Optimal stretching for lattice points and eigenvalues

Richard Laugesen and Shiya Liu

University of Illinois at Urbana–Champaign

Shape Optimization and Isoperimetric and Functional Inequalities

Conference at CIRM Luminy

November 22, 2016

What shape minimizes n-th eigenvalue λ_n?

How does the spectrum (analysis) constrain the domain (geometry). . . ?

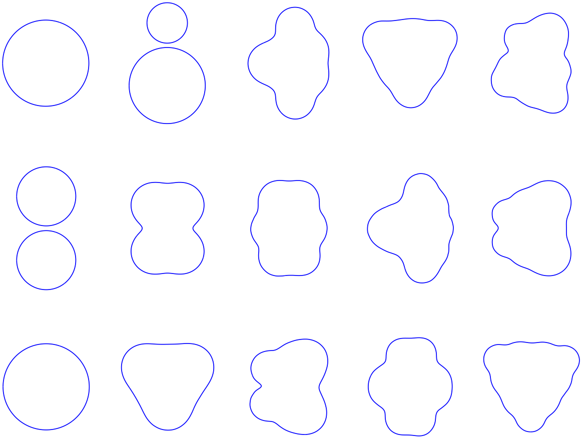


Figure: Minimizers of the first 15 Dirichlet eigenvalues (numerical work by Oudet and later Antunes–Freitas).

What shape minimizes n-th eigenvalue λ_n?

Conjecture (Antunes and Freitas)

If Ω_n minimizes λ_n among planar domains of area π, then Ω_n converges to a unit disk as $n \to \infty$.

Equivalently, let

$$N(t) = \text{counting function} = \#\{n : \lambda_n \leq t\}.$$

Then:

Conjecture

If Ω_t maximizes $N(t)$ among planar domains of area π, then Ω_t converges to a unit disk as $t \to \infty$.

Motivation for disk to maximize high freq. counting fn.

Two-term Weyl Asymptotic

For domain Ω in 2-dimensions,

$$N(t) = a|\Omega|t - b|\partial\Omega|\sqrt{t} + o(\sqrt{t}) \quad \text{as } t \to \infty,$$

where $a, b > 0$ are constants.

Putting isoperimetric ineq. $|\partial\Omega| > |\partial\mathbb{D}|$ into Weyl implies for $\Omega \neq \mathbb{D}$ that

$$N(t) < N_{\mathbb{D}}(t) \qquad \forall \text{ large } t.$$

Why does this argument not prove the conjecture?
Because we fixed Ω and then let $t \to \infty$.
Instead we must minimize w.r.t. Ω **before** letting $t \to \infty$!

Three step heuristic for attacking the problem (Antunes–Freitas)

1. Show optimizing domains $\{\Omega_t\}$ form a compact family.
2. Control Weyl remainder uniformly for such families.
3. Conclude from quantitative isoperimetric ineq. that Ω_t approaches disk.

Special case — Dirichlet rectangles

For $t > 0$, let

$$N_s(t) = \text{eigenvalue counting function for rectangle of area 1 with sidelengths } s \text{ and } 1/s$$

$$s^*(t) = s\text{-value maximizing } N_s(t)$$

Theorem (Antunes–Freitas 2013)

Optimal rectangles converge to unit square:

$$s^*(t) \to 1 \qquad \text{as } t \to \infty.$$

Proof sketch

Express eigenvalue counting as lattice point counting (see next page).
Weyl asymptotic with controlled remainder is known from number theory (like Gauss Circle Problem).
Implement the three-step heuristic. . .

Express Dirichlet rectangle eigenvalues using lattice points

$$\frac{\text{eigenvalue}}{\pi^2} \equiv \left(\frac{j}{1/s}\right)^2 + \left(\frac{k}{s}\right)^2 \leq r^2 \quad \text{where } j,k>0$$

$$\iff \quad (j,k) \text{ lies inside ellipse of semiaxes } r/s \text{ and } rs.$$

So eigenvalues are counted by

$$N_s(r) = \#\{\text{lattice points } (j,k) \text{ inside } r\Gamma(s)\}$$

where

$$\Gamma(s) = \text{ellipse of area 1 with semiaxes } 1/s \text{ and } s.$$

Let

$$s^*(r) = s\text{-value maximizing } N_s(r).$$

Antunes–Freitas theorem says $s^*(r) \to 1$ as $r \to \infty$
(rectangle minimizing n-th eigenvalue tends to a square)
i.e., ellipse maximizing the first-quadrant lattice count tends to a circle.

Circle ($s = 1$) is generally not optimal at a finite radius r!!!

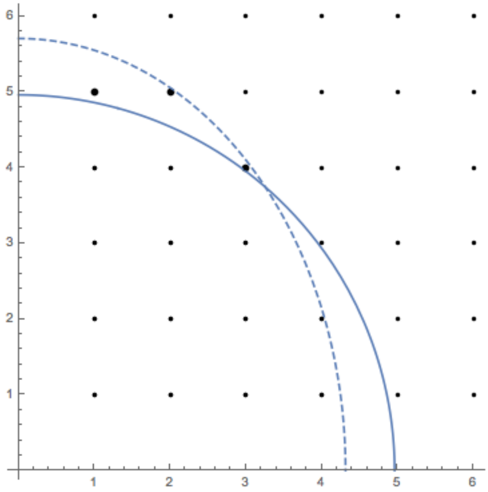


Figure: Compare circle $s = 1$ (solid) with ellipse $s = 1.15$ (dashed), for $r = 4.96$. Ellipse encloses more points than circle: $N_1(4.96) = 13$ and $N_{1.15}(4.96) = 16$.

General case: optimal stretching for lattice point counting

Γ: strongly concave C^2-curve decreasing from $(0,1)$ to $(1,0)$, with "monotonic 2nd deriv."

$\Gamma(s)$: "generalized ellipse" obtained from Γ, with semiaxes s and $1/s$

Theorem (Laugesen–Liu 2016, on ArXiv)

Optimal shape enclosing most lattice points is "balanced" in the limit:

$$s^*(r) \to 1 \qquad \text{as } r \to \infty.$$

Special case: $\Gamma =$ *quarter circle gives Antunes–Freitas Theorem.*

Proof Step 1 (compact family) — Obtain two-term **inequality**

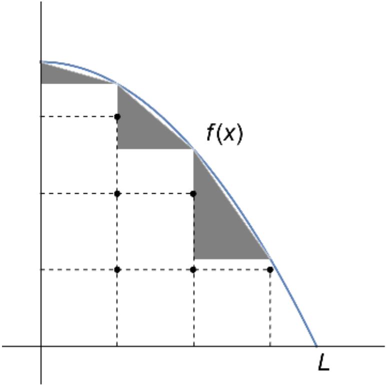
$$N_s(r) \leq \text{Area}\left(r\Gamma\right) - \widetilde{c}rs$$

by elementary geometry, where constant $\widetilde{c} = \widetilde{c}(\Gamma) > 0$ is not sharp. Then

$$\begin{aligned} \text{Area}\left(r\Gamma\right) - cr + o(r) = N_1(r) \quad & \text{by counting fn. } \textbf{asymptotic} \\ \leq N_{s^*(r)}(r) \leq \text{Area}\left(r\Gamma\right) - \widetilde{c}rs^*(r). \end{aligned}$$

Hence $\limsup_{r\to\infty} s^*(r) \leq c/\widetilde{c} < \infty$, so stretch factor is bounded.

Idea for proving the two-term inequality:



$$
\begin{aligned}
&\text{number of lattice points} \\
&\leq \text{area of Riemann rectangles} \\
&\leq (\text{area under curve}) - (\text{area of triangles})
\end{aligned}
$$

and

$$(\text{area of triangles}) \geq \frac{1}{2}\big(f(0) - f(L/2)\big)$$

General curves, cont.

Proof Step 2 (uniform counting fn. remainder) — Sharp two-term counting fn. asymptotic with controlled remainder:

$$N_s(r) = \text{Area}\left(r\Gamma\right) - cr(s + 1/s) + o(r)\left(s^2 + 1/s^2\right).$$

Remainder is uniformly controlled since s and $1/s$ are bounded by Step 1.
Method for Step 2:
Euler–Maclaurin summation, exponential sums, and van der Corput lemma from number theory (following Krätzel 2004). Or use Huxley's estimates.

Proof Step 3 (like a quantitative isoperimetric inequality) — replace non-sharp inequality in Step 1 with sharp asymptotic from Step 2, deduce

$$\limsup_{r\to\infty}\left(s^*(r) + \frac{1}{s^*(r)}\right) \leq 2.$$

Conclude $s^*(r) \to 1$, so optimal stretch factor for general concave region tends to 1.

Optimal p-ellipses for lattice point counting

Γ: p-circle, $|x|^p + |y|^p = 1$ $\Gamma(s)$: p-ellipse with semiaxes s, $1/s$

Theorem (Laugesen–Liu 2016)

Optimal p-ellipse approaches p-circle:

$$s^*(r) \to 1 \qquad \text{as } r \to \infty,$$

for each $1 < p < \infty$.

Previous theorem does *not* apply, since p-ellipse behaves badly at axes:

$$f''(0) = \begin{cases} -\infty & 1 < p < 2, \\ 0 & 2 < p < \infty. \end{cases}$$

So p-ellipse ($p \neq 2$) is either not C^2-smooth or else not strongly concave. We avoid these bad points by counting lattice points (j,k) directly when $0 < j, k < r^{1-1/p}$.

Optimal 1-ellipses for lattice point counting

Γ: 1-circle, $|x| + |y| = 1$ Γ(s): 1-ellipse with semiaxes s, $1/s$

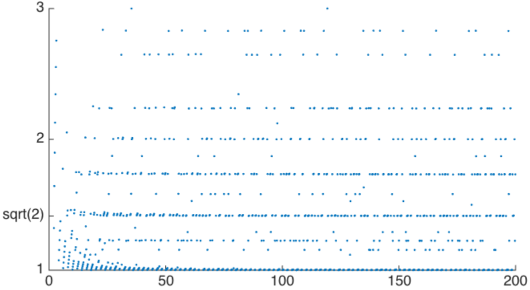


Figure: Graph of $s = s^*(r)$ for 1-ellipses

Optimal 1-ellipse does **not** approach 1-circle, numerically, as $r \to \infty$.
i.e. optimal right triangle does **not** approach 45–45-90° triangle.

Odd integer lattice points exhibit same phenomenon.
Spectral interpretation: minimize λ_n w.r.t. family of harmonic oscillator potentials $(sx)^2 + (y/s)^2$? Optimal s does **not** converge to 1 as $n \to \infty$.

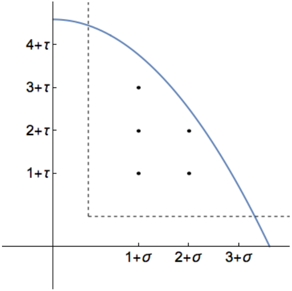
Current work (PhD thesis for Shiya Liu)

Maxmize counting fn. for *shifted* and *deformed* lattices, hence minimize λ_n for separated Schrödinger potentials.

1. Shifted lattice $(j+\sigma, k+\tau)$ with $j,k>0$

   Positive shift: if $\sigma,\tau \geq 0$ then $s^*(r) \to \sqrt{(\tau+1/2)/(\sigma+1/2)}$

   Negative shift: if $\sigma<0$ or $\tau<0$ then $\exists$ curve Γ such that $s^*(r) \to \infty$



2. Deformed lattice (in progress)

Related work

Higher dimensions

van den Berg and Gittins (2016) extend Antunes–Freitas to rectangular boxes in 3 dimensions.
Optimal box for minimizing λ_n approaches a cube as $n \to \infty$.
Open in dimension ≥ 4.

Riesz means of eigenvalues

Larson (last week on ArXiv) treats convex domains (some restrictions). He maximizes the Riesz mean $\sum_{n=1}^{\infty}(\Lambda - \lambda_n)_+^{\gamma}$ where $\Lambda > 0$ and $\gamma \geq 3/2$.

Neumann eigenvalue counting

van den Berg, Buçur and Gittins (2016) prove Neumann analogue of Antunes–Freitas: the optimal rectangle for maximizing μ_n approaches square as $n \to \infty$. That is, they minimize the count of nonnegative integer lattice points $(j, k \geq 0)$ inside ellipses.

Laugesen and Liu (2016) extend from ellipses to general concave curves.

Conclusion

Evidence is mounting for symmetry of eigenvalue optimizers as $n \to \infty$.