Regularity for functionals involving perimeter

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Shape optimization, Isoperimetric and Functional Inequalities

Statement of the problem

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  - a spectral functional of the form $F(\lambda_1(\Omega),\dots,\lambda_k(\Omega))$, where $(\lambda_1(\Omega),\dots,\lambda_k(\Omega))$ are the first k eigenvalues of the Dirichlet-Laplacian and $F:\mathbb{R}^k\to\mathbb{R}$ is locally Lipschitz continuous and increasing in each variable.

Examples

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- If $\mathcal{G} = E_1$ or λ_1. Isoperimetric + Saint-Venant/Faber-Krahn inequalities. The ball is solution.

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Numerical results by Bogosel-Oudet 2014

$$\min\left\{P(\Omega)+\lambda_k(\Omega),\ \Omega\subset\mathbb{R}^2\right\}\quad \text{no volume constraint}$$

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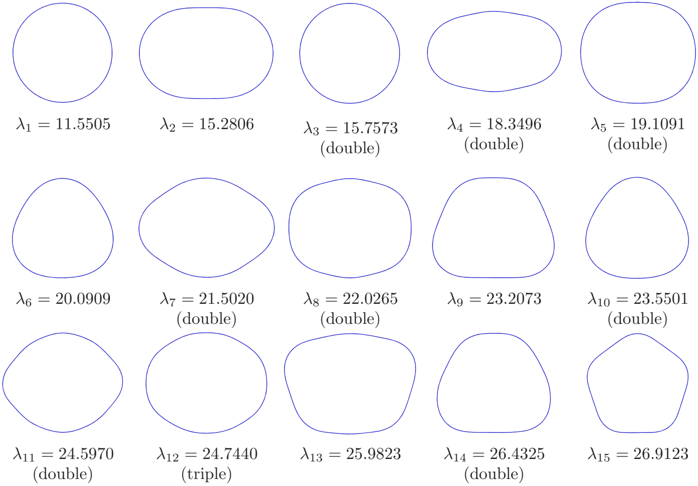

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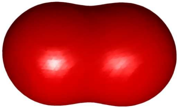
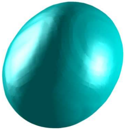
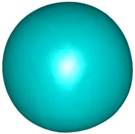
$$\min\left\{P(\Omega)+\lambda_k(\Omega),\ \Omega\subset\mathbb{R}^3\right\}.$$



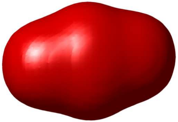
$\lambda_2(\Omega)\,\mathrm{Per}(\Omega)=223.63$



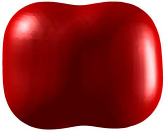
$\lambda_3(\Omega)\,\mathrm{Per}(\Omega)=251.91$



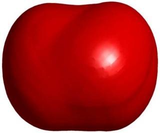
$\lambda_4(\Omega)\,\mathrm{Per}(\Omega)=255.56$



$\lambda_5(\Omega)\,\mathrm{Per}(\Omega)=343.75$

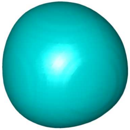


$\lambda_6(\Omega)\,\mathrm{Per}(\Omega)=394.77$

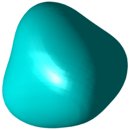


$\lambda_7(\Omega)\,\mathrm{Per}(\Omega)=421.20$

$\lambda_8(\Omega)\,\mathrm{Per}(\Omega)=439.80$



$\lambda_9(\Omega)\,\mathrm{Per}(\Omega)=446.58$



$\lambda_{10}(\Omega)\,\mathrm{Per}(\Omega)=510.00$

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- $\mathcal{G} = \lambda_k$ but no volume constraint: [De Philippis-Velichkov 2014]. Use of sub/sup-solutions [Bucur].

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Consequence: $C^{1,\beta}$-regularity up to a singular set of dimension less than $d - 8$.

Sharpness of the hypotheses

Non-existence situations

- $f \in L^\infty(\mathbb{R}^d)$ such that $0 \leq f < 1$ and $f(x) \to_{|x|\to\infty} 1$, $D = \mathbb{R}^d$.

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Representatives of sets

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- Regularity of $\partial\Omega$ is meant for

$$\partial\Omega := \{x \in \mathbb{R}^d, \forall r > 0, \ \ 0 < |\Omega \cap B_r(x)| < |B_r|\}.$$

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Solutions of

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  - Compactness (good situation)
  - Compactness at infinity (only if $\mathcal{G}=E_f$: hypotheses on f)
  - Vanishing (easy to exclude)
  - Dichotomy: difficult to exclude. We use boundedness of solutions, which relies on weak regularity theory.

Elements of proof III

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for some $\mu\in\mathbb{R}_+$.

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- General approach based on first order shape derivatives and lipschitz estimates: for $\Phi\in W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d)$,

$$\left|\widetilde{\mathcal{G}}(\Phi(\Omega))-\widetilde{\mathcal{G}}(\Omega)\right|\leq C_{d,|\Omega|,\mathcal{G}}\|\Phi-Id\|_{W^{1,\infty}}.$$

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Then

- $H^1_0(\Omega^*) = \widetilde{H}^1_0(\Omega^*)$,
- w_{Ω^*} is locally Lipschitz continuous on $\mathbb{R}^d$.

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Regularity I: Supersolutions for $P + \mu|\cdot|$

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$$P(\Omega^*) + \mu|\Omega^*| \leq P(\Omega) + \mu|\Omega|, \quad \text{for every } \Omega \text{ with } \Omega^* \subset \Omega.$$

Then

- $H^1_0(\Omega^*) = \widetilde{H}^1_0(\Omega^*)$,
- w_{Ω^*} is locally Lipschitz continuous on $\mathbb{R}^d$.

Based on

- Exterior estimates:

$$\forall x \in \partial\Omega, \ \ \frac{|B_r(x) \cap \Omega^c|}{|B_r|} \leq c < 1,$$

Elements of proof IV

Regularity I: Supersolutions for $P + \mu|\cdot|$

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- $H_0^1(\Omega^*) = \widetilde{H}_0^1(\Omega^*)$,
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Based on

- Exterior estimates:

$$\forall x \in \partial\Omega, \quad \frac{|B_r(x) \cap \Omega^c|}{|B_r|} \leq c < 1,$$

- Mean curvature bounds in the viscosity sense:

$$\Big[H_\Omega \geq -\mu\Big] \;\Rightarrow\; \text{regularity for elliptic PDE.}$$

Elements of proof V

Regularity II: Subsolutions

Assume

$$P(\Omega^*) + \widetilde{\mathcal{G}}(\Omega^*) + \mu|\Omega^*| \le P(\Omega) + \widetilde{\mathcal{G}}(\Omega) + \mu|\Omega|, \ \forall \Omega \subset \Omega^*,$$

and w_{Ω^*} is lipschitz continuous on $\mathbb{R}^d$.

Elements of proof V

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Elements of proof V

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Then Ω^* is a quasi-minimizer for interior perturbations.

Based on

- control variations of $\widetilde{\mathcal{G}}$ by variations of $\widetilde{E_1}$.
- Lipschitz continuity of w_{Ω^*}.

Last remarks

- Further regularity: classical if $\mathcal{G} = E_f$. More involved if $\mathcal{G} = \lambda_k$: see [Bogosel,Oudet 2014] and [Bogosel 2016].

Last remarks

- Further regularity: classical if $\mathcal{G} = E_f$. More involved if $\mathcal{G} = \lambda_k$: see [Bogosel,Oudet 2014] and [Bogosel 2016].

- Similar problem [van den Berg]:

$$\min\left\{\lambda_k(\Omega),\ \Omega \subset \mathbb{R}^d,\ |\Omega| = m,\ P(\Omega) = p\right\}.$$

  Existence if $\leq$ instead of $=$ [Bucur, Mazzoleni 2015]. Regularity?