Shape optimization with Robin conditions and free discontinuity problems

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November 2016

The first eigenvalue of the Robin-Laplacian

Let $\Omega \subseteq \mathbb{R}^N$ be open, bounded and with a Lipschitz boundary, and let $\beta > 0$. We set

$$\lambda_{1,\beta}(\Omega) := \min_{u \in H^1(\Omega), u \neq 0} \frac{\int_\Omega |\nabla u|^2 \, dx + \beta \int_{\partial\Omega} u^2 \, d\mathcal{H}^{N-1}}{\int_\Omega u^2 \, dx}.$$

The value $\lambda_{1,\beta}$ is characterized by the elliptic boundary value problem

$$\begin{cases} -\Delta u = \lambda_{1,\beta} u & \text{in } \Omega \\ \frac{\partial u}{\partial n} + \beta u = 0 & \text{on } \partial\Omega \\ u > 0 & \text{in } \Omega. \end{cases}$$

The Faber-Krahn inequality

Faber-Krahn inequality (Bossel (1986) and Daners (2007))

We have

$$\lambda_{1,\beta}(B) \leq \lambda_{1,\beta}(\Omega),$$

where B is a ball such that $|B| = |\Omega|$. Equality holds if and only if Ω is a ball.

- Admissible functions to compute $\lambda_{1,\beta}(\Omega)$ do not vanish on $\partial\Omega$: $\rightsquigarrow$ no trivial rearrangements.
- Bossel and Daners' approach is based on a direct comparison between Ω and B through a dearrangement.

Trying to apply the direct method of the Calculus of Variations

- compactness properties on the domains (in some topology) are not obvious;
- concerning the eigenfunctions, after an extension to 0 outside the domain we have

$$
\begin{aligned}
&\|u^2\|_{BV} \\
&= \int_{\mathbb{R}^N} |2u\nabla u| \, dx + \int_{J_u} |\gamma_1(u)^2 - \gamma_2(u)^2| \, d\mathcal{H}^{N-1} + \int_{\mathbb{R}^N} u^2 \, dx \\
&\quad \leq \int_\Omega u^2 \, dx + \int_\Omega |\nabla u|^2 \, dx + \int_{\partial\Omega} u^2 \, d\mathcal{H}^{N-1} + \int_\Omega u^2 \, dx \\
&\qquad \leq C(1 + \lambda_{1,\beta}(\Omega)) \int_\Omega u^2 \, dx.
\end{aligned}
$$

Some compactness for the eigenfunctions is available...

A free discontinuity framework

We concentrate the free discontinuity functional

$$R_\beta(u) := \frac{\int_{\mathbb{R}^N} |\nabla u|^2\, dx + \beta \int_{J_u} [\gamma_1(u)^2 + \gamma_2(u)^2]\, d\mathcal{H}^{N-1}}{\int_{\mathbb{R}^N} u^2\, dx}$$

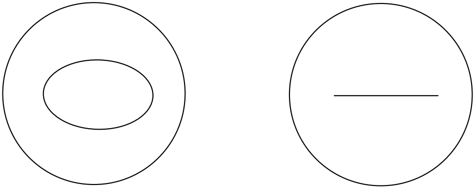
on a suitable class of functions of bounded variation. If u is the first eigenfunction of Ω, we have

$$\lambda_{1,\beta}(\Omega) = R_\beta(u)$$

The surface term

$$\beta \int_{J_u} [\gamma_1(u)^2 + \gamma_2(u)^2] \, d\mathcal{H}^{N-1} \tag{1}$$

is somehow *unusual*: several forms would be admissible in connection with our problem, since $\gamma_2(u) = 0$ for the *eigenfunctions*. The sum is natural if we want to deal with a minimization, as suggested by the picture



$\Omega_n \to \Omega$

The Faber-Krahn inequality

The problem $\inf_{|\Omega|=m} \lambda_{1,\beta}(\Omega)$ leads to

$$\inf_{u \in SBV^{\frac{1}{2}}(\mathbb{R}^N), |supp(u)|=m} R_\beta(u).$$

where

$$SBV^{\frac{1}{2}}(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N) \,:\, u \geq 0, u^2 \in SBV(\mathbb{R}^N)\}.$$

Theorem (Bucur-G. (ARMA 2010 and ARMA 2015))

The free discontinuity problem admits a solution. Every solution is of the form $u = \psi 1_B$, where B is a ball with $|B| = |\Omega|$ and ψ is the associated first classical eigenfunction.

If we consider

$$\tilde{R}_\beta(u) := \frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \beta \int_{J_u} [\gamma_1(u)^2 + \gamma_2(u)^2]\,d\mathcal{H}^{N-1}}{\left(\int_{\mathbb{R}^N} u\,dx\right)^2},$$

we can show that $T_\beta(\Omega) \geq T_\beta(B)$ where

$$T_\beta(\Omega) := \min_{u \in H^1(\Omega), u \neq 0} \frac{\int_\Omega |\nabla u|^2\,dx + \beta \int_{\partial\Omega} u^2\,d\mathcal{H}^{N-1}}{\left(\int_\Omega |u|\,dx\right)^2},$$

so that we have a Saint-Venant inequality for the torsional rigidity under Robin conditions.

If we consider

$$\hat{R}_\beta(u) := \frac{\left(\int_{\mathbb{R}^N} |\nabla u|^2\, dx\right)^{1/2} + \left(\int_{J_u} [\gamma_1(u)^2 + \gamma_2(u)^2]\, d\mathcal{H}^{N-1}\right)^{1/2}}{\left(\int_{\mathbb{R}^N} u^2\, dx\right)^{1/2}},$$

we can prove $C_2(\Omega) \geq C_2(B)$, where

$$C_2(\Omega) := \min_{u \in W^{1,2}(\Omega), u \neq 0} \frac{\|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\partial\Omega)}}{\|u\|_{L^2(\Omega)}}.$$

We thus recover the optimal constant in the Poincaré inequality with traces

$$\|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\partial\Omega)} \geq C_2(B)\|u\|_{L^2(\Omega)}$$

- Existence of a minimizer follow by the direct method modulo a concentration compactness alternative.
- By means of a regularity analysis, the support of u is shown to be an open and connected set Ω with $\partial\Omega$ which is rectifiable with $\mathcal{H}^{N-1}(\partial\Omega) < +\infty$. Moreover u is smooth on Ω with

$$-\Delta u = \lambda_u u,$$

where $\lambda_u = R_\beta(u)$.
- By suitable reflection arguments, one shows that

$$\Omega = B \qquad \text{and} \qquad u = \psi 1_B$$

obtaining the Faber-Krahn inequality.

A free discontinuity functional for λ_k

Recall that

$$\lambda_{k,\beta}(\Omega) = \min_{S \in \mathcal{S}_k} \max_{u \in S \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 \, dx + \beta \int_{\partial\Omega} u^2 \, d\mathcal{H}^{N-1}}{\int_\Omega u^2 \, dx},$$

where $\mathcal{S}_k$ denotes the family of k-dimensional subspaces of $H^1(\Omega)$.

We thus consider for $u = (u_1, \dots, u_k)$ the functional

$$R_{k,\beta}(u) := \max_{v \in V(u)} \frac{\int_{\mathbb{R}^N} |\nabla v|^2 \, dx + \beta \int_{J_u} [\gamma_1(v)^2 + \gamma_2(v)^2] \, d\mathcal{H}^{N-1}}{\int_{\mathbb{R}^N} v^2 \, dx}$$

where

$$u \in SBV^{\frac{1}{2}}_{\pm}(\mathbb{R}^N;\mathbb{R}^k) := \Big\{ u = (u_1, \dots, u_k) \, : \, u_i^{\pm} \in SBV^{\frac{1}{2}}(\mathbb{R}^N), \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{J_u} [|\gamma_1(u)|^2 + |\gamma_2(u)|^2] \, d\mathcal{H}^{N-1} < +\infty \Big\},$$

$V(u)$ is the vector space generated by the components of u, and $dim V(u) = k$. We denote the space with $\mathcal{F}_k(\mathbb{R}^N)$.

Theorem (Bucur-G. (2016))

For every $k \geq 1$ the free discontinuity problem admits a solution with bounded support. Moreover

$$\min_{u \in \mathcal{F}_k(\mathbb{R}^N), |supp(u)| = m} R_{k,\beta}(u) = \inf_{|\Omega| = m} \lambda_{k,\beta}(\Omega). \tag{2}$$