Isoperimetric inequalities for spectrum of Laplacian on surfaces

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Let M be a closed surface and g be a Riemannian metric on M. Let us consider the Laplace-Beltrami operator

$$\Delta f = -\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left(\sqrt{|g|} g^{ij} \frac{\partial f}{\partial x^j} \right),$$

and its eigenvalues

$$0 = \lambda_0(M, g) < \lambda_1(M, g) \leqslant \lambda_2(M, g) \leqslant \lambda_3(M, g) \leqslant \dots \tag{1}$$

Let us denote by $m(M, g, \lambda_i)$ the multiplicity of the eigenvalue $\lambda_i(M, g)$, Let us consider a functional

$$\bar{\lambda}_i(M, g) = \lambda_i(M, g) Area(M, g),$$

where $Area(M, g)$ is the area of M with respect to metric g.

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It follows that the functionals $\bar{\lambda}_i(M,g)$ are bounded from above and it is a natural question to find for a given compact surface M and number $i \in \mathbb{N}$ the quantity

$$\Lambda_i(M) = \sup_g \bar{\lambda}_i(M,g),$$

where the supremum is taken over the space of all Riemannian metrics g on M.

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$\Lambda_1(M)$ is also known for $\mathbb{T}^2, \mathbb{K}^2, \mathbb{T}^2 \# \mathbb{T}^2$: N.N., Jakobson-N.N.-Polterovich, ElSoufi-Giacomini-Jazar, Jakobson-Levitin-N.N.-Nigam-Polterovich,

Theorem (N.N., Y.Sire)

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Multiplicity of eigenvalues

For eigenvalue λ_i of Δ on (M, g) denote by m_i its multiplicity

Cheng proved (1976)

$$m_i \leq C(i, \chi(M))$$

on $\mathbb{S}^2$

$$m_i \leq 2i + 1$$

Sharp bounds for m_1 are known for $\chi(M) \geq -3$:
Besson, N.N., Sévennec

Theorem (Colin de Verdière)

$$\sup_g m_i \geq chr(M) - 1 = [\frac{1}{2}(7 + \sqrt{49 - 24\chi(M)})] - 1$$

Conjecture. (Colin de Verdière)

$$\sup_g m_i = chr(M) - 1$$

Theorem (M. and T. Hoffmann-Ostenhof, N.N)

On $\mathbb{S}^2$ the inequalities hold

$$m_2 \leq 3, \quad m_3 \leq 5$$

Theorem (N.N., A.Penskoi)

On $\mathbb{R}P^2$ the inequality holds

$$m_2 \leq 6$$

Conjecture.

$$m_i \leq C(\chi(M))\sqrt{i}$$

Theorem (Courant Nodal Domain Theorem)

An eigenfunction corresponding to the eigenvalue λ_i has at most $i+1$ nodal domains.

Proposition Let u be an eigenfunction corresponding to the eigenvalue λ_i. Let x_j, $j = 1, \ldots, n$, be zeroes of u of order $m_j > 1$. Then

$$i + 1 \geqslant \chi(M) - n + \sum_{j=1}^{n} m_j.$$

Theorem (Bers, 1955)

Let (M, g) be a compact 2-dimensional closed Riemannian manifold and x_0 is a point on M. Then there exist its neighbourhood chart U with coordinates $x = (x^1, x^2) \in U \subset \mathbb{R}^2$ centered at x_0 such that for any eigenfunction u of the Laplace-Beltrami operator on M there exists an integer $n \geqslant 0$ and a non-trivial homogeneous harmonic polynomial $P_n(x)$ of degree n on the Euclidean plane $\mathbb{R}^2$ such that $u(x) = P_n(x) + O(|x|^{n+1})$

Maximizing metrics and bubbling phenomen

Theorem. (N.N., Y.Sire) Let (M, g) be a Riemannian surface. For any $k \geqslant 1$ and a sequence of metrics $\{g'_i\}_{i \geqslant 1} \in [g]$ of the form $g'_i = \mu'_i g$ such that

$$\lim_{i \to \infty} \lambda_k(g'_i) = \Lambda_k(M, [g])$$

there exists a subsequence of metrics $\{g_n\}_{n \geqslant 1} = \{g'_{i_n}\}_{n \geqslant 1} \in [g]$, where $g_n = \mu_n g$, such that

$$\lim_{n \to \infty} \lambda_k(g_n) = \Lambda_k(M, [g])$$

and a probability measure μ such that

$$\mu_n \rightharpoonup^* \mu \ \text{ weakly in measure as } n \to +\infty.$$

Moreover, the following decomposition holds,

$$\mu = \mu_r + \mu_s$$

where μ_r is a nonnegative C^∞ function and μ_s is the singular part given, if not trivial, by the formula

$$\mu_s = \sum_{i=1}^{K} c_i \delta_{x_i}$$

for some $K \geqslant 1$, $c_i \geqslant 0$ and some "bubbling points" $x_i \in M$. Furthermore, the number K satisfies the bound

$$K \leqslant k - 1.$$

If we denote by U the eigenspace of the Laplace-Beltrami operator on $(M, \mu_r g)$ associated to the eigenvalue $\Lambda_k(M, [g])$, then there exists a family of eigenvectors $\{u_1, \ldots, u_l\} \subset U$ such that the map

$$\varphi = (u_1, \ldots, u_l) : M \to \mathbb{R}^l$$

is a harmonic immersion into the sphere $\mathbb{S}^{l-1}$.

Definition

Let (M, g) and (N, h) be Riemannian manifolds. A smooth map $f : M \longrightarrow N$ is called harmonic if f is an extremal for the energy functional $E[f] = \int_M |df(x)|^2 \, dVol_g$.

Proposition Let M, N be Riemannian manifolds. If $f : M \looparrowright N$ is an isometric immersion, then f is harmonic if and only if f is minimal.
Proposition A harmonic map $\mathbb{S}^2 \looparrowright \mathbb{S}^n$ is automatically conformal and hence minimal in the induced metric.

Theorem (Takahashi1966)

An isometric immersion

$$f : M \looparrowright \mathbb{S}^n_R \subset \mathbb{R}^{n+1}, \quad f = (f^1, \ldots, f^{n+1}),$$

is a minimal isometric immersion of a manifold M into the sphere $\mathbb{S}^n_R$ of radius R, if and only if f^i are eigenfunctions of the Laplace-Beltrami operator Δ,

$$\Delta f^i = \lambda f^i,$$

with the same eigenvalue λ given by formula

$$\lambda = \frac{\dim M}{R^2},$$

Theorem (Calabi1967, Barbosa1975)

Let $F : \mathbb{S}^2 \longrightarrow \mathbb{S}^n$ be a harmonic immersion with branch points such that the image is not contained in a hyperplane. Then

(i) *the area of $\mathbb{S}^2$ with respect to the induced metric $(\mathbb{S}^2, F^*g)$ is an integer multiple of 4π;*

(ii) *n is even, $n = 2m$, and*

$$Area(\mathbb{S}^2, F^*g) \geq 2\pi m(m+1).$$

Definition

If $Area(\mathbb{S}^2, F^*g) = 4\pi d$, then we say that F is of harmonic degree d.

We obtain immediately a lower bound for the harmonic degree.

Proposition Let $F : \mathbb{S}^2 \longrightarrow \mathbb{S}^{2m}$ be a harmonic immersion with branch points such that the image is not contained in a hyperplane. Then $d \geqslant \frac{m(m+1)}{2}$.

Proposition It is sufficient for our goals to consider harmonic immersions with branch points $F : \mathbb{S}^2 \longrightarrow \mathbb{S}^4$ (such that the image is not contained in a hyperplane) of harmonic degree $d \geqslant 3$ and $F : \mathbb{S}^2 \longrightarrow \mathbb{S}^2$.

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Definition (Penrose twistor map)

$$T : \mathbb{CP}^3 \longrightarrow \mathbb{HP}^1 \cong \mathbb{S}^4, \quad T([z_0 : z_1 : z_2 : z_3]) = [z_0 + z_1 j : z_2 + z_3 j].$$

Let z be a conformal parameter on $\mathbb{S}^2$.

Definition

Let us call a curve

$$f : \mathbb{S}^2 \longrightarrow \mathbb{CP}^3, \quad f(z) = [f_0(z) : f_1(z) : f_2(z) : f_3(z)],$$

horizontal if

$$f_1' f_2 - f_1 f_2' + f_3' f_4 - f_3 f_4' = 0.$$

Theorem (Bryant 1982)

For each harmonic immersion with branch points $F : \mathbb{S}^2 \longrightarrow \mathbb{S}^4$ *there exist either holomorphic or antiholomorphic horizontal curve* $f : \mathbb{S}^2 \longrightarrow \mathbb{CP}^3$, *such that* $T \circ f = F$,
For each (anti)holomorphic horizontal curve $f : \mathbb{S}^2 \longrightarrow \mathbb{CP}^3$ *the map* $F = T \circ f : \mathbb{S}^2 \longrightarrow \mathbb{S}^4$ *is a harmonic immersion with branched points.*
If a harmonic immersion $F : \mathbb{S}^2 \longrightarrow \mathbb{S}^4$ *has a holomorphic (antiholomorphic) horizontal curve* $f : \mathbb{S}^2 \longrightarrow \mathbb{CP}^3$, *then* $A \circ F : \mathbb{S}^2 \longrightarrow \mathbb{S}^4$ *has an antiholomorphic (holomorphic) horizontal curve.*

Definition

An (anti)holomorphic horizontal curve f appearing in Bryant Theorem is called the lift of an harmonic immersion F.

Theorem (Bryant 1982)

Let $F : \mathbb{S}^2 \longrightarrow \mathbb{S}^4$ *be a harmonic immersion with branched points of harmonic degree* d *with holomorphic lift* $f : \mathbb{S}^2 \longrightarrow \mathbb{CP}^3$. *Then* $f : \mathbb{S}^2 \longrightarrow \mathbb{CP}^3$ *is an algebraic curve of degree* d.

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Theorem (Bolton-Woodward 2001)

For a linearly full holomorphic horisontal curve in $\mathbb{CP}^3$ of degree d if $d = 3$ then F does not have neither branch points nor umbilics, if $d > 3$ then F has at least one branch point or umbilic.

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Theorem (Loo 1989 Kotani 1994)

The set $\mathcal{M}$ *of all minimal immersions from* $\mathbb{S}^2$ *into* $\mathbb{S}^4$ *is connected for fixed area.*