A stability result for the first eigenvalue of the p-Laplacian

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The Faber-Krahn inequality

$\Omega \subset \mathbb{R}^n$ a bounded domain, $p > 1$.

The first Dirichlet eigenvalue of p-Laplacian

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Since $\lambda_p(r\Omega) = r^{-p}\lambda_p(\Omega)$ for $r > 0$, Faber-Krahn becomes

$$|\Omega|^{\frac{p}{n}} \lambda_p(\Omega) \geq |B|^{\frac{p}{n}} \lambda_p(B) \qquad B \text{ is the unit ball}$$

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- Hansen, Nadirashvili (1991) considered the case when Ω is convex and proved that

$$\lambda_2(\Omega) - \lambda_2(B) \geq \gamma(n)(1 - r_i(\Omega))^\alpha$$

where $r_i(\Omega)$ is the inner radius and

$$\alpha(n) = \begin{cases} 3 & \text{if } n = 2 \\ \text{any number} > 3 & \text{if } n = 3 \\ \frac{n+3}{2} & \text{if } n \geq 4 \end{cases}$$

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Melas (1992), proved a similar estimate in the convex case, using the inner and the outer radius of Ω

For a general open set a natural way to measure the distance from a ball is to consider the Fraenkel asymmetry

$$\mathcal{A}(\Omega) := \min_{x \in \mathbb{R}^n} \left\{ \frac{|\Omega \Delta B_r(x)|}{|B_r|} : \ |\Omega| = |B_r| \right\}$$

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Conjecture by Hansen, Nadirashvili (1991) and by Bhattacharya, Weitsman (1996) for the first eigenvalue of the Laplacian

$$|\Omega|^{\frac{2}{n}} \lambda_2(\Omega) - |B|^{\frac{2}{n}} \lambda_2(B) \geq \gamma(n) \mathcal{A}(\Omega)^2$$

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We cannot expect a smaller exponent. Take the ellipsoids

$$\Omega_\varepsilon = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \ |x'|^2 + (1+\varepsilon) x_n^2 \leq 1\} \qquad \text{for } 0 < \varepsilon << 1$$

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Then one can show that

$$\mathcal{A}(\Omega_\varepsilon) \approx \varepsilon \qquad\qquad |\Omega_\varepsilon|^{\frac{2}{n}} \lambda_2(\Omega_\varepsilon) - |B|^{\frac{2}{n}} \lambda_2(B) \approx \varepsilon^2$$

Conjecture:

$$(*) \qquad |\Omega|^{\frac{2}{n}}\lambda_2(\Omega) - |B|^{\frac{2}{n}}\lambda_2(B) \geq \gamma(n)\mathcal{A}(\Omega)^2$$

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$$|\Omega|^{\frac{p}{n}}\lambda_p(\Omega) - |B|^{\frac{p}{n}}\lambda_p(B) \geq \gamma(n,p)\mathcal{A}(\Omega)^{p+2}$$

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Brasco, De Philippis and Velichkov solved the conjecture (*) in 2013!! (in any dimension)

Proof based on the celebrated regularity results for free boundary problems involving the Laplacian:

- Alt, Caffarelli (1981)
- Alt, Caffarelli, Friedman (1984)

Our (N. F. & Y. Zhang) result (2016)

Let $n \geq 2$ and $p > 1$. Then there exists a constant $\gamma(n, p)$ s.t.

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The issue here is the "2"!!

With "2" replaced by "3" see Brasco, De Philippis (2016)

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From now on we shall assume $|\Omega| = |B|$
and we write $\lambda(\Omega)$ instead of $\lambda_p(\Omega)$ (p will be fixed)

Step 1 ($p > 1$ fixed and $|\Omega| = |B|$)

Following BDV, to prove $\lambda(\Omega) - \lambda(B) \geq \gamma\, \mathcal{A}(\Omega)^2$

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$$E(\Omega) := \min\left\{\frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \int_\Omega u\,dx\ :\ u \in W_0^{1,p}(\Omega)\right\}$$

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If f_Ω is the first eigenfunction of the p-Laplacian then

$$-\mathrm{div}\left(|\nabla f_\Omega|^{p-2} \nabla f_\Omega\right) = \lambda(\Omega) |f_\Omega|^{p-2} f_\Omega \qquad f_\Omega = 0 \text{ on } \partial\Omega$$

If u_Ω is minimizer of $E(\Omega)$ then

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$$E(\Omega) - E(B) = \frac{1-p}{p}\left(\int_\Omega |\nabla u_\Omega|^p - \int_B |\nabla u_B|^p\right)$$

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and then we use the following extension of the Kohler-Jobin inequality (Brasco, 2014):

$$\frac{\lambda(\Omega)}{\lambda(B)} \geq \left(\frac{E(B)}{E(\Omega)}\right)^{\alpha(p,n)}$$

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If $|\Omega| = |B|$ and $\lambda(\Omega) \leq 2\lambda(B)$ then

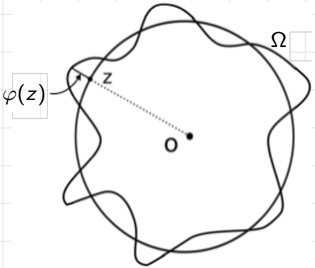
$$\lambda(\Omega) - \lambda(B) \geq c\big[E(\Omega) - E(B)\big]$$

Step 2 $(|\Omega| = |B|)$

To prove

$$E(\Omega) - E(B) \geq \gamma\, \mathcal{A}(\Omega)^2$$

we first assume that Ω is a bounded open set very close to B.



Given $\varphi \in C^{2,\alpha}(\partial B)$, $|\varphi| < 1$

We say that Ω is a nearly spherical set parameterized by φ if

$$\partial\Omega = \{x = z(1 + \varphi(z)) : \ z \in \partial B\}$$

Theorem

There exist δ, γ_0 such that if Ω is a nearly spherical set of class $C^{2,\alpha}$ parametrized by φ, with $\|\varphi\|_{C^{2,\alpha}(\partial B)} \leq \delta$, the barycenter of Ω is at the origin and $|\Omega| = |B|$, then

$$E(\Omega) - E(B) \geq \gamma_0 \|\varphi\|^2_{H^{\frac{1}{2}}(\partial B)}$$

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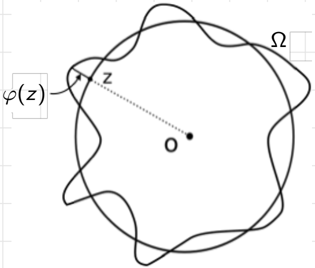

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$$E(\Omega) - E(B) \geq \gamma_0 \|\varphi\|^2_{H^{\frac{1}{2}}(\partial B)} \geq \gamma_0 \|\varphi\|^2_{L^2(\partial B)} \geq c\mathcal{A}(\Omega)^2$$

Step 3 (Reduction to bounded sets $|\Omega| = |B|$)

Since $\mathcal{A}(\Omega) < 2$ to prove that

$$D(\Omega) := E(\Omega) - E(B) \geq \gamma\, \mathcal{A}(\Omega)^2$$

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Lemma

There exist $C, \delta_0, R > 0$, such that $|\Omega| = |B|$ and $D(\Omega) \leq \delta_0$, one can find another open set $\widetilde{\Omega}$ with $|\widetilde{\Omega}| = |B|$ and $\widetilde{\Omega} \subset B_R$ with the property that

$$\mathcal{A}(\Omega) \leq \mathcal{A}(\widetilde{\Omega}) + CD(\Omega), \qquad D(\widetilde{\Omega}) \leq CD(\Omega)$$

Step 4 (Reduction to nearly spherical sets via regularity)

We have now to show that if $|\Omega| = |B|, \Omega \subset B_R$ then

$$D(\Omega) := E(\Omega) - E(B) \geq \gamma \, \mathcal{A}(\Omega)^2$$

To prove this inequality we replace the Fraenkel asymmetry

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Lemma

$$\Omega \subset B_R \quad \Longrightarrow \quad \alpha(\Omega) \geq c_1\, \mathcal{A}(\Omega)^2$$

If Ω *is a nearly spherical set parametrized by* φ, $\|\varphi\|_{L^\infty(\partial B)} \leq \delta$,

$$\alpha(\Omega) \leq c_2 \|\varphi\|^2_{L^2(\partial B)}$$

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Thus we need to show that if $|\Omega| = |B|\,, \Omega \subset B_R, \,, D(\Omega) \leq \delta$, then

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first proposed by Cicalese, Leonardi (2012)

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Danielli-Petrosjan (2005) studied the free boundary problem for the p-Laplacian equation with right hand side $= 0$

Back to Step 2

Theorem

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$$(*) \qquad E(\Omega) - E(B) \geq \gamma_0 \|\varphi\|^2_{H^{\frac{1}{2}}(\partial B)}$$

Recall that

$$E(\Omega) = \min\left\{\frac{1}{p}\int_\Omega |\nabla u|^p \, dx - \int_\Omega u \, dx : \; u \in W_0^{1,p}(\Omega)\right\}$$

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To prove (*) we try a second variation argument.
We construct an autonomous vector field $X \in C^{2,\alpha}(\mathbb{R}^n)$, s.t. $\operatorname{div} X = 0$ in a nhood of ∂B and consider the flow

$$\frac{\partial \Phi_t}{\partial t}(x) = X(\Phi_t(x)), \quad \Phi_0(x) = x, \quad \text{for } (x,t) \in \mathbb{R}^n \times [0,1]$$

with $|\Phi_t(B)| = |B|$, $\Phi_1(B) = \Omega$

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Setting $\Omega_t := \Phi_t(B)$ and $e(t) := E(\Omega_t)$ we would like to write

$$E(\Omega) - E(B) = e(1) - e(0) = e'(0) + \frac{1}{2}e''(0) + \int_0^1 (1-t)(e''(t) - e''(0))dt$$

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$|\Phi_t(B)| = |B| \quad \Phi_1(B) = \Omega$

Setting $\Omega_t := \Phi_t(B)$ and $e(t) := E(\Omega_t)$, we would like to write

$$E(\Omega)-E(B) = e(1)-e(0) = e'(0)+\frac{1}{2}e''(0)+\int_0^1 (1-t)(e''(t)-e''(0))dt$$

Then we would like to prove that

$$(*) \qquad E(\Omega) - E(B) \geq \gamma_0 \|\varphi\|^2_{H^{\frac{1}{2}}(\partial B)}$$

where $$E(\Omega) = \min\left\{\frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \int_\Omega u\,dx \,:\, u \in W_0^{1,p}(\Omega)\right\}$$

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$e'(t)$ is OK, but due to the degeneracy of the p-Laplacian,

$e''(t)$ does not exist......

Thus, for $\kappa \geq 0$ and $t \in (0,1)$ we set

$$e_\kappa(t) = E_\kappa(\Omega_t) = \min_{u \in W_0^{1,p}(\Omega)} \left\{ \frac{1}{p} \int_\Omega (\kappa^2 + |\nabla u|^2)^{\frac{p}{2}} \, dx - \int_\Omega u \, dx \right\}$$

Note that $e_0(t) = e(t)$.

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For $\kappa > 0$ we calculate

$$e'_\kappa(t) = \frac{1}{p} \int_{\Omega_t} \mathrm{div}\big((\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{p}{2}} X\big) - \int_{\Omega_t} \mathrm{div}\big((\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}} |\nabla u_{\kappa,t}|^2 X\big)$$

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$$\begin{cases} -\operatorname{div}\left((\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}} \nabla u_{\kappa,t}\right) = 1 & \text{in } \Omega_t \\ u_{\kappa,t} = 0 & \text{on } \partial\Omega_t. \end{cases}$$

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$C^{1,\alpha}$ estimates for the p-laplacian in $\overline{\Omega}_t$

$\Longrightarrow u_{\kappa,t}$ converge in $C^{1,\alpha}$ to u_t as $\kappa \to 0$, uniformly w.r.t. t

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$C^{1,\alpha}$ estimates for the p-laplacian in $\overline{\Omega}_t$

$\Longrightarrow$ $u_{\kappa,t}$ converge in $C^{1,\alpha}$ to u_t as $\kappa \to 0$, uniformly w.r.t. t

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$$\Longrightarrow \lim_{\kappa \to 0} e'_\kappa(t) = e'_0(t) \quad \text{uniformly w.r.t. } t$$

For $\kappa > 0,\ t \in [0,1]$ we calculate

$$e''_\kappa(t) = \int_{\partial\Omega_t} (\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}} (\nabla W_{\kappa,t} \cdot \nu_{\Omega_t}) W_{\kappa,t} d\mathcal{H}^{n-1}$$

$$+ (p-2) \int_{\partial\Omega_t} (\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-4}{2}} (\nabla W_{\kappa,t} \cdot \nabla u_{\kappa,t})(\nabla u_{\kappa,t} \cdot \nu_{\Omega_t}) W_{\kappa,t} d\mathcal{H}^{n-1}$$

$$- \int_{\partial\Omega_t} (\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}} (\nabla^2 u_{\kappa,t}[\nabla u_{\kappa,t}] \cdot X_\tau)(X \cdot \nu_{\Omega_t}) d\mathcal{H}^{n-1}$$

$$- (p-2) \int_{\partial\Omega_t} (\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-4}{2}} |\nabla u_{\kappa,t}|^2 (\nabla^2 u_{\kappa,t}[\nabla u_{\kappa,t}] \cdot X_\tau)(X \cdot \nu_{\Omega_t}) d\mathcal{H}^{n-1}$$

$$+ \int_{\partial\Omega_t} |\nabla u_{\kappa,t}| \Big((\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}} |\nabla u_{\kappa,t}| H_{\Omega_t} - 1 \Big) (X \cdot \nu_{\Omega_t})^2 d\mathcal{H}^{n-1},$$

where $W_{\kappa,t}$ is the unique solution in $H^1(\Omega_t)$ of the equation

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e''_\kappa(t) &= \int_{\partial\Omega_t} (\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}} (\nabla W_{\kappa,t}\cdot \nu_{\Omega_t}) W_{\kappa,t}\, d\mathcal{H}^{n-1} \\
&+ (p-2)\int_{\partial\Omega_t} (\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-4}{2}} (\nabla W_{\kappa,t}\cdot \nabla u_{\kappa,t})(\nabla u_{\kappa,t}\cdot \nu_{\Omega_t}) W_{\kappa,t}\, d\mathcal{H}^{n-1} \\
&- \int_{\partial\Omega_t} (\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}} (\nabla^2 u_{\kappa,t}[\nabla u_{\kappa,t}]\cdot X_\tau)(X\cdot \nu_{\Omega_t})\, d\mathcal{H}^{n-1} \\
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\begin{cases}
\operatorname{div}\left[(\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}\left(\nabla W_{\kappa,t} + (p-2)\dfrac{\nabla u_{\kappa,t}\cdot \nabla W_{\kappa,t}}{\kappa^2+|\nabla u_{\kappa,t}|^2}\nabla u_{\kappa,t}\right)\right] = 0 & \text{in } \Omega_t \\
\dot u_{\kappa,t} = -\nabla u_{\kappa,t}\cdot X & \text{on } \partial\Omega_t
\end{cases}
$$

Passing to the limit in

$$E_\kappa(\Omega) - E_\kappa(B) = e'_\kappa(0) + \frac{1}{2}e''_\kappa(0) + \int_0^1 (1-t)(e''_\kappa(t) - e''_\kappa(0))dt$$

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$e''_\kappa(t)$ is quadratic in $\nabla W_{\kappa,t}$ and linear in $\nabla^2 u_{\kappa,t}[\nabla u_{\kappa,t}]$

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and $$\begin{cases} -\operatorname{div}\left((\kappa^2+|\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}}\nabla u_{\kappa,t}\right) = 1 & \text{in } \Omega_t \\ u_{\kappa,t} = 0 & \text{on } \partial\Omega_t. \end{cases}$$

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We know

$$u_{\kappa,0} \to c\left(1 - |x|^{\frac{p}{p-1}}\right) \quad \text{in } C^{1,\alpha}, \qquad \left(\kappa^2 + |\nabla u_{\kappa,0}|^2\right)^{\frac{p-2}{2}} \to c'|x|^{\frac{p-2}{p-1}}$$

$$E_\kappa(\Omega) - E_\kappa(B) = e_\kappa'(0) + \frac{1}{2}e_\kappa''(0) + \int_0^1 (1-t)(e_\kappa''(t) - e_\kappa''(0))dt$$

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Lemma 1 (hard!)

$$n^{\frac{p-2}{p-1}} \lim_{\kappa\to 0} e''_\kappa(0) = \int_B |x|^{\frac{p-2}{p-1}} \left(|\nabla W|^2 + (p-2) \left| \frac{x}{|x|} \cdot \nabla W \right|^2 \right) dx - \int_{\partial B} W^2 d\mathcal{H}^{n-1}$$

where W *is the unique weak solution in* $H^1(B;\mu)$ *of the equation*

$$\begin{cases} \operatorname{div}\left(|x|^{\frac{p-2}{p-1}} \nabla W + (p-2)|x|^{\frac{p-2}{p-1}} \left(\frac{x}{|x|} \cdot \nabla W \right) \frac{x}{|x|} \right) = 0 & \text{in } B \\ W = n^{-\frac{1}{p-1}} X \cdot \nu_B & \text{on } \partial B \end{cases}$$

and the weight $\mu = |x|^{\frac{p-2}{p-1}}$

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Lemma 2 (harder!) *For $\kappa, t \in (0,1]$ we have*

$$|e''_\kappa(t) - e''_\kappa(0)| \leq \omega(\|\varphi\|_{C^{2,\alpha}} + \kappa)\|X\cdot\nu_B\|^2_{H^{\frac{1}{2}}(\partial B)}$$

$$E_\kappa(\Omega) - E_\kappa(B) = e_\kappa'(0) + \frac{1}{2}e_\kappa''(0) + \int_0^1 (1-t)(e_\kappa''(t) - e_\kappa''(0))dt$$

Using the two previous lemmas and $e_0'(0) = 0$, let us take $\kappa \to 0^+$ and then we get

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$$\begin{aligned} E(\Omega) - E(B) &\geq \gamma_{n,p} \int_B |x|^{\frac{p-2}{p-1}} \left(|\nabla W|^2 + (p-2) \left| \frac{x}{|x|} \cdot \nabla W \right|^2 \right) dx \\ &\quad - \gamma_{n,p} \int_{\partial B} W^2 d\mathcal{H}^{n-1} - \omega(\|\varphi\|_{C^{2,\alpha}(\partial B)}) \|X \cdot \nu_B\|^2_{H^{1/2}(\partial B)} \\ &\geq \qquad \text{(writing } W \text{ in terms of spherical harmonics)} \\ &\geq \gamma' \|X \cdot \nu_B\|^2_{H^{1/2}(\partial B)} - \omega(\|\varphi\|_{C^{2,\alpha}(\partial B)}) \|X \cdot \nu_B\|^2_{H^{1/2}(\partial B)} \end{aligned}$$

$$E_\kappa(\Omega) - E_\kappa(B) = e'_\kappa(0) + \frac{1}{2} e''_\kappa(0) + \int_0^1 (1-t)(e''_\kappa(t) - e''_\kappa(0))dt$$

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&\geq \qquad \text{(writing } W \text{ in terms of spherical harmonics)} \\
&\geq \gamma' \|X \cdot \nu_B\|^2_{H^{1/2}(\partial B)} - \omega(\|\varphi\|_{C^{2,\alpha}(\partial B)}) \|X \cdot \nu_B\|^2_{H^{1/2}(\partial B)}
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Thus if $\|\varphi\|_{C^{2,\alpha}(\partial B)}$ is small we conclude

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Thus if $\|\varphi\|_{C^{2,\alpha}(\partial B)}$ is small we conclude

$$E(\Omega) - E(B) \geq c \|X \cdot \nu_B\|^2_{H^{1/2}(\partial B)} \geq c \|\varphi\|^2_{H^{1/2}(\partial B)}$$

Back to Step 4 (Reduction to nearly spherical sets via regularity)

Thus we need to show that if $|\Omega| = |B|, \Omega \subset B_R, , D(\Omega) \leq \delta$, then

$$D(\Omega) := E(\Omega) - E(B) \geq \gamma_0 \, \alpha(\Omega)$$

where

$$\alpha(\Omega) = \int_{\Omega \Delta B_1(x_\Omega)} \big| 1 - |x - x_\Omega| \big| \, dx$$

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Assume that there exists a sequence $\Omega_h \subset B_R, |\Omega_h| = |B|$ with

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$$(*) \quad \mathrm{Inf}\left\{ E(U) + \sqrt{\delta_h^2 + \sigma^2(\alpha_h(U) - \delta_h)^2} + P(|U| - |B|) : \; U \subset B_R \right\}$$

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We show that $(*)$ has a minimizer U_h, that $D(U_h) \to 0$ and

$$(**) \qquad E(U_h) - E(B) \le c(n,p)\sigma\, \alpha(U_h)$$

Back to Step 4 (Reduction to nearly spherical sets via regularity)

Thus we need to show that if $|\Omega| = |B|, \Omega \subset B_R, , D(\Omega) \leq \delta$, then

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....There exists a nonnegative Borel function q_{u_h} such that

$$-\mathrm{div}(|\nabla u_h(x)|^{p-2}\nabla u_h(x)) = \chi_{U_h}(x) - q_{u_h}(x)^{p-1}\mathcal{H}^{n-1}\llcorner\partial U_h \qquad \text{in } B_R$$

where $U_h = \{u_h > 0\}$ and for all $x \in \partial U_h$

$$\frac{1}{C} \leq q_{u_j}(x) \leq C$$