Regularity of the optimal sets for spectral functionals: Part II, some generalizations

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The sum functional

We are dealing with the shape optimization problem:

$$\min\left\{\lambda_1(\Omega)+\cdots+\lambda_k(\Omega)\ :\ \Omega\subset\mathbb{R}^d,\ open,\ |\Omega|=1\right\},$$

which is equivalent with a scaling argument to

$$\min\left\{\lambda_1(\Omega)+\cdots+\lambda_k(\Omega)+\Lambda|\Omega|\ :\ \Omega\subset\mathbb{R}^d,\ open\right\},$$

for some Lagrange multiplier $\Lambda>0$.

The free boundary formulation

For a vector $V=(v_1,\dots,v_k)\in H^1(\mathbb{R}^d,\mathbb{R}^k)$, we consider the energy functional:

$$\begin{aligned}\mathcal{F}_0(V) &:= \int_{\mathbb{R}^d} |\nabla V|^2\,dx + \Lambda|\{|V|>0\}| \\ &= \sum_{i=1}^k \int_{\mathbb{R}^d} |\nabla v_i|^2\,dx + \Lambda|\{v_1^2+\cdots+v_k^2>0\}|.\end{aligned}$$

The vector of normalized eigenfunctions $U=(u_1,\dots,u_k)$ for the shape optimization problem for the sum of eigenvalues solves:

$$\min\left\{\mathcal{F}_0(V)\ :\ V\in H^1(\mathbb{R}^d,\mathbb{R}^k),\ \int_{\mathbb{R}^d} v_i v_j\,dx=\delta_{ij}\right\}.$$

Regularity of optimal shapes: the sum functional

Theorem (M., Terracini, Velichkov)

Let Ω^* be an optimal set for the problem:

$$\min\Big\{\lambda_1(\Omega)+\cdots+\lambda_k(\Omega)\ :\ \Omega\subset\mathbb{R}^d,\ \textit{open},\ |\Omega|=1\Big\},$$

then Ω^* is connected and its topological boundary is the disjoint union of a regular part $Reg(\partial\Omega^*)$ and of a singular set $Sing(\partial\Omega^*)$. $Reg(\partial\Omega^*)$ is relatively open and C^∞ regular, while $Sing(\partial\Omega^*)$ is relatively closed and such that

- *If $d<d^*$, then $Sing(\partial\Omega^*)=\emptyset$,*
- *If $d=d^*$, then $Sing(\partial\Omega^*)$ is made by a finite number of isolated points,*
- *If $d>d^*$, then $dim_H(Sing(\partial\Omega^*))\le d-d^*$.*

Key steps in the proof

- Pass to a free-boundary problem
- Nondegeneracy
- Monotonicity formula
- Blow-up analysis
- Optimality condition in a viscosity sense
- Boundary Harnack and reduction to the one-phase case

We wish to extend our results to the more general case:

$$\min\Big\{F(\lambda_1(\Omega),\dots,\lambda_k(\Omega))\ :\ \Omega\subset\mathbb{R}^d,\ \text{quasi-open},\ |\Omega|=1\Big\},$$

for functionals $F\colon\mathbb{R}^k\to\mathbb{R}$ increasing in each variable and lower semicontinuous.
The case that we have in mind (and we are currently not able to treat) is

$$\min\Big\{\lambda_k(\Omega)\ :\ \Omega\subset\mathbb{R}^d,\ \text{quasi-open},\ |\Omega|=1\Big\},$$

What are the main problems?

- Free boundary equivalent?
- Lagrange multiplier?
- Optimality condition (even formally)?
- And when λ_1 is missing?

Symmetric and regular functions

Definition

We say that $\varphi \in \mathcal{G}$ if $\varphi \colon \mathcal{S}_k(\mathbb{R}) \to \mathbb{R}$ is C^1 in $\mathcal{S}_k(\mathbb{R}) \setminus \{0\}$ and

$$\varphi(M) = \varphi(P^T M P), \qquad \text{for all } M \in \mathcal{S}_k(\mathbb{R}),\ P \in \mathcal{O}_k(\mathbb{R}).$$

Moreover we consider the restriction ψ of φ to the space of the diagonal matrices, that is, $\psi(a_1, \dots, a_k) = \varphi(diag(a_1, \dots, a_k))$ and we require the following conditions:

- $\frac{\partial \psi}{\partial a_i} > 0$ on $(\mathbb{R}^+)^k$ for all $i = 1, \dots, k$,
- For each i and $\bar{a}_1, \dots \bar{a}_{i-1}, \bar{a}_{i+1}, \dots, \bar{a}_k > 0$, we have

$$\psi(\bar{a}_1, \dots \bar{a}_{i-1}, a_i, \bar{a}_{i+1}, \dots, \bar{a}_k) \to \infty, \qquad \text{as } a_i \to \infty.$$

Examples of $\varphi \in \mathcal{G}$:

- $\varphi(M) = (trace(M^p))^{1/p}$, then $\psi(a_1, \ldots, a_k) = \left(\sum_{i=1}^k a_i^p\right)^{1/p}$
- $\varphi(M) = det(M)$, then $\psi(a_1, \ldots, a_k) = \prod_{i=1}^k a_i$

Corresponding eigenvalues functionals:

$$\left(\sum_{i=1}^k \lambda_i(\cdot)^p\right)^{1/p}, \qquad \prod_{i=1}^k \lambda_i(\cdot)$$

Why the class $\mathcal{G}$ is nice?

Given $U \in H^1_0(\Omega, \mathbb{R}^k)$, we define the $k \times k$ symmetric matrix

$$M(U) = \left(\int_\Omega \nabla u_i \cdot \nabla u_j \, dx \right)_{i,j=1,\dots,k}$$

Our goal is to study the problem

$$\min \left\{ \varphi(M(U)) \ : \ U \in H^1_0(\Omega, \mathbb{R}^k), \ \int_\Omega u_i u_j \, dx = \delta_{ij} \right\}$$

Remark (Ramos, Tavares, Terracini): If $\varphi \in \mathcal{G}$, then the minimum is achieved for a U such that $M(U)$ is a *diagonal matrix*.

In other words, for $\varphi \in \mathcal{G}$:

$$\min\left\{\varphi(M(U)) \;:\; U\in H^1_0(\Omega,\mathbb{R}^k),\ \int_\Omega u_iu_j\,dx=\delta_{ij}\right\}$$
$$=\min\left\{\varphi(M(U)) \;:\; U\in H^1_0(\Omega,\mathbb{R}^k),\ \int_\Omega u_iu_j\,dx=\delta_{ij}\right.$$
$$\left.\text{and}\int_\Omega \nabla u_i\cdot\nabla u_j\,dx=0,\ \text{for } i\neq j\right\}$$

Remark (Ramos, Tavares, Terracini): If $\varphi(M)=\varphi(P^TMP)$ for all $M\in\mathcal{S}_k(\mathbb{R})$ and $P\in\mathcal{O}_k(\mathbb{R})$, then

$$\frac{\partial\varphi}{\partial\xi_{ij}}(D)=0,$$

for all diagonal matrix D when $i<j$.

PDE solved by U

For each $U \in H^1_0(\Omega, \mathbb{R}^k)$ optimal for the minimum problem above with $\varphi \in \mathcal{G}$, we have

$$-a_i \Delta u_i = \sum_{j=1}^{k} \mu_{ij} u_j, \qquad \forall\, i = 1, \dots, k,$$

where

$$a_i := \frac{\partial \varphi}{\partial \xi_{ii}}(M(U)) > 0,$$

and, since $M(U)$ is a diagonal matrix, also $(\mu_{ij})_{i,j}$ is diagonal, more precisely

$$\mu_{ij} = \mu_{ji} := \delta_{ij} a_i \int_\Omega |\nabla u_i|^2 = \delta_{ij} a_i \lambda_i(\Omega).$$

In the end, the equation that we get is

$$-a_i \Delta u_i = \mu_{ii} u_i, \qquad \forall\, i = 1, \dots, k.$$

Optimality condition and monotonicity functional

The monotonicity functional becomes:

$$\phi(x_0, r) := \frac{1}{r^d}\left(\int_{B_r(x_0)} \Big(\sum_{i=1}^{k} a_i |\nabla u_i|^2\Big)\,dx + \Lambda |\{|U|^2 > 0\} \cap B_r(x_0)|\right) - \frac{1}{r^{d+1}} \int_{\partial B_r(x_0)} \sum_{i=1}^{k} a_i u_i^2 \, d\mathcal{H}^{d-1}$$

Optimality condition and monotonicity functional

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The optimality condition reads as:

$$\sum_{i=1}^k a_i|\nabla u_i|^2 = \Lambda, \qquad \text{on } \partial\{|U| > 0\}.$$

A small class of functionals?

Problem: not all smooth functions of eigenvalues are smooth (C^1) symmetric functions.
Good examples are:

$$\lambda_1 + \cdots + \lambda_k, \qquad (\lambda_1^p + \cdots + \lambda_k^p)^{1/p}, \qquad \prod_{i=1}^{k} \lambda_i.$$

But...

$$\lambda_k = \max_{i=1,\dots k} \lambda_i = \lim_{p\to\infty} (\lambda_1^p + \cdots + \lambda_k^p)^{1/p},$$

is only Lipschitz continuous.
So an approximation argument is needed when dealing with the regularity for the functional λ_k!

An intermediate step

An easier problem than λ_k could be the study of functionals:

$$F\colon \mathbb{R}^k \to \mathbb{R},$$

(locally) Lipschitz continuous, non-decreasing in each variable and strictly increasing in the first variable.

Examples: $\lambda_1 + \lambda_k$, $\lambda_1^2 + \lambda_k$.

Problems also in this case?

- We need also an approximation in the measure: no clear Lagrange multiplier!
- Approximation of the functional from the class $\mathcal{G}$.
- Lipschitz continuity not of all eigenfunctions.

Thanks!