Asymptotic optimal sets for the eigenvalues of the Laplacian

Katie Gittins

Université de Neuchâtel

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Dirichlet eigenvalues

Let $\Omega \subset \mathbb{R}^m$, $m \geq 2$, be an open set of finite Lebesgue measure $|\Omega| < \infty$. **Dirichlet** eigenvalues of the Laplacian on Ω, $\lambda_k(\Omega) \in \mathbb{R}$, satisfy

$$\begin{cases} -\Delta u_k(x) = \lambda_k(\Omega) u_k(x) & x \in \Omega, \\ u_k(x) = 0 & x \in \partial\Omega, \end{cases}$$

and form a non-decreasing sequence, counted with multiplicities,

$$\lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \cdots \leq \lambda_k(\Omega) \leq \ldots$$

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$$\lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \cdots \leq \lambda_k(\Omega) \leq \ldots$$

Goal

For each $k \in \mathbb{N}$, determine an open set $\Omega_k^* \subset \mathbb{R}^m$ such that, for prescribed $c \in \mathbb{R}$, $c > 0$,

$$\lambda_k(\Omega_k^*) = \inf\{\lambda_k(\Omega) : \Omega \subset \mathbb{R}^m \text{ open, } |\Omega| = c\}.$$

New approach

Antunes and Freitas:
If, for each $k \in \mathbb{N}$, a minimiser Ω_k^* for λ_k exists in a chosen collection of sets in $\mathbb{R}^m$, then determine the asymptotic minimal set as $k \to \infty$, i.e. the limit of a sequence of minimisers $(\Omega_k^*)_k$ as $k \to \infty$.

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Colbois and El Soufi:
The following are equivalent

- λ_k^* is asymptotically equal to $4\pi^2(c\omega_m)^{-2/m}k^{2/m}$ as $k \to \infty$, where ω_m is the measure of a ball of radius 1 in $\mathbb{R}^m$.
- Pólya's Conjecture: for all bounded, open sets $\Omega \subset \mathbb{R}^m$ of measure c,

$$\lambda_k(\Omega) \geq 4\pi^2(c\omega_m)^{-2/m}k^{2/m}.$$

Rectangles in $\mathbb{R}^2$ of unit measure

For $b \geq 1$, let

$$R_b = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < b,\, 0 < x_2 < b^{-1}\}.$$

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For each $k \in \mathbb{N}$, there is a minimising rectangle $R_{b_k^*}$ such that

$$\lambda_k^* = \lambda_k(R_{b_k^*}) = \inf\{\lambda_k(R_{b_k}) : |R_{b_k}| = 1\}.$$

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Theorem (Antunes, Freitas, 2013)

Any sequence of minimising rectangles $(R_{b_k^*})_k$ converges to the unit square as $k \to \infty$.

Cuboids in $\mathbb{R}^3$ of unit measure

For $0 < a_1 \leq a_2 \leq a_3$ with $a_1 a_2 a_3 = 1$, let

$$R_{a_1,a_2,a_3} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < x_1 < a_1, 0 < x_2 < a_2, 0 < x_3 < a_3\}.$$

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Dirichlet eigenvalues of the Laplacian on R_{a_1,a_2,a_3}

$$\frac{\pi^2 i_1^2}{a_1^2} + \frac{\pi^2 i_2^2}{a_2^2} + \frac{\pi^2 i_3^2}{a_3^2}, \quad i_1, i_2, i_3 \in \mathbb{N}.$$

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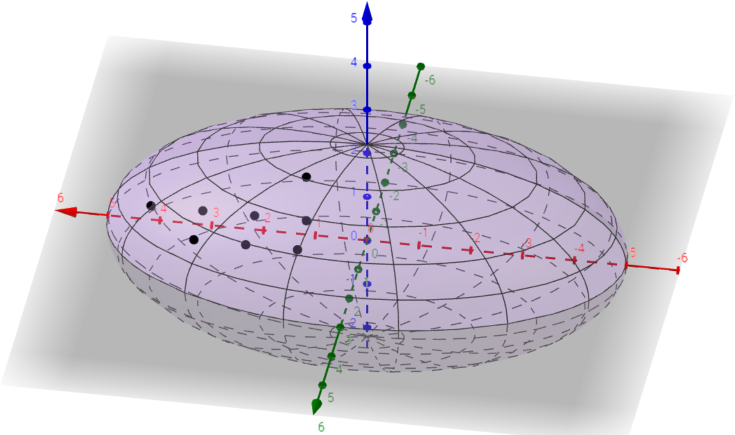
Let $\lambda \in \mathbb{R}$, $\lambda \geq 0$, and $a_1, a_2, a_3 \in \mathbb{R}$ such that $a_1 a_2 a_3 = 1$.
Define the ellipsoid

$$E(\lambda, a_1, a_2) := \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} \leq \frac{\lambda}{\pi^2}\right\}$$

with volume $|E(\lambda, a_1, a_2)| = \frac{4}{3\pi^2}\lambda^{3/2}$.

Cuboids in $\mathbb{R}^3$ of unit measure

Dirichlet eigenvalues $\lambda_1(R_{a_1,a_2,a_3}), \ldots, \lambda_k(R_{a_1,a_2,a_3})$ correspond to the integer lattice points that are inside or on the ellipsoid $E(\lambda_k, a_1, a_2)$ in the first octant.



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The following are equivalent:

- Determining a minimising cuboid for λ_k among all cuboids of unit measure in $\mathbb{R}^3$.
- Determining the 3-dimensional ellipsoid of minimal volume which encloses k integer lattice points in the first octant.

Cuboids in $\mathbb{R}^3$ of unit measure

For each $k \in \mathbb{N}$, there is a minimising cuboid $R_{a^*_{1,k},a^*_{2,k},a^*_{3,k}}$ such that

$$\lambda^*_k = \lambda_k(R_{a^*_{1,k},a^*_{2,k},a^*_{3,k}}) = \inf\{\lambda_k(R_{a_1,a_2,a_3}) : a_1 \leq a_2 \leq a_3, a_1a_2a_3 = 1\}.$$

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Theorem (van den Berg, Gittins, 2016)

Any sequence of minimising cuboids $\left(R_{a^*_{1,k},a^*_{2,k},a^*_{3,k}}\right)_k$ converges to the unit cube in $\mathbb{R}^3$ as $k \to \infty$.

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Any sequence of minimising cuboids $\left(R_{a^*_{1,k},a^*_{2,k},a^*_{3,k}}\right)_k$ converges to the unit cube in $\mathbb{R}^3$ as $k \to \infty$. Furthermore,

$$a^*_{3,k} \leq 1 + O(k^{-(2-\beta)/6}),\ k \to \infty,$$

where β is an exponent of the remainder in

$$\#\{(i_1, i_2, i_3) \in \mathbb{Z}^3 : i_1^2 + i_2^2 + i_3^2 \leq R^2\} - \frac{4\pi}{3}R^3 = O(R^\beta)\,, R \to \infty.$$

Cuboids in $\mathbb{R}^3$ of unit measure

Sketch of proof:

- Relate the number of integer lattice points inside or on $E(\lambda_k^*, a_{1,k}^*, a_{2,k}^*)$ to the number of such points in the first octant.

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Sketch of proof:

- Relate the number of integer lattice points inside or on $E(\lambda_k^*, a_{1,k}^*, a_{2,k}^*)$ to the number of such points in the first octant.
- Use well-known estimates for the number of integer lattice points inside or on an ellipsoid $E(\lambda, a_1, a_2)$.

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The side-lengths $\{a_{1,k}^*, a_{2,k}^*, a_{3,k}^*\}$ must remain bounded as $k \to \infty$.

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This depends upon an upper bound for the counting function

$$
\begin{aligned}
N(\lambda) &= \#\{(i_1, i_2, i_3) \in \mathbb{N}^3 \cap E(\lambda, a_1, a_2)\} \\
&\leq \sum_{i_1 \in \mathbb{N}} \sum_{i_2 \in \mathbb{N}} \left\lfloor \left(\frac{a_3^2}{\pi^2}\lambda - \frac{a_3^2}{a_1^2} i_1^2 - \frac{a_3^2}{a_2^2} i_2^2 \right)_+^{1/2} \right\rfloor.
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Q: Does an analogous result hold in higher dimensions?

Neumann eigenvalues

Let $\Omega \subset \mathbb{R}^m$ be a bounded, open set with Lipschitz boundary.
Neumann eigenvalues of the Laplacian on Ω, $\mu_k(\Omega) \in \mathbb{R}$, satisfy

$$\begin{cases} -\Delta u_k(x) = \mu_k(\Omega) u_k(x) & x \in \Omega, \\ \frac{\partial u_k(x)}{\partial \vec{n}} = 0 & x \in \partial\Omega, \end{cases}$$

where $\vec{n}$ is the outward pointing unit normal vector to $\partial\Omega$, and form a non-decreasing sequence, counted with multiplicities,

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Goal

For each $k \in \mathbb{N}$, determine an open set $\Omega_k^* \subset \mathbb{R}^m$ such that, for prescribed $c \in \mathbb{R}$, $c > 0$,

$$\mu_k(\Omega_k^*) = \sup\{\mu_k(\Omega) : \Omega \subset \mathbb{R}^m \text{ bounded, open, Lipschitz, } |\Omega| = c\}.$$

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For $R \subset \mathbb{R}^2$ a rectangle,

$$\mu_k(R) = \frac{4\pi k}{|R|} - \frac{2\pi^{1/2}\mathrm{Per}(R) k^{1/2}}{|R|^{3/2}} + o(k^{1/2}), k \to \infty.$$

Rectangles in $\mathbb{R}^2$ of unit measure

Theorem (van den Berg, Bucur, Gittins, 2016)

Any sequence of maximising rectangles $(R_{b_k^*})$ converges to the unit square as $k \to \infty$. Moreover,

$$b_k^* = 1 + O(k^{(\theta-1)/4}), k \to \infty,$$

where θ is an exponent of the remainder in Gauss' circle problem

$$\#\{(i_1, i_2) \in \mathbb{Z}^2 : i_1^2 + i_2^2 \leq R^2\} - \pi R^2 = O(R^\theta)\,, R \to \infty.$$

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Q: Does an analogous result hold in higher dimensions?

Rectangles in $\mathbb{R}^2$ of prescribed perimeter

Let

$$R_{a,c} = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < c, 0 < x_2 < a\}.$$

Consider

$$\inf\{\mu_k(R_{a,c}) : \mathrm{Per}(R_{a,c}) = 2(a + c) = 4\}.$$

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Theorem (van den Berg, Bucur, Gittins, 2016)

(i) *If $k = 1$, then this problem does not have a minimiser, and the infimum equals $\frac{\pi^2}{4}$.*

(ii) *If $k \geq 2$, then a minimising rectangle, $R_{a_k^*, c_k^*}$ with $a_k^* + c_k^* = 2$, for μ_k exists. Any sequence of minimising rectangles converges to the unit square as $k \to \infty$.*

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For $k \in \mathbb{N}$, there is a unique maximising rectangle $R_{a_k^*,c_k^*}$ with $a_k^* = \frac{2}{k+1} \in (0,1]$ and $c_k^* = 2 - a_k^*$ such that

$$\mu_k(R_{a_k^*,2-a_k^*}) = \frac{\pi^2 k^2}{(2-a_k^*)^2} = \frac{\pi^2}{(a_k^*)^2} = \frac{\pi^2(k+1)^2}{4},$$

i.e. $\mu_k^* = \mu_k(R_{a_k^*,2-a_k^*})$ is realised by the pairs $(k,0)$ and $(0,1)$.

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i.e. $\mu_k^* = \mu_k(R_{a_k^*,2-a_k^*})$ is realised by the pairs $(k,0)$ and $(0,1)$.

The sequence of maximising rectangles collapses as $k \to \infty$.

Higher dimensions with surface area constraint

For $k \in \mathbb{N}$ and $m \geq 3$, the minimisation problem

$$\inf\{\mu_k(R) : R \text{ is a cuboid in } \mathbb{R}^m, \mathrm{Per}(R) = 4\}$$

does not have a solution.

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The infimum is equal to zero via a sequence of cuboids with at least two vanishing edges.

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Either

- there is a non-degenerate maximising sequence for μ_k, or
- there is a maximising sequence $\left(R_{a_1^{(n)},\dots,a_m^{(n)}}\right)_n$ for μ_k with one vanishing side-length $a_1^{(n)} \to 0$ and, for all $i \in \{1, \dots, m\}$, $a_i^{(n)} \to a_i$ as $n \to \infty$. The perimeter constraint becomes $a_2 a_3 \dots a_m = 2$.

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The eigenvalues of $R_{a_1,\dots,a_m}$ are the eigenvalues of the $(m-1)$-dimensional cuboid with edges of length $a_2, a_3, \dots, a_m$ and measure 2.

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The eigenvalues of $R_{a_1,\dots,a_m}$ are the eigenvalues of the $(m-1)$-dimensional cuboid with edges of length $a_2, a_3, \dots, a_m$ and measure 2.

For $k \in \mathbb{N}$ and $m \geq 3$, the maximisation problem

$$\sup\{\mu_k(R) : R \text{ is a cuboid in } \mathbb{R}^{m-1}, |R| = 2\}$$

has a solution.

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Q: What is the limit of a sequence of maximising cuboids in $\mathbb{R}^m$ as $k \to \infty$?

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Q: What is the limit of a sequence of maximising cuboids in $\mathbb{R}^m$ as $k \to \infty$?

For a maximising cuboid, μ_k^* behaves like $k^{2/m}$ as $k \to \infty$.

On a degenerating sequence which collapses towards a fixed $(m-1)$-dimensional cuboid, μ_k^* behaves like $k^{2/(m-1)}$ as $k \to \infty$.

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Q: What is the limit of a sequence of maximising cuboids in $\mathbb{R}^m$ as $k \to \infty$?

For a maximising cuboid, μ_k^* behaves like $k^{2/m}$ as $k \to \infty$.

On a degenerating sequence which collapses towards a fixed $(m-1)$-dimensional cuboid, μ_k^* behaves like $k^{2/(m-1)}$ as $k \to \infty$.

Any sequence of maximisers must collapse as $k \to \infty$.

Related open problems

Let $T_{a,b}$ denote the flat torus obtained from the parallelogram in $\mathbb{R}^2$ with vertices $(0,0)$, $(1,0)$, (a,b), $(a+1,b)$ by identifying parallel edges.
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$$\sup\{b \cdot \lambda_k(a,b) : (a,b) \in \mathbb{R}^2\}.$$

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Conjecture (Kao, Lai and Osting, 2016)

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Q: Is this true?
Q: Optimal cylinders for the Dirichlet and Neumann eigenvalues with a measure constraint?

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Optimisation of Dirichlet eigenvalues: perimeter constraint

For $\ell \in \mathbb{R}$, $\ell > 0$,

$$\lambda_k(\Omega_k^*) = \inf\{\lambda_k(\Omega) : \Omega \subset \mathbb{R}^m \text{ open}, |\Omega| < \infty, \mathrm{Per}(\Omega) = \ell\}.$$

De Philippis and Velichkov: a minimiser exists, is bounded and connected.

Bucur and Freitas: any sequence of minimisers $\Omega_k^* \subset \mathbb{R}^2$ of λ_k with perimeter ℓ converges to the disc of perimeter ℓ as $k \to \infty$.

van den Berg: for each $k \in \mathbb{N}$, there exists $\Omega_k^* \subset \mathbb{R}^m$, $m \geq 2$, such that

$$\lambda_k(\Omega_k^*) = \inf\{\lambda_k(\Omega) : \Omega \subset \mathbb{R}^m \text{ open, convex}, |\Omega| < \infty, \mathrm{Per}(\Omega) = \ell\}.$$

For any sequence of minimisers, there exists a sequence of isometries of these minimisers which converges to a ball of perimeter ℓ as $k \to \infty$.

Antunes and Freitas: any sequence of m-dimensional minimising cuboids converges to the m-dimensional unit cube as $k \to \infty$.

Optimal cylinders

Cylinder $C(r,\ell) = S_r \times [0,\ell]$, where S_r is a circle of radius r.

For $b > 0$, set $\ell = b$ and $r = b^{-1}$ so $|C(b^{-1}, b)| = 2\pi$.

Dirichlet eigenvalues on $C(b^{-1}, b)$:

$$\frac{\pi^2 i^2}{b^2} + j^2 b^2,\ i \in \mathbb{N},\ j \in \mathbb{Z}.$$

$\inf\{\lambda_k(C(b^{-1}, b)) : b > 0\} = 0$ via a sequence of cylinders with $(b_k^\ell)_\ell$ such that $b_k^\ell \to \infty$ as $\ell \to \infty$.

$\sup\{\lambda_k(C(b^{-1}, b)) : b > 0\} = \infty$ via a sequence of cylinders with $(b_k^\ell)_\ell$ such that $b_k^\ell \to 0$ as $\ell \to \infty$.

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Neumann eigenvalues on $C(b^{-1}, b)$:

$$\frac{\pi^2 i^2}{b^2} + j^2 b^2,\ i \in \mathbb{N} \cup \{0\},\ j \in \mathbb{Z}.$$

$\inf\{\mu_k(C(b^{-1}, b)) : b > 0\} = 0$ via a sequence of cylinders with $(b_k^\ell)_\ell$ such that $b_k^\ell \to \infty$ as $\ell \to \infty$, and via a sequence of cylinders with $(b_k^\ell)_\ell$ such that $b_k^\ell \to 0$ as $\ell \to \infty$.

A maximising cylinder for μ_k exists.

Q: What is it? What is the asymptotic maximal cylinder as $k \to \infty$?