# 2 dimensions are easier*

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*Results with L. Esposito, V. Ferone, T. Lachand-Robert, C. Nitsch, G. Sweers, C. Trombetti & C. Weber

# Isoperimetric Inequality

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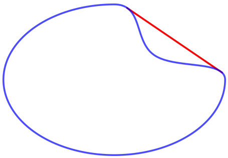

In two dimensions it suffices even to consider convex sets only. They can be parametrized by a function $r(\theta)$, and then we maximize the area

$$A(\Omega) = \int_0^{2\pi} \int_0^{r(\theta)} \rho d\rho d\theta = \frac{1}{2} \int_0^{2\pi} r^2(\theta)\, d\theta$$

subject to given length

$$L(\partial\Omega) := \int_0^{2\pi} \sqrt{r^2(\theta) + r_\theta^2(\theta)}\, d\theta.$$

So among periodic functions $r(\theta)$ we can look at the functional $A + \lambda L$ with Euler-Lagrange equation

$$r + \frac{\lambda r}{\sqrt{r^2 + r_\theta^2}} - \frac{d}{d\theta}\left(\frac{\lambda r_\theta}{\sqrt{r^2 + r_\theta^2}}\right) = 0.$$

After integration

$$(\kappa =)\frac{r\, r_{\theta\theta} - 2r_\theta^2 - r^2}{(r^2 + r_\theta^2)^{3/2}} = \frac{1}{\lambda}.$$

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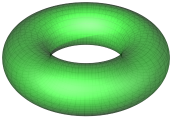
Notice that in 3 dimensions convex hull does not reduce perimeter (and unbounded sets can have finite perimeter).



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6. The Tertiary period
7. The Quatenary period
8. A Summary
9. Cui prodest? The tallest column

# Problem 1: Cheeger sets

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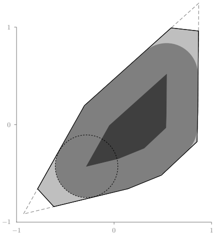
When $\Omega = (-a, a)^2$ is a square, the corresponding Cheeger set is a rounded square which can be easily calculated.
The Cheeger constant $\frac{|\partial \Omega_C|}{|\Omega_C|}$ is $h(\Omega) = (\sqrt{\pi} + 2)/(2a)$ and the circular arcs have radius $1/h(\Omega)$.

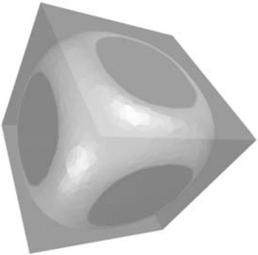


# Problem 1: Cheeger sets

## Cheeger sets model earth slides

# Open Problem 1a: Cheeger sets

When $\Omega$ is a cube, no analytical description of its Cheeger set has been given, other than that it is convex and that the free parts of its boundary have constant mean curvature $h(\Omega) = \frac{|\partial\Omega_C|}{|\Omega_C|}$. A numerical approximation and visualization was given by Lachand-Robert and Oudet in 2005.



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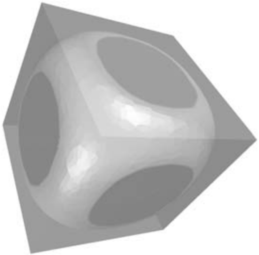


Give an analytical representation of the bright rounded edges.

# Problem 1b: Cheeger sets

When $\Omega$ is convex, so is its (unique) Cheeger set.

(When $\Omega$ is not convex, there are examples of nonuniqueness and nonconvexity of the Cheeger set.)

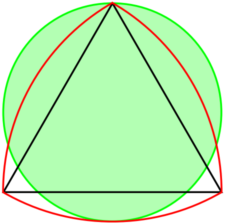
Convexity and uniqueness for convex $\Omega$ is fairly easy to prove in 2d.: One sweeps the inside of $\Omega$ with a disc of radius $1/h(\Omega)$, but the proof is much trickier in higher dimensions.
There are proofs of Caselles, Chambolle & Novaga (2007) and of Alter & Caselles (2009).

Simplify the convexity proof for $d \geq 3$.

# Open Problem 2: Convex sets of constant width

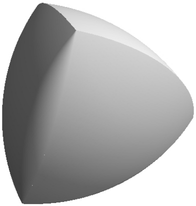
have been studied for more than a century. A nice exposition can be found in the book “Geometry and the Imagination” by Hilbert and Cohn-Vossen.

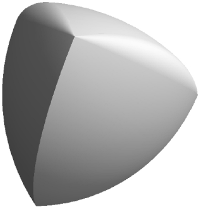
Among all two-dimensional convex sets of constant width $d$ the disk with radius $d/2$ maximizes area and the **Reuleaux-triangle** minimizes area. A Reuleaux-triangle is the intersection of three disks with centers in the corners of an equilateral triangle.



# Open Problem 2: Convex sets of constant width

In three dimensions it has been shown that the ball maximizes volume among all convex sets of given width $d$, and is has been conjectured that the **Meissner-bodies** minimize volume. These are obtained from a small modification of the Meissner-tetrahedron, which is the intersection of four balls of radius $d/2$ with centers in the four corners of a regular tetrahedron.





# Open Problem 3: A Fencing Problem

Imagine a convex piece of land that you want to cut into two subsets of equal area with a minimal cut. Given the total area (but not the shape) of the initial set, which shape renders the longest shortest cut?

This problem was posed by Polyá in 1958, and his conjecture that the answer is a disk was not confirmed until 2012 in *"The longest shortest cut"*.

The proof is quite technical and the three-dimensional analogue, that a ball and a bisecting plane will serve the same purpose, seems to be a difficult open problem.

# Open Problem 4: Ulam floating

When a ball of specific weight 1/2 is dropped into water, in contrast to a cube or ellipsoid it swims semistable in any direction.

Is the ball the only shape that has this property,
known as Ulam floating?

Although the problem was widely circulated in the 1930's, there are still opposing convictions as to how to answer this question.

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There is also a two-dimensional analogue. Some trees with convex cross-section have a preferred orientation in water.


**Fig. 1.** An equilibrium position (left) and a non-equilibrium position (right).

Ulam: Are there any cross sections different from the disc that don't have a preferred direction?

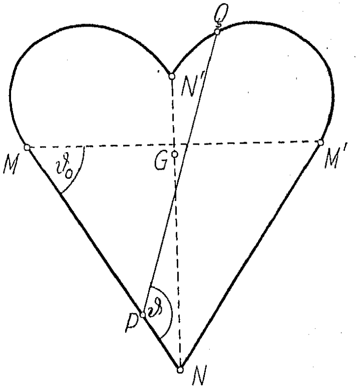
H. AUERBACH 1934. *Sur un problème de M. Ulam concernant l'équilibre des corps flottants.* Studia Math. **7** (1938), 121–142.

Yes, any convex plane Zindler set has Ulam's floating property.

By definition a *Zindler set* has the property that any area bisecting chord has the same length.

There are many Zindler sets . . .

nonconvex ones, like the heart from Auerbach's paper, . . .



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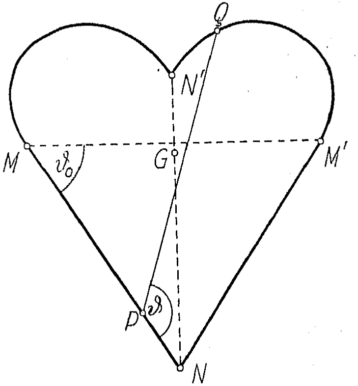


and many many convex ones, e.g. the Auerbach triangle.

# Problem 4: Ulam floating for cylinders

Among convex Zindler sets of given area one can look for the one with the longest water-line dividing it into two sets of equal area.

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Among convex Zindler sets of given area one can look for the one with the longest water-line dividing it into two sets of equal area.

In view of Open Problem 3 of the longest shortest fence one should suspect the disk as optimal, but this is wrong.

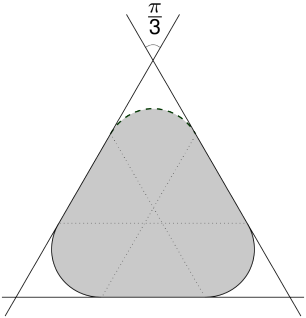
In fact, the Auerbach triangle is optimal. This is surprising even to experts in shape optimization. The result was conjectured by Auerbach in 1934, and N.Fusco and A.Pratelli were able to prove it not until 2011.

Finally, in 2012 in the paper on longest shortest fences, the assumption of being a Zindler set was removed.

# Auerbach triangle

$$\begin{cases} x(t) = \dfrac{e^{4t}-1}{e^{4t}+1} - t \\[2em] y(t) = 2\dfrac{e^{2t}}{e^{4t}+1} \end{cases}$$

$$t \in [-(\log 3)/4, (\log 3)/4]$$



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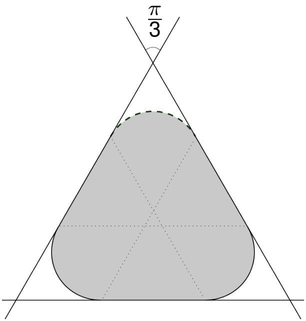


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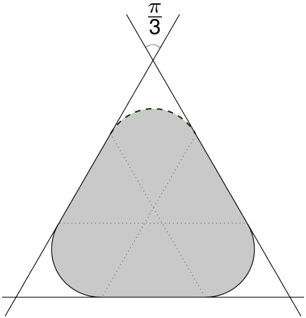
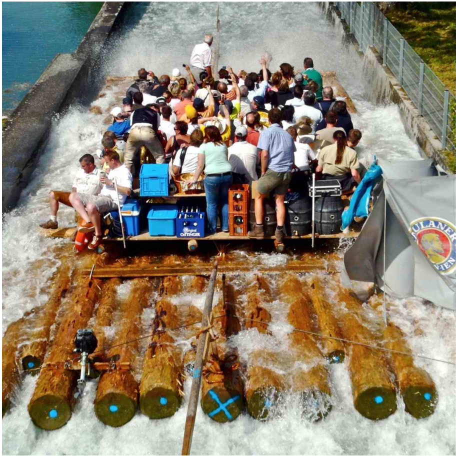



All bisecting chords of the Auerbach triangle have the same constant length which is bigger than the diameter 2 of the circle with the same area $\pi$.
Moreover the chords bisect also the perimeter.
Note that the shortest arc bisecting the area has length less than 2.
Therefore the Auerbach triangle does not contradict the result on the longest shortest cut.

## Ulam floating is semistable, but there are better floats.



# Problem 5: Euler elastica

are curves $\gamma$, whose bending energy is measured in terms of their curvature $\kappa$ as

$$E(\gamma) = \int_\gamma \tfrac{1}{2}\kappa^2(s)\, ds.$$

**Remark:** Among all simple regular closed curves of given length $L$, only the circle minimizes elastic energy. This follows from a reduction to curves which bound convex sets. In fact $L\int_\gamma \kappa^2\, ds$ is scale-invariant, so convexifying a curve decreases the product and this remains so under rescaling.

Then one can apply Hölder's inequality

$$\int_\gamma \kappa^2\, ds \geq \left(\int_\gamma \kappa\, ds\right)^2 L^{-1} \geq \frac{(2\pi)^2}{L}$$

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Analogous problems in three dimensions could be:
Show that among all simply-connected open three-dimensional sets with boundary $\gamma$ the ball minimizes the Willmore energy $\int_\gamma H^2\ ds$ ($H$ denoting mean curvature) for given surface area (or perimeter) $|\gamma|$ or for given enclosed volume $|\Omega|$.

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However, for $n = 3$ the Willmore energy alone is scale invariant, so prescribing the perimeter or enclosed volume provides no restriction. This was already noticed in 1924 by Thomsen, and a simple proof was given by Willmore in 1965 that the minimizing shape must be a sphere.

Another generalization to three and more dimensions might be the study of the scale-invariant functional $J(\gamma) = \int_\gamma H^n ds\, |\Omega|^{1/n}$.
At least among convex sets $\Omega \subset \mathbb{R}^n$ with boundary $\gamma$ and for $n \geq 2$ one can easily show that

$$\int_\gamma H^n\, ds \geq \int_\gamma K^{\frac{n}{n-1}}\, ds \geq \left(\int_\gamma K\, ds\right)^{\frac{n}{n-1}} |\gamma|^{\frac{-1}{n-1}} = (n\omega_n)^{\frac{n}{n-1}} |\gamma|^{\frac{-1}{n-1}}.$$

Here $K$ denotes Gauss curvature and $n\omega_n$ the $(n-1)$-dimensional perimeter of the unit sphere in $\mathbb{R}^n$. The first inequality uses the geometric-algebraic mean inequality, the second one Hölder's, and for given perimeter $|\gamma|$ the estimate becomes sharp if and only if $\gamma$ is a sphere.

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For nonconvex $\Omega$ and $n = 3$ a counterexample to the estimate was just recently given by Ferone, Nitsch and Trombetti.

# Open Problem 6: Farthest Point Distance Function

For $n = 1$ $\quad u(x) = \frac{1}{2}|x - y|$ is harmonic off $y$ and $\Delta u = \delta_y$.

For $n = 2$ $\quad u(x) = \frac{1}{2\pi} \log |x - y|$ is harmonic off $y$ and $\Delta u = \delta_y$.

For $n > 2$ $\quad u(x) = -\frac{1}{n(n-2)\omega_n}|x - y|^{2-n}$ is harmonic off $y$ and $\Delta u = \delta_y$.

In all three cases $u(x) = \phi(|x - y|)$
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and $\Delta u$ is a (nonnegative) probability measure with support in $y$.

For $E \neq \emptyset$ compact we define the farthest point distance function to $E$:

$$d_E(x) := \max_{y \in E} |x - y|.$$

# The measure of $E$

What happens to $\Delta u$ when $u(x) = \phi(d_E(x))$ and when $E$ consists of more than one point $y$?

$$\phi(d_E(x)) = \phi(\max_{y\in E} |x-y|) = \max_{y\in E} \phi(|x-y|)$$

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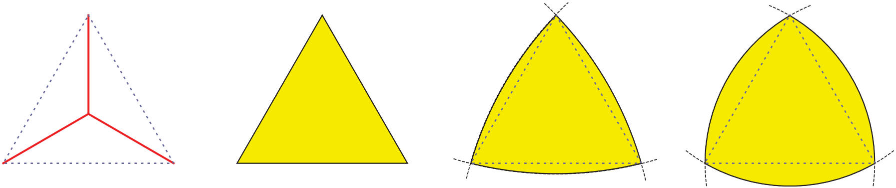

$E_1$ $E_2$ $E_3$ $E_4$

Figura : *Four compact sets $E_i$ with $\sigma_{E_i}(E_i) = \frac{1}{2}$. Only the last one, the Reuleaux triangle, is of constant width. For all four we have $E_i^* = E_4$.*

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but if $E$ is a ball of radius $R$ then $d_E(x) = R + |x|$
and one can calculate that $\sigma_B(B) = 2^{1-n} < 1$.

**Conjecture** (Laugesen & Pritsker, 2003)
For any compact $E$ with more than one point we have $\sigma_E(E) \leq 2^{1-n}$.

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**Theorem** (Kawohl, Nitsch & Sweers 2014)
Conjecture holds for $n > 2$ and $E$ of constant width,
but in this class equality $\sigma_E(E) = 2^{1-n}$ holds only for the ball.

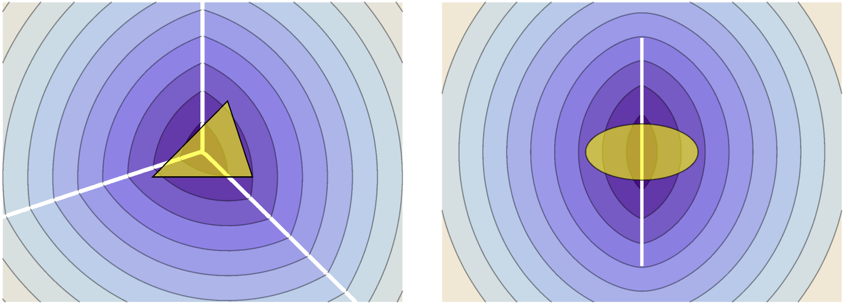

Figura : *Level lines of $d_E$ for a triangle and an ellipse; the white lines show where $d_E$ is not $C^1$ and where $\sigma_E$ is not absolutely continuous with respect to the Lebesgue measure.*

Formally

$$\sigma_E(E) = \int_E \Delta\phi(d_E(x))\, dx,$$

but $\sigma_E$ is not absolutely continuous with respect to Lebesgue measure. However, the following integrals are well defined and we were able to show

$$\sigma_E(E) \leq \int_{\partial E} \frac{\partial\phi(d_E(x))}{\partial\nu} ds = \frac{1}{n\omega_n} \int_{\partial E} d_E^{1-n} \frac{\partial d_E}{\partial\nu}\, ds \qquad (1)$$

For a ball of radius $R$ we have $d_E = 2R$ and the rhs becomes

$$\frac{1}{n\omega_n(2R)^{n-1}} \int_{\partial E} ds = \frac{1}{n\omega_n(2R)^{n-1}} |\partial E| = \frac{n\omega_n R^{n-1}}{n\omega_n(2R)^{n-1}} = 2^{1-n}.$$

For a set of constant width $w_E$ we have $d_E = w_E$ on $\partial E$ so that

$$\sigma_E(E) \leq \frac{1}{n\omega_n\, w_E^{n-1}} |\partial E|.$$

For sets of constant width

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There are other partial results,
e.g. for polyhedra (Wise 2014)
or point symmetric sets (K, Nitsch, Sweers)

that take more time to explain. . . .

# Problem 7: Hadwiger's inequality

is a relative isoperimetric inequality. Let $Q_n$ be the open unit cube in $\mathbb{R}^d$ and $A$ any measurable subset of $Q_n$. Then

$$Per(A; Q_n) \geq 4\,|A|\,(1 - |A|).$$

The result holds in any dimension and there are proofs by Hadwiger (1972) and Ambrosio, Bourgain, Brezis & Figalli (2016).

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I tried my own proof. After repeated reflections and Steiner symmetrizations one may assume that the optimal $A$ is starshaped with respect to zero and that its boundary can be parametrized by a function $y(x_1, \ldots, x_{n-1})$. Given the volume of $A$, we minimize surface area, so $\partial A \cap Q_n$ must have constant mean curvature, and the natural boundary condition for this geometric variational problem tells us that $\partial A$ runs orthogonally into $\partial Q_n$.

# Problem 7: Hadwiger's inequality

Again 2*d* are easier. The only boundaries that can occur are circular arcs (for small or large |A|) or straight line segments (for intermediate |A|), running vertically into the sides of a square.
The optimal shape occurs for $|A| = 1/2$ and it is a rectangle.

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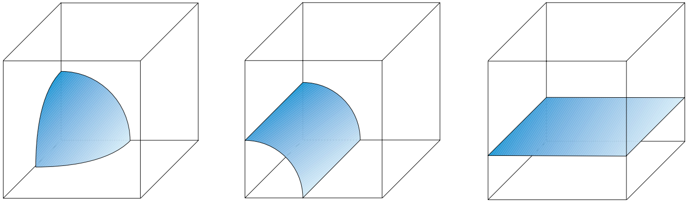
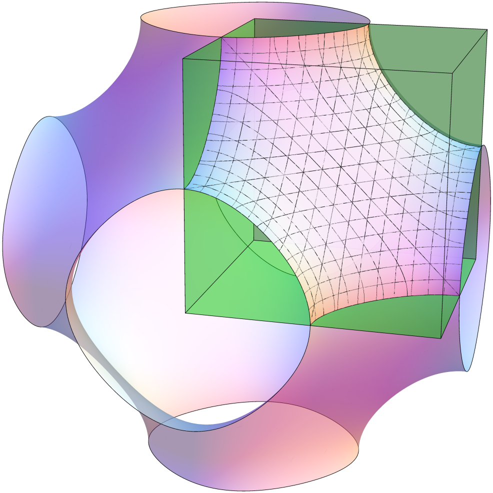
For $d \geq 3$ the optimal $\partial A$ has constant mean curvature and runs vertically into $\partial Q_n$. Here are some cmc sets which qualify for the competition in 3*d*, taken from Ritore & Ross (2002)



Show that these are the only interesting ones and rule out others

# Problem 7: Hadwiger's inequality

like the Schwarz surface . . .



Thank you for your patience and attention.

The paper **“2 dimensions are easier”**

is open access and has all the references for Problems 1a and 2–6.