Symmetry breaking for a problem in optimal insulation

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Shape optimization and Isoperimetric and Functional Inequalities

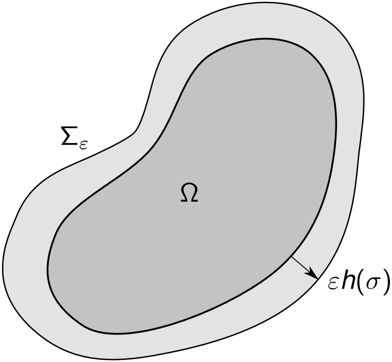
Joint work with...

Joint work with...

Joint work with...

The model

Conducting body Ω of conductivity 1.
Insulator $\Sigma_\varepsilon = \left\{\sigma + t\nu(\sigma)\ :\ \sigma \in \partial\Omega,\ 0 \leq t < \varepsilon h(\sigma)\right\}$ of conductivity $\delta << 1$



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A problem of insulation (stationary case)

The temperature u of the conducting body Ω, with heat sources $f \in L^2(\Omega)$ is given by the solution of the PDE

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ -\Delta u = 0 & \text{in } \Sigma_\varepsilon \\ u = 0 & \text{on } \partial(\Omega \cup \Sigma_\varepsilon) \\ \dfrac{\partial u^-}{\partial \nu} = \delta \dfrac{\partial u^+}{\partial \nu} & \text{on } \partial\Omega. \end{cases}$$

or equivalently by the minimization of the functional

$$F_{\varepsilon\delta}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\delta}{2}\int_{\Sigma_\varepsilon} |\nabla u|^2\,dx - \int_\Omega fu\,dx$$

in the Sobolev space $H_0^1(\Omega \cup \Sigma_\varepsilon)$

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If $\varepsilon \approx \delta$ we get something different...

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One way to see what happens, is to look at the functional

$$F_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\varepsilon}{2}\int_{\Sigma_\varepsilon} |\nabla u|^2\,dx - \int_\Omega fu\,dx \quad u \in H_0^1(\Omega \cup \Sigma_\varepsilon)$$

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and notice that it Γ-converges to

$$E(u,h) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_{\partial\Omega} \frac{u^2}{h}\,d\mathcal{H}^{d-1} - \int_\Omega fu\,dx \quad u \in H^1(\Omega)$$

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Therefore this time we have the heat equation with Robin boundary conditions.

Optimization of the energy

We fix the total mass of insulator to be $m > 0$

$$\mathcal{H}_m = \left\{ h : \partial\Omega \to \mathbb{R} \text{ measurable}, \, h \geq 0, \, \int_{\partial\Omega} h \, d\mathcal{H}^{d-1} = m \right\}.$$

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the case $f \equiv 1$ means that we are maximizing the total heat.

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Therefore, the optimization problem can be rewritten as

$$\min \left\{ \frac{1}{2}\int_\Omega |\nabla u|^2\, dx + \frac{1}{2m}\left(\int_{\partial\Omega} |u|\, d\mathcal{H}^{d-1}\right)^2 - \int_\Omega fu\, dx \ :\ u\in H^1(\Omega) \right\}.$$

Existence and uniqueness

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Moreover

Proposition

Assume Ω is connected. Then the functional

$$u\mapsto F(u)=\frac{1}{2}\int_{\Omega}|\nabla u|^2\,dx+\frac{1}{2m}\Big(\int_{\partial\Omega}|u|\,d\mathcal{H}^{d-1}\Big)^2$$

is strictly convex on $H^1(\Omega)$, hence for every $f\in L^2(\Omega)$ the minimization problem admits a unique solution.

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Therefore the optimal insulation is given by a constant thickness:

$$h=\frac{m}{d\omega_d R^{d-1}}$$

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- If $R_1 > R_2$ then $c_1 = 0$ and $c_2 = \frac{m}{d^2 \omega_d R_2^{d-2}}$. All the insulator must be uniformly distributed around the smaller ball, while the larger ball remains not insulated.

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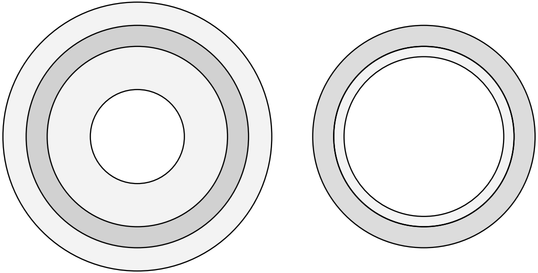

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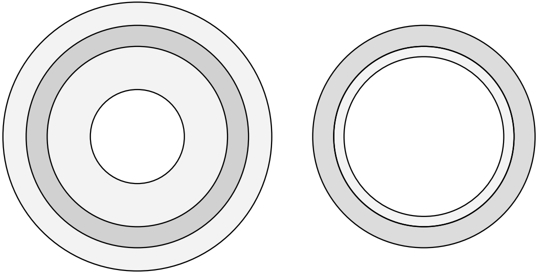

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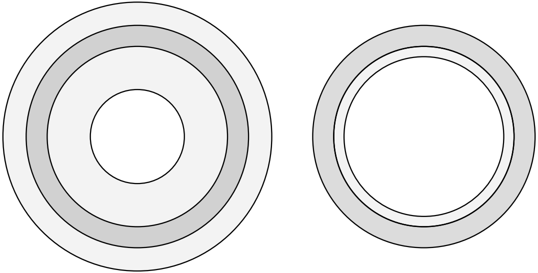

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A problem of insulation (non-stationary case)

In the non stationary case we consider the parabolic heat eqution

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Again we consider the limit $\varepsilon, \delta \to 0$.

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If $\varepsilon = \delta$ we get again Robin Laplacian eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ h\dfrac{\partial u}{\partial \nu} + u = 0 & \text{on } \partial\Omega \end{cases}$$

Optimization of the eigenvalue

The first Robin eigenvalue $\lambda(h)$ is also characterized by

$$\lambda(h) = \inf \left\{ \frac{\int_\Omega |\nabla u|^2 \, dx + \int_{\partial\Omega} h^{-1} u^2 \, d\mathcal{H}^{d-1}}{\int_\Omega u^2 \, dx} \ : \ u \in H^1(\Omega), \ u \neq 0 \right\}.$$

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And again interchanging the minimization we get...

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Let u be a minimizer, again the optimal density h_{opt} is proportional to u

$$h_{opt}=m\frac{u}{\int_{\partial\Omega}u\,d\mathcal{H}^{d-1}}\,.$$

Optimization on the ball

Theorem

Let Ω be a ball. Then there exists $m_0 > 0$ such that the solution of the variational problem is radial if $m > m_0$, while the solution is not radial for $0 < m < m_0$. As a consequence, the optimal density h_{opt} is not constant if $m < m_0$.

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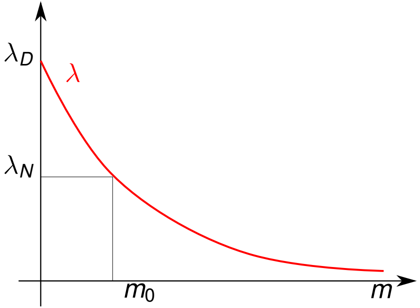
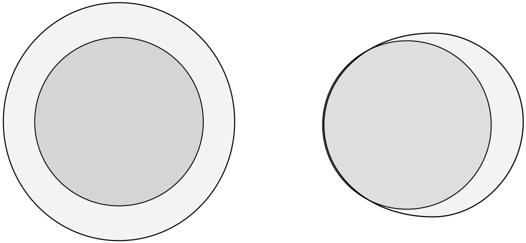


Optimization on the ball

The picture is therefore the following



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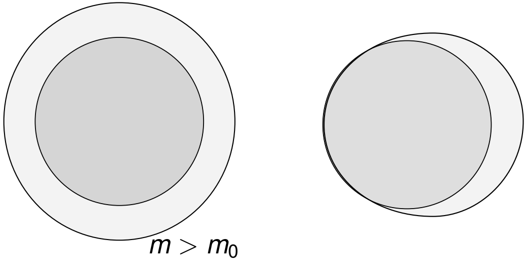

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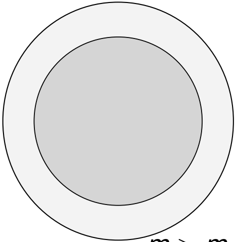

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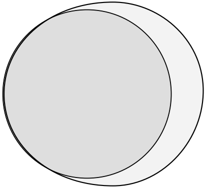

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Thank you!