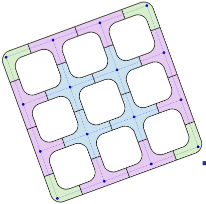

The Steklov spectrum and coarse discretization

with B. Colbois and B. Raveendran

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Plan

1. Introduction/Motivation
2. Coarse discretizations of manifolds with bounded geometry
3. Spectral comparison
4. Applications

1. The Steklov eigenvalue problem

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$$\Delta u = 0 \quad \text{in } \Omega,$$

$$\partial_n u = \sigma u \quad \text{on } M = \partial\Omega.$$

2. Isoperimetry on surfaces

Theorem (Weinstock, 1954)
If $\Omega \subset \mathbb{R}^2$ is a simply-connected domain, then

$$\sigma_1 L \leq 2\pi.$$

Theorem (Fraser-Schoen, 2013)
If Ω is a compact surface of genus γ with b boundary components, then

$$\sigma_1 L \leq 2\pi(\gamma + b).$$

Theorem (Kokarev, 2014)
If Ω is a compact surface of genus γ, then

$$\sigma_1 L \leq 8\pi(\gamma + 1).$$

Theorem (Colbois– Girouard, 2014)

There exists a sequence Ω_N of surfaces with the genus $1+N$ and such that

$$\sigma_1(\Omega_N)L(\partial\Omega_N) \geq CN,$$

for some universal constant $C > 0$.

Strategy:

1. Consider a regular graph Γ of degree 4
2. Use a «fundamental piece» to build a surface Ω_Γ
3. Prove a spectral comparison estimate with $\lambda_1(\Gamma)$
4. Consider expanding sequence of graphs

QED

Question: How many boundary components are required?

2. Discretization of manifolds with boundary

Let $\kappa \geq 0$ and $r_0 \in (0, 1)$.

A n-dimensional compact manifold M is in $\mathcal{M} = \mathcal{M}(\kappa, r_0, n)$ if

H1) The boundary Σ admits a neighbourhood which is isometric to the cylinder $[0, 1] \times \Sigma$, with the boundary corresponding to $\{0\} \times \Sigma$;

H2) The Ricci curvature of M is bounded below by $-(n-1)\kappa$;

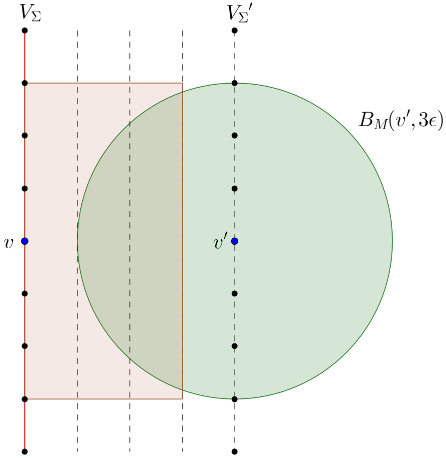
H3) The Ricci curvature of Σ is bounded below by $-(n-2)\kappa$;

H4) For each point $p \in M$ such that $d(p, \Sigma) > 1$, $\mathrm{inj}_M(p) > r_0$;

H5) For each point $p \in \Sigma$, $\mathrm{inj}_\Sigma(p) > r_0$.

Discretization of $M \in \mathcal{M}(n, \kappa, r_0)$

Fix $\epsilon \in (0, r_0/4)$.



$V_\Sigma \subset \Sigma$: maximal ϵ-separated.

$V'_\Sigma := \{4\epsilon\} \times V_\Sigma \subset M$.

$V'_\Sigma \subset V_I$: maximal ϵ-separated in $M \setminus [0, 4\epsilon[\times \Sigma$.

The set $V = V_\Sigma \cup V_I$ is given the structure of a graph Γ:

- any two $v, w \in V$ adjacent if $d_M(v, w) < 3\epsilon$;
- any $v \in V_\Sigma$ adjacent to $v' = (4\epsilon, v) \in V'_\Sigma \subset V_I$.

This graph $\Gamma = (V, E)$ is a $\underline{\epsilon\text{-discretization of } M}$.

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This graph $\Gamma = (V, E)$ is a <u>ϵ-discretization of M</u>.

Lemma

For any $0 < \epsilon < r_0/4$, and any ϵ-discretization (Γ, V_Σ) of M, the natural inclusion $V \subset M$ is a rough isometry:

$$\frac{\epsilon}{4} d_\Gamma(x, y) - 10 \leq d_M(x, y) \leq 4\epsilon d_\Gamma(x, y) + 10.$$

Rough isometries

A rough isometry between two metric spaces X and Y is a map $\Phi : X \to Y$ such that, there exist constants $a \geq 1, b \geq 0, \tau \geq 0$ satisfying

$$a^{-1}d(x_1, x_2) - b \leq d(\Phi(x_1), \Phi(x_2)) \leq a\, d(x_1, x_2) + b \qquad (1)$$

for every $x_1, x_2 \in X$ and which satisfies

$$\bigcup_{x \in X} B(\Phi(x), \tau) = Y.$$

3. Comparison estimates

Let $\Gamma = (V, E)$ be a graph with boundary $B \subset V$.

The Dirichlet energy of a function $f : V \to \mathbb{R}$ is

$$q(f) := \sum_{v \sim w} \big(f(v) - f(w)\big)^2.$$

For each $j < |B|$, the j-th Steklov eigenvalue of the (Γ, B) is

$$\sigma_j(\Gamma, B) = \min_E \max_{f \in E} \frac{q(f)}{\|f\|_B^2}$$

the minimum is over $j + 1$-dimensional subspaces E of $\ell^2(V)$.

Main theorem (Colbois–Girouard–Raveendran)

Let $\epsilon \in (0, r_0/4)$.

There exist numbers $a, b > 0$ depending on κ, r_0, n and ϵ such that any ϵ-discretization (Γ_M, V_Σ) of $M \in \mathcal{M}(\kappa, r_0, n)$ satisfies

$$a < \frac{\sigma_1(M)}{\sigma_1(\Gamma, V_\Sigma)} < b.$$

4. Two applications

Application 1

There exists a sequence of domains $\Omega_N \subset \mathbb{R}^2$, such that

1. The isoperimetric ratio $I(\Omega_N) \to \infty$ as $N \to \infty$;
2. There exists a constant $c > 0$, such that $\sigma_1(\Omega_N)|\Sigma_N| \geq c$.

Application 2

There exist a sequence $\{\Omega_N\}_{N\in\mathbb{N}}$ of compact surfaces with connected boundary and a constant $C > 0$ such that, $\text{genus}(\Omega_N) = 1 + N$, and

$$\sigma_1(\Omega_N)L(\partial\Omega_N) \geq CN.$$

Application 1

Theorem (Colbois–Girouard–El Soufi, 2011)

There exists $C = C(n)$ such that

$$\sigma_k(\Omega)|\partial\Omega|^{1/n} \leq \frac{C}{I(\Omega)^{(n-1)/n}} k^{2/n}.$$

Where the isoperimetric ratio $I(\Omega) := \frac{|\partial\Omega|}{|\Omega|^{n/(n+1)}} \geq I(\mathbb{B})$.

classical isoperimetric inequality

Corollary

If $n \geq 3$ then any sequence of domains $\Omega_N \subset \mathbb{R}^n$ such that

$$\lim_{N\to\infty} I(\Omega_N) = +\infty$$

satisfies

$$\lim_{N\to\infty} \sigma_k(\Omega_N)|\partial\Omega_N| = 0.$$

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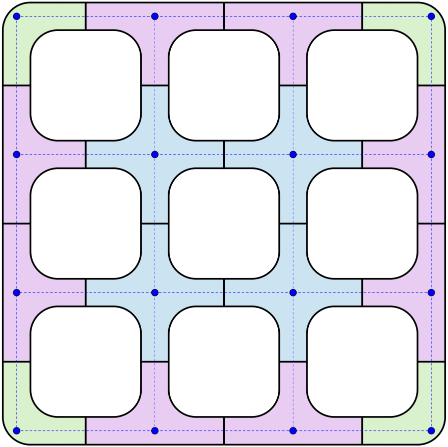

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