\alpha = \frac{p}{p+1}$ and using the p-concavity of h combined with the Brunn-Minkowski inequality we get the conclusion.

Quantitative BM inequalities for τ

Quantitative BM for τ (Ghilli-S. 2015)

Let Ω_0 and Ω_1 be convex bodies in $\mathbb{R}^n$, then the following hold:

$$\tau(\Omega_\lambda) \geq \mathcal{M}_{\frac{1}{n+2}}(\tau(\Omega_0), \tau(\Omega_1), \lambda) + \beta\, H_0(\Omega_0, \Omega_1)^{3(n+1)}\,, \tag{0.16}$$

$$\tau(\Omega_\lambda) \geq \mathcal{M}_{\frac{1}{n+2}}(\tau(\Omega_0), \tau(\Omega_1), \lambda) + \delta\, A(\Omega_0, \Omega_1)^{6}\,, \tag{0.17}$$

where β and δ are constants depending on n, λ, $\mathcal{M}_{\frac{p}{np+1}}(\tau(\Omega_0, \tau(\Omega_1), \lambda)$ and the diameters and the measures of Ω_0 and Ω_1.

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where β and δ are constants depending on n, λ, $\mathcal{M}_{\frac{p}{np+1}}(\tau(\Omega_0, \tau(\Omega_1), \lambda)$ and the diameters and the measures of Ω_0 and Ω_1.

Notice that from any BM inequalities for a (rotation and translation invariant) functional, it is possible to derive an Urysohn's inequality (ball is optimal for fixed mean width) for the same functional and then we can obtain stability for these kind of inequalities too. (If I have time, at the end will show in some detail the case of torsional rigidity)

Stabiliy of BM without convexity

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Figalli-Jerison (2014)

Let $n \geq 2$, and $A, B \subset \mathbb{R}^n$ be measurable sets with $|A| = |B| = 1$. Let $\lambda \in (0,1)$, set $\tau = \min\{\lambda, 1-\lambda\}$ and $S = (1-\lambda)A + \lambda B$. If

$$|S| \ \leq \ 1 + \delta$$

for some $\delta \leq e^{-M_n(\tau)}$, then there exists a convex $K \subset \mathbb{R}^n$ such that, up to a translation,

$$A, B \subseteq K \qquad \text{and} \qquad |K \setminus A| + |K \setminus B| \leq \tau^{-N_n} \delta^{\sigma_n(\tau)}.$$

The constant N_n can be explicitly computed and we can take
$M_n(\tau) = \frac{2^{3^{n+2}} n^{3^n} |\log \tau|^{3^n}}{\tau^{3^n}}, \qquad \sigma_n(\tau) = \frac{\tau^{3^n}}{2^{3^{n+1}} n^{3^n} |\log \tau|^{3^n}}.$

Stabiliy of BBL without concavity restrictions

Exploiting the result of Figalli-Jerison, we can obtain a stability for BBL without any p-concavity assumption, proving that near equality in BBL is possible if and only if the involved functions are close to coincide up to homotheties of their graphs and they are also nearly p-concave, in a suitable sense.

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Since our main result regards the case

$$p = \frac{1}{s} \quad \text{with } s \in \mathbb{N} ,$$

let me first restate BBL in this specific case.

Stabiliy of BBL without concavity restrictions

BBL for $p = 1/s$

Let $s > 0$ and f, g be as said above. Let $\lambda \in (0,1)$ and h be a nonnegative function belonging to $L^1(\mathbb{R}^n)$ such that

$$h((1-\lambda)x + \lambda y) \geq \left((1-\lambda)u_0^{1/s} + \lambda u_1^{1/s}\right)^s \tag{0.18}$$

for every $x \in \mathrm{sprt}(u_0)$, $y \in \mathrm{sprt}(u_1)$.
Then

$$\int_{\mathbb{R}^n} h\,dx \geq \mathcal{M}_{\frac{1}{n+s}}(I_0, I_1; \lambda)\ . \tag{0.19}$$

Moreover equality holds in (0.19) if and only if there exists a nonnegative concave function ϕ such that

$$u(x)^s = a_0\, u_0(b_0 x - \bar{x}_0) = a_1\, u_1(b_1 x - \bar{x}_1) = a_2\, h(bx_2 - \bar{x}_2) \quad \text{a.e. } x \in \mathbb{R}^n\,, \tag{0.20}$$

for some $\bar{x}_0, \bar{x}_1, \bar{x}_2 \in \mathbb{R}^n$ and suitable $a_i, b_i > 0$ for $i = 0, 1, 2$.

Rossi-S. 2016, to appear in GAFA Seminar Notes

Let f, g, h satisfying the assumption of the previous theorem (BBL for $p = 1/s$) with

$$0 < s \in \mathbb{N}\,.$$

Assume that

$$\int_{\mathbb{R}^n} h\, dx \leq \mathcal{M}_{\frac{1}{n+s}}(F, G; \lambda) + \epsilon \tag{0.21}$$

for some $\epsilon > 0$ small enough. Then there exist a $\frac{1}{s}$-concave function $u : \mathbb{R}^n \longrightarrow [0, +\infty)$ and two functions $\hat{f}$ and $\hat{g}$, coinciding with f and g up to suitable homotheties, such that the following hold:

$$u \geq \hat{f}\,, \qquad u \geq \hat{g}\,, \tag{0.22}$$

$$\int_{\mathbb{R}^n}(u - \hat{f})\, dx \;+\; \int_{\mathbb{R}^n}(u - \hat{g})\, dx \;\leq\; C_{n+s}\left(\frac{\epsilon}{\mathcal{M}_{\frac{1}{n+s}}(F, G; \lambda)}\right)\,, \tag{0.23}$$

where $C_{n+s}(\eta)$ is an infinitesimal function for $\eta \longrightarrow 0$ (whose expression is explicitely given).

Rossi-S. 2016, to appear in GAFA Seminar Notes

Let f, g, h satisfying the assumption of the previous theorem (BBL for $p = 1/s$) with

$$0 < s \in \mathbb{N}\,.$$

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Proof

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$$K_0 = \{(x,t)\,;\, x \in \mathrm{sprt}(u_0),\, 0 \le t \le u_0(x)\}$$

$$K_1 = \{(x,t)\,;\, x \in \mathrm{sprt}(u_1),\, 0 \le t \le u_1(x)\}$$

$$K_\lambda = (1-\lambda)K_0 + \lambda K_1 = \{(x,t)\,;\, x \in \mathrm{sprt}(u^*_{p,\lambda}),\, 0 \le t \le u^*_{p,\lambda}(x)\}$$

Proof

Following B. Klartag's ideas (2007), for $1 < s \in \mathbb{N}$ and for any nonnegative function $f \in L^1(\mathbb{R}^n)$, we define the following set in $\mathbb{R}^{n+s}$:

$$K_{f,s} = \{(x,y) \in \mathbb{R}^{n+s} = \mathbb{R}^n \times \mathbb{R}^s : \ x \in \mathrm{sprt}(f),\ |y| \leq f(x)^{1/s}\}, \tag{0.24}$$

Note: $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^s$.

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Note: $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^s$.

In other words, $K_{f,s}$ is the subset of $\mathbb{R}^{n+s}$ obtained as union of the s-dimensional closed balls of center $(x,0)$ and radius $f(x)^{1/s}$, for x belonging to the support of f, or, if you prefer, the set in $\mathbb{R}^{n+s}$ obtained by rotating with respect to $y=0$ the $(n+1)$-dimensional set
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Observe that $K_{f,s}$ is convex if and only if f is $(1/s)$-concave (that is for us a function f having compact convex support such that $f^{1/s}$ is concave on $\mathrm{sprt}\partial f$). Moreover, thanks to Fubini's Theorem, it holds

$$|K_{f,s}| = \int_{\mathrm{sprt}(f)} \omega_s \cdot \left[f(x)^{1/s}\right]^s \, dx = \omega_s \int_{\mathbb{R}^n} f(x) \, dx. \qquad (0.25)$$

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$$K_{u^*,s} = (1-\lambda)K_{u_0,s} + \lambda K_{u_1,s}\,. \tag{0.26}$$

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By applying BM inequality to K_*, K_0, K_1 we get

$$|K_{h,s}|^{\frac{1}{n+s}} \geq |K_*|^{\frac{1}{n+s}} \geq (1-\lambda)|K_0|^{\frac{1}{n+s}} + \lambda|K_1|^{\frac{1}{n+s}}, \tag{0.27}$$

Finally (0.25) yields

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thus dividing (0.27) by $\omega_s^{\frac{1}{n+s}}$ we get the desired BBL ineq.
And it is easy to understand as any stability result for BM can be translated in a stability result for BBL....with a little work.

Urysohn's inequality for τ

Given a convex set Ω, we say that $\Omega^\sharp_m$ is a *rotation mean* of Ω if there exist a number $m \in \mathbb{N}$ and $\rho_1, \dots, \rho_m \in SO(n)$ such that

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The following theorem is due to Hadwiger.

Theorem (Hadwiger)

Given an open bounded convex set Ω, there exists a sequence of rotation means of Ω converging in Hausdorff metric to a ball $\Omega^\sharp$ with diameter equal to the mean width $w(\Omega)$ of Ω.

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Notice that in the plane the mean width of a convex set coincides essentially with its perimeter. Precisely: $w(\Omega) = |\partial\Omega|/\pi$. Then $\Omega^\sharp$ is a circle with the same perimeter as Ω.

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By the BM inequality for torsional rigidity, we get

$$\tau(\Omega_m^\sharp) \geq \tau(\Omega) \quad \text{for evey } m\,,$$

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Quantitative Urysohn's inequalities for τ

(Ghilli-S. 2015)

Let Ω be an open bounded convex set of $\mathbb{R}^n$, $n \geq 2$ with centroid in the origin. Let $\Omega^\sharp$ be the ball with the same mean-width of Ω with center in the origin. Then the following hold

$$\tau(\Omega^\sharp) \geq \tau(\Omega)\left(1 + \mu H^{3(n+1)}\right), \tag{0.28}$$

$$\tau(\Omega^\sharp) \geq \tau(\Omega)\left(1 + \nu A^{6}\right), \tag{0.29}$$

where $H = H(\Omega, \Omega^\sharp)$ and $A = \max\{A(\Omega, \Omega_\rho) : \rho \text{ rotation in } \mathbb{R}^n\}$ are small enough, μ and ν are constants, the former depending on n, $\tau(\Omega)$ and the diameter of Ω, the latter depending only on n and $\tau(\Omega)$.

Possible extensions

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Then most of the above arguments, showed for the case of torsional rigidity, may be repeated for other functionals (in fact we have already treated the case of the Monge-Ampère eigenvalue with D. Ghilli in a previous paper (2014)).

The end

THANKS!