Geometrical properties of resources optimal arrangements for species survival (current work)

Jimmy LAMBOLEY, Antoine LAURAIN, Grégoire NADIN & Yannick PRIVAT

CNRS, LJLL, Univ. Paris 6

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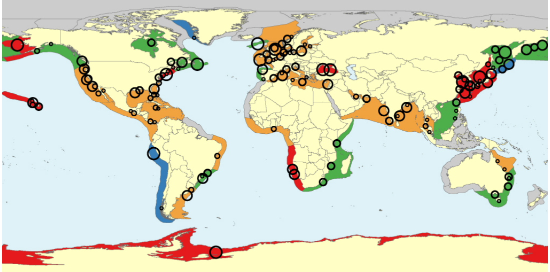
Outline

1. Modeling issues : toward a shape optimization problem
2. Analysis of optimal resources domains
   - Known results
   - New results
3. Conclusion and open problems



Outline

Biological model : population dynamic

Logistic diffusive equation (Fisher-Kolmogorov 1937, Fleming 1975, Cantrell-Cosner 1989)

Introduce

- $\rightsquigarrow$ $\Omega \subset \mathbb{R}^N$: bounded domain with Lipschitz boundary (habitat)
- $\rightsquigarrow$ ω : positive parameter
- $\rightsquigarrow$ $u(t,x)$: density of a species at location x and time t
- $\rightsquigarrow$ $m(x)$: intrinsic growth rate of species at location x and
  - $\Omega \cap \{m > 0\}$ (resp. $\Omega \cap \{m < 0\}$) is the favorable (resp. unfavorable) part of habitat
  - $\int_\Omega m$ measures the total resources in the spatially heterogeneous environment Ω
  - After renormalization, one is allowed to assume that

$$-1 \leq m(x) \leq \kappa \quad \text{with } \kappa > 0 \quad \text{and } m \text{ changes sign.}$$

Biological model

$$\begin{cases} u_t = \Delta u + \omega u[m(x) - u] & \text{in } \Omega \times \mathbb{R}_+, \\ u(0,x) \geq 0, \quad u(0,x) \not\equiv 0 & \text{in } \overline{\Omega}, \end{cases}$$

Biological model : population dynamic

Choice of boundary conditions

$$\partial_n u + \beta u = 0 \qquad \text{on } \partial\Omega \times \mathbb{R}^+,$$

where β is a non-negative parameter standing for inhospitableness of the region surrounding Ω.

- $\rightsquigarrow$ Case $\beta = 0$: no-flux boundary condition (the boundary acts as a barrier)
- $\rightsquigarrow$ Case $\beta = +\infty$: Dirichlet condition (deadly boundary)
- $\rightsquigarrow$ Intermediate case $\beta > 0$: Ω is surrounded by a partially inhospitable region

Biological model : population dynamic

The complete model

$$\begin{cases} u_t = \Delta u + \omega u[m(x) - u] & \text{in } \Omega \times \mathbb{R}_+, \\ \partial_n u + \beta u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(0,x) \geq 0, \quad u(0,x) \not\equiv 0 & \text{in } \overline{\Omega}, \end{cases}$$

($\rightsquigarrow$ takes into account effects of dispersal and partial heterogeneity)

Analysis of the model : extinction/survival condition

The complete model

$$\begin{cases} u_t = \Delta u + \omega u[m(x) - u] & \text{in } \Omega \times \mathbb{R}_+, \\ \partial_n u + \beta u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(0,x) \geq 0, \quad u(0,x) \not\equiv 0 & \text{in } \overline{\Omega}, \end{cases}$$

Introduce the eigenvalue problem

$$\lambda(m) = \inf \left\{ \frac{\int_\Omega |\nabla \varphi|^2 + \beta \int_{\partial\Omega} \varphi^2}{\int_\Omega m\varphi^2}, \quad \varphi \in H^1(\Omega), \ \int_\Omega m\varphi^2 > 0 \right\}.$$

Theorem (Cantrell-Cosner 1989, Berestycki-Hamel-Roques 2005)

Let u^* be the unique positive steady solution of the logistic equation above. One has

- $\omega \leq \lambda(m) \quad \Longrightarrow \quad u(t,x) \underset{t \to \infty}{\longrightarrow} 0,$
- $\omega > \lambda(m) \quad \Longrightarrow \quad u(t,x) \underset{t \to \infty}{\longrightarrow} u^*(x).$

Comments on the eigenvalue problem (with a sign changing weight m)

Another characterization of $\lambda(m)$

$\lambda(m)$ is the unique principal ($\Leftrightarrow \varphi > 0$) positive eigenvalue of the problem :

$$\left\{ \begin{array}{ll} \Delta\varphi + \lambda m\varphi = 0 & \text{in } \Omega, \\ \partial_n\varphi + \beta\varphi = 0 & \text{on } \partial\Omega, \end{array} \right.$$

Moreover,

- in the Robin and Dirichlet case ($0 < \beta \leq +\infty$), 2 principal eigenvalues : $\lambda^- < 0 < \lambda^+$
- in the critical case $\beta = 0$, 2 principal eigenvalues : 0 and λ and one has

$$\lambda > 0 \Longleftrightarrow \int_\Omega m < 0.$$

Optimal arrangements of resources

Conclusion of this part

The species can be maintained iff $\omega > \lambda(m)$.

Hence, the smaller $\lambda(m)$ is, the more likely the species can survive

$\rightsquigarrow$ among all weights m, which of them yields the smallest principal eigenvalue $\lambda(m)$?

Optimal arrangements of resources

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Infinite dimensional shape optimization problem

$$\inf_{m\in\mathcal{M}_{m_0,\kappa}} \lambda(m), \tag{P}$$

with

$$\mathcal{M}_{m_0,\kappa} = \left\{ m \in L^\infty(\Omega,[-1,\kappa]),\ |\{m>0\}|>0,\ \int_\Omega m \leq -m_0|\Omega| \right\}$$

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Bang-bang property of minimizers

Proposition (Lou-Yanagida 2006, Derlet-Gossez-Takac 2010)

Problem (P) has a solution. Moreover, every minimizer m satisfies

$$\int_\Omega m = -m_0|\Omega| \qquad \text{and} \qquad m = \kappa \mathbb{1}_E - \mathbb{1}_{\Omega\setminus E}.$$

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Shape optimization version of the problem

Consequence : the two following problems

$$\inf\left\{\lambda(m), \quad m \in L^\infty(\Omega, [-1,\kappa]),\ |\{m>0\}|>0,\ \int_\Omega m \leq -m_0|\Omega|\right\} \tag{1}$$

and

$$\inf\left\{\lambda(E) := \lambda(\kappa\mathbb{1}_E - \mathbb{1}_{\Omega\setminus E}), \quad |E| = c|\Omega|\right\}, \tag{2}$$

where $c = c(m_0) \in (0,1)$, are equivalent. Moreover, each infimum is in fact a minimum.

State of the art
Highly non-exhaustive

- Case $\beta = \infty$, with no sign changement : symmetrization, regularity in case of symmetry [Krein 1955, Friedland 1977, Cox 1990]
- Periodic case : [Hamel-Roques 2007]
- 1D case, $\beta = 0$: solved [Lou-Yanagida 2006]
- 1D case, $\beta > 0$: optimization among intervals [Hintermüller-Kao-Laurain 2012]
- 2D case : regularity [Chanillo-Kenig-To 2008]
- Numerics : [Cox, Hamel-Roques, Hintermüller-Kao-Laurain]

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New results : complete solution in dim. 1

$$\inf \left\{ \lambda(E) := \lambda(\kappa \mathbb{1}_E - \mathbb{1}_{\Omega \setminus E}), \quad |E| = c|\Omega| \right\} \tag{P}$$

Theorem (Lamboley, Laurain, Nadin, YP 2016)

If $\Omega =]0,1[$ and E^* is a solution, then E^* is an interval.

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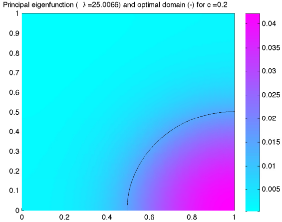
Theorem (Lamboley, Laurain, Nadin, YP 2016)

If $\Omega =]0,1[$ and E^* is a solution, then E^* is an interval.

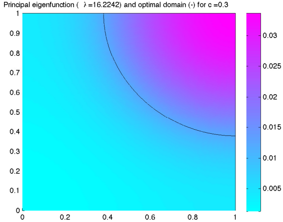
Consequence : there exists $\beta^* = \beta^*(\kappa, c)$ such that

- if $\beta > \beta^*$, same solution as $\beta = \infty$,
- if $\beta < \beta^*$, same solution as $\beta = 0$,
- if $\beta = \beta^*$, solutions are all the intervals of length c.

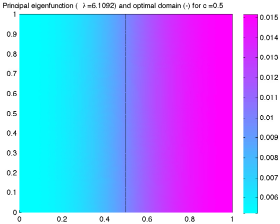
Higher dimensions : $\Omega = (0,1)^2$, $\kappa = 0.5$, $\beta = 0$



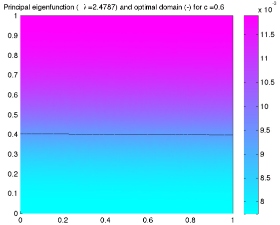
(a) $c = 0.2$



(b) $c = 0.3$

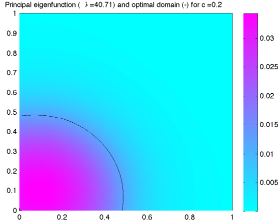


(c) $c = 0.5$

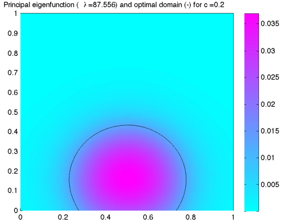


(d) $c = 0.6$

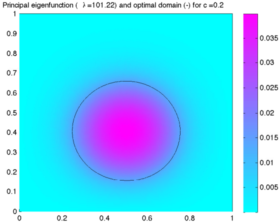
Higher dimensions : $\Omega = (0,1)^2$, $\kappa = 0.5$, $c = 0.2$



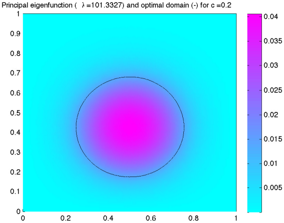
(a) $\beta = 1$



(b) $\beta = 5$



(c) $\beta = 50$



(d) $\beta = 1000$

New results : in dimension $N \geq 2$, is the solution a ball ?

$$\inf \left\{ \lambda(E) := \lambda(\kappa \mathbb{1}_E - \mathbb{1}_{\Omega \setminus E}), \quad |E| = c|\Omega| \right\} \tag{P}$$

Theorem (Lamboley, Laurain, Nadin, YP 2016)

Let assume that $N \geq 2$ and $\partial\Omega$ is connected and C^1.
Assume E or $\Omega \backslash E$ iS a union of concentric rings and has a finite number of connected components.

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- E is critical $\Rightarrow$ Ω is a centered ball

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- E is critical $\Rightarrow$ Ω is a centered ball
- If β is large enough,

  E is a minimum $\Rightarrow$ E and Ω are concentric balls.

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Let assume that $N \geq 2$ and $\partial\Omega$ is connected and C^1.
Assume E or $\Omega \setminus E$ iS a union of concentric rings and has a finite number of connected components. Then

- E is critical $\Rightarrow$ Ω is a centered ball
- If β is large enough,

  E is a minimum $\Rightarrow$ E and Ω are concentric balls.

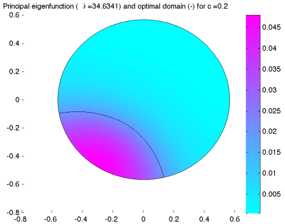
Steps of the proof :

$\rightsquigarrow$ φ is radial in E

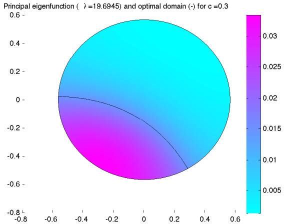
$\rightsquigarrow$ φ is radial in Ω

$\rightsquigarrow$ Ω is a centered ball.
Easy if $\beta = \infty$; study the contact with the inscribed and circumscribed balls otherwise.

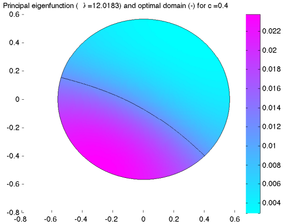
Other numerical computations : $\Omega = B(0,1), \beta = 0, \kappa = 0.5$



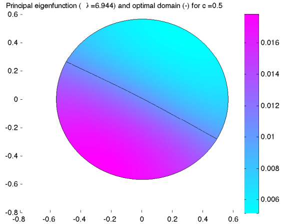
(e) $c = 0.2$



(f) $c = 0.3$



(g) $c = 0.4$



(h) $c = 0.5$

New results : Neumann case in dimension $N = 2, 3, 4$: non-optimality of the centered ball in a ball

$$\inf \left\{ \lambda(E) := \lambda(\kappa \mathbb{1}_E - \mathbb{1}_{\Omega \setminus E}), \quad |E| = c|\Omega| \right\} \tag{P}$$

Theorem (Lamboley, Laurain, Nadin, YP 2016)

Let $N \in \{2, 3, 4\}$, $\beta = 0$ and $\Omega = B(0, 1) \subset \mathbb{R}^N$.
Then the centered ball of volume $c|\Omega|$ is not a minimizer for Problem (P).

Proof : Disymmetrization procedure

Outline

Conclusion and open questions

$$\text{On the problem } \inf\left\{\lambda(E) := \lambda(\kappa \mathbb{1}_E - \mathbb{1}_{\Omega\setminus E}), \quad |E| = c|\Omega|\right\} \qquad (P)$$

- If Ω is a ball, is E a concentric ball ?

  $\rightsquigarrow$ Solved if $N=1$: yes if β is large enough, no else.
  $\rightsquigarrow$ Yes if $\beta=\infty$, No if $\beta=0$ and $N\in\{2,3,4\}$

- Can $\partial E\cap\Omega$ be a piece of sphere ?

  $\rightsquigarrow$ No if $\beta=0$ and Ω is a square/cube

- Optimality of strips ?

  $\rightsquigarrow$ Expected to hold for some values of m_0 if $\beta=0$ and Ω is a square/cube

- Find sufficient conditions so that $\partial E\cap\partial\Omega\neq\emptyset$,

  $\rightsquigarrow$ Expected to be always true if $\beta=0$

Conclusion and open questions

Ongoing work (1)

Same problem with an improved model

$\leadsto$ We enrich the model by adding an advection term along the gradient of the habitat quality (Belgacem and Cosner)

$$\begin{cases} \partial_t u = \operatorname{div}(\nabla u - \alpha u \nabla m) + \lambda u(m-u) & \text{in } \Omega \times (0,\infty), \\ e^{\alpha m}(\partial_n u - \alpha u \partial_n m) + \beta u = 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases}$$

This models the tendency of the population to move up along the gradient of m.

New shape optimization problem

$$\inf_{m \in \mathcal{M}_{m_0,\kappa}} \lambda(m),$$

with

$$\lambda(m) = \inf_{\varphi \in \mathcal{S}_0} \frac{\int_\Omega e^{\alpha m} |\nabla \varphi|^2}{\int_\Omega m e^{\alpha m} \varphi^2} \quad \text{and} \quad \mathcal{S}_0 = \{\varphi \in H^1(\Omega), \int_\Omega m e^{\alpha m} \varphi^2 > 0\}$$

Conclusion and open questions

Ongoing work (2)

Effects of dispersal and spatial heterogeneity of the environment on the total population size

$\rightsquigarrow$ Consider the steady-state

$$\begin{cases} \mu\Delta\bar{u} + \bar{u}(m-\bar{u}) = 0 & \text{in } \Omega, \\ \partial_n\bar{u} + \beta\bar{u} = 0 & \text{on } \partial\Omega, \end{cases} \qquad (\mu = \text{migration rate})$$

This problem has a unique positive solution in $W^{2,p}(\Omega)$, for every $p \geq 1$.

New optimization problem

$$\sup_{m\in\mathcal{M}_{m_0,\kappa}} \int_\Omega \bar{u}(x)\,dx, \quad \text{(total population size of the species)}$$

or

$$\sup_{m\in\mathcal{M}_{m_0,\kappa}} \int_\Omega \bar{u}(x)^3\,dx, \quad \text{(natural energy of the population)}$$

$\rightsquigarrow$ Ph.D. thesis of I. Mazari (univ. Paris 6)

Thank you for your attention