A stability result for the first eigenvalue of the *p*-Laplacian

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Shape Optimization and Isoperimetric and Functional Inequalities

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The Faber-Krahn inequality

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 a bounded domain, $p>1$.

The first Dirichlet eigenvalue of *p*-Laplacian

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 with '=' iff Ω is a ball

Since $\lambda_p(r\Omega) = r^{-p}\lambda_p(\Omega)$ for r > 0, Faber-Krahn becomes

$$|\Omega|^{\frac{\rho}{n}}\lambda_{p}(\Omega)\geq |B|^{\frac{\rho}{n}}\lambda_{p}(B)$$
 B is the unit ball

The stability problem

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- Hansen, Nadirashvili (1991) considered the case when Ω is convex and proved that

$$\lambda_2(\Omega) - \lambda_2(B) \ge \gamma(n)(1 - r_i(\Omega))^{\alpha}$$

where $r_i(\Omega)$ is the inner radius and

$$\alpha(n) = \begin{cases} 3 & \text{if } n = 2\\ \text{any number} > 3 & \text{if } n = 3\\ \frac{n+3}{2} & \text{if } n \ge 4 \end{cases}$$

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Melas (1992), proved a similar estimate in the convex case, using the inner and the outer radius of $\boldsymbol{\Omega}$

For a general open set a natural way to measure the distance from a ball is to consider the Fraenkel asymmetry $4(\Omega) := \min \left\{ \frac{|\Omega \Delta B_r(x)|}{|\Omega| = |B_r|} \right\}$

$$\mathcal{A}(\Omega) := \min_{\mathbf{x} \in \mathbb{R}^n} \left\{ \frac{|\Omega \Delta B_r(\mathbf{x})|}{|B_r|} : |\Omega| = |B_r| \right\}$$

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Conjecture by Hansen, Nadirashvili (1991) and by Bhattacharya, Weitsman (1996) for the first eigenvalue of the Laplacian

$$|\Omega|^{\frac{2}{n}}\lambda_2(\Omega) - |B|^{\frac{2}{n}}\lambda_2(B) \geq \gamma(n)\mathcal{A}(\Omega)^2$$

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We cannot expect a smaller exponent. Take the ellipsoids

$$\Omega_{\varepsilon} = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x'|^2 + (1+\varepsilon)x_n^2 \le 1\}$$
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Then one can show that

$$\mathcal{A}(\Omega_{arepsilon})pprox arepsilon \qquad |\Omega_{arepsilon}|^{rac{2}{n}}\lambda_2(\Omega_{arepsilon}) - |B|^{rac{2}{n}}\lambda_2(B)pprox arepsilon^2$$

 $|\Omega|^{\frac{2}{n}}\lambda_2(\Omega) - |B|^{\frac{2}{n}}\lambda_2(B) \ge \gamma(n)\mathcal{A}(\Omega)^2$



(*)
$$|\Omega|^{\frac{2}{n}} \lambda_2(\Omega) - |B|^{\frac{2}{n}} \lambda_2(B) \ge \gamma(n) \mathcal{A}(\Omega)^2$$

Bhattacharya (2001) proved for p>1: if $\Omega\subset \mathbb{R}^2$

$$|\Omega|^{\frac{p}{n}}\lambda_p(\Omega)-|B|^{\frac{p}{n}}\lambda_p(B)\geq \gamma(p)\mathcal{A}(\Omega)^3$$

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Bhattacharya (2001) proved for p>1: if $\Omega\subset {\rm I\!R}^2$

$$|\Omega|^{\frac{\rho}{n}}\lambda_{\rho}(\Omega) - |B|^{\frac{\rho}{n}}\lambda_{\rho}(B) \geq \gamma(\rho)\mathcal{A}(\Omega)^{3}$$

F., Maggi and Pratelli (2009) proved that if $\Omega \subset \mathbb{R}^n$, p > 1

$$|\Omega|^{\frac{\rho}{n}}\lambda_{\rho}(\Omega) - |B|^{\frac{\rho}{n}}\lambda_{\rho}(B) \ge \gamma(n,\rho)\mathcal{A}(\Omega)^{\rho+2}$$

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2013!! (in any dimension)

Proof based on the celebrated regularity results for free boundary

Brasco, De Philippis and Velichkov solved the conjecture (*) in

- problems involving the Laplacian:
 Alt, Caffarelli (1981)
- Alt, Caffarelli, Friedman (1984)

Let $n \ge 2$ and p > 1. Then there exists a constant $\gamma(n, p)$ s.t.

$$|\Omega|^{\frac{p}{n}}\lambda_p(\Omega) - |B|^{\frac{p}{n}}\lambda_p(B) \ge \gamma \mathcal{A}(\Omega)^2$$

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The issue here is the "2"!!

With "2" replaced by "3" see Brasco, De Philippis (2016)

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Strategy of the proof:

Similar to the one of Brasco, De Philippis and Velichkov

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From now on we shall assume $|\Omega| = |B|$ and we write $\lambda(\Omega)$ instead of $\lambda_p(\Omega)$ (p will be fixed)

Step 1
$$(p>1$$
 fixed and $|\Omega|=|B|)$

Following BDV, to prove
$$\lambda(\Omega) - \lambda(B) \ge \gamma \mathcal{A}(\Omega)^2$$
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 $E(\Omega) - E(B) \ge \gamma A(\Omega)^2$ where

$$\mathcal{L}(\mathcal{D}) \geq \gamma \mathcal{A}(\Omega)$$
 where

$$\int |\nabla u|^p dy = \int u dy \cdot u dy$$

$$E(\Omega) := \min \left\{ \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx - \int_{\Omega} u \, dx : u \in W_0^{1,p}(\Omega) \right\}$$

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If f_{Ω} is the first eigenfunction of the p-Laplacian then

$$-\operatorname{div}(|\nabla f_{\Omega}|^{p-2}\nabla f_{\Omega}) = \lambda(\Omega)|f_{\Omega}|^{p-2}f_{\Omega} \qquad f_{\Omega} = 0 \text{ on } \partial\Omega$$

If
$$u_{\Omega}$$
 is minimizer of $E(\Omega)$ then
$$-{
m div}(|\nabla u_{\Omega}|^{p-2}\nabla u_{\Omega})=1 \qquad u_{\Omega}=0 \ {
m on} \ \partial \Omega$$

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$$\lambda(\Omega)-\lambda(B)=\int_{\Omega}|\nabla f_{\Omega}|^{p}-\int_{\Omega}|\nabla f_{B}|^{p}$$

then
$$I_{\Omega} = I_{\Omega} = I_{\Omega}$$

 $E(\Omega) - E(B) = \frac{1-p}{p} \left(\int_{\Omega} |\nabla u_{\Omega}|^p - \int_{B} |\nabla u_{B}|^p \right)$

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olacian then
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cian then
$$f_0 = 0$$
 on $\partial \Omega$

$$dx: u \in W_0^{-n}(\Omega)$$

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$$E(\Omega)$$
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and then we use the following extension of the Kohler-Jobin

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inequality (Brasco, 2014):

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inequality (Brasco, 2014): $\frac{\lambda(\Omega)}{\lambda(B)} \ge \left(\frac{E(B)}{F(\Omega)}\right)^{\alpha(p,n)}$

$$\overline{\lambda(B)} \ge \left(\overline{E(\Omega)}\right)$$

If $|\Omega| = |B|$ and $\lambda(\Omega) \le 2\lambda(B)$ then

$$\lambda(\Omega) - \lambda(B) \ge c [E(\Omega) - E(B)]$$

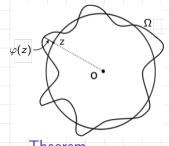
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Step 2
$$(|\Omega| = |B|)$$

To prove

$$E(\Omega) - E(B) \ge \gamma \mathcal{A}(\Omega)^2$$

we first assume that Ω is a bounded open set very close to B.



Given
$$\varphi \in C^{2,\alpha}(\partial B)$$
, $|\varphi| < 1$

We say that Ω is a nearly spherical set parameterized by φ if

$$\partial\Omega = \{x = z(1 + \varphi(z)) : z \in \partial B\}$$

Theorem

There exist δ, γ_0 such that if Ω is a nearly spherical set of class $C^{2,\alpha}$ parametrized by φ , with $\|\varphi\|_{C^{2,\alpha}(\partial B)} \leq \delta$, the barycenter of Ω is at the origin and $|\Omega| = |B|$, then

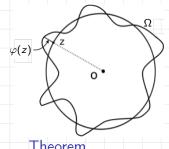
$$E(\Omega) - E(B) \ge \gamma_0 \|\varphi\|_{H^{\frac{1}{2}}(\partial B)}^2$$

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$$E(\Omega) - E(B) \ge \gamma_0 \|\varphi\|_{H^{\frac{1}{2}}(\partial B)}^2 \ge \gamma_0 \|\varphi\|_{L^2(\partial B)}^2 \ge c\mathcal{A}(\Omega)^2$$

Step 3 (Reduction to bounded sets
$$|\Omega| = |B|$$
)

Since $\mathcal{A}(\Omega) < 2$ to prove that

$$D(\Omega) := E(\Omega) - E(B) \ge \gamma A(\Omega)^2$$

it is enough to deal with the case that $D(\Omega) \leq \delta_0$

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Lemma

There exist $C, \delta_0, R > 0$, such that $|\Omega| = |B|$ and $D(\Omega) \leq \delta_0$, one can find another open set $\widetilde{\Omega}$ with $|\widetilde{\Omega}| = |B|$ and $\widetilde{\Omega} \subset B_R$ with the property that

$$\mathcal{A}(\Omega) \leq \mathcal{A}(\widetilde{\Omega}) + \mathit{CD}(\Omega), \qquad \mathit{D}(\widetilde{\Omega}) \leq \mathit{CD}(\Omega)$$

We have now to show that if $|\Omega| = |B|$, $\Omega \subset B_R$ then

$$D(\Omega) := E(\Omega) - E(B) \ge \gamma A(\Omega)^2$$

To prove this inequality we replace the Fraenkel asymmetry

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with the following (almost equivalent, but smoother asymmetry)

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Lemma

$$\Omega \subset B_R \implies \alpha(\Omega) \geq c_1 \mathcal{A}(\Omega)^2$$

If Ω is a nearly spherical set parametrized by φ , $\|\varphi\|_{L^{\infty}(\partial B)} \leq \delta$,

$$\alpha(\Omega) \leq c_2 \|\varphi\|_{L^2(\partial B)}^2$$

Thus we need to show that if $|\Omega|=|B|$, $\Omega\subset B_R$, $D(\Omega)\leq \delta$, then $D(\Omega):=E(\Omega)-E(B)\geq \gamma_0\,\alpha(\Omega)$

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Strategy: A contradiction argument based on regularity first proposed by Cicalese, Leonardi (2012) then modified by Acerbi, F., Morini (2013)

Step 4 (Reduction to nearly spherical sets via regularity)

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Strategy: A contradiction argument based on regularity first proposed by Cicalese, Leonardi (2012)

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To implement this strategy one has to adapt the theory of Alt, Caffarelli to a free boundary problem for the *p*-Laplacian equation

Danielli-Petrosjan (2005) studied the free boundary problem for the p-Laplacian equation with right hand side = 0

Theorem

There exist δ, γ_0 such that if Ω is a nearly spherical set of class $C^{2,\alpha}$ parametrized by φ , with $\|\varphi\|_{C^{2,\alpha}(\partial B)} \leq \delta$, the barycenter of Ω is at the origin and $|\Omega| = |B|$, then

(*)
$$E(\Omega) - E(B) \ge \gamma_0 \|\varphi\|_{H^{\frac{1}{2}}(\partial B)}^2$$

Recall that

$$E(\Omega) = \min \left\{ \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx - \int_{\Omega} u \, dx : u \in W_0^{1,p}(\Omega) \right\}$$

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To prove (*) we try a second variation argument. We construct an autonomous vector field $X \in C^{2,\alpha}(\mathbb{R}^n)$, s.t. $\operatorname{div} X = 0$ in a nhood of ∂B and consider the flow

$$\frac{\partial \Phi_t}{\partial t}(x) = X(\Phi_t(x)), \quad \Phi_0(x) = x, \quad \text{for } (x, t) \in \mathbb{R}^n \times [0, 1]$$

with
$$|\Phi_t(B)| = |B|$$
, $\Phi_1(B) = \Omega$

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with $|\Phi_t(B)| = |B|$, $\Phi_1(B) = \Omega$

Setting $\Omega_t := \Phi_t(B)$ and $e(t) := E(\Omega_t)$ we would like to write

$$E(\Omega) - E(B) = e(1) - e(0) = e'(0) + \frac{1}{2}e''(0) + \int_0^1 (1-t)(e''(t) - e''(0))dt$$

$$(*) E(\Omega) - E(B) \ge \gamma_0 \|\varphi\|_{H^{\frac{1}{2}}(\partial B)}^2$$

where
$$E(\Omega) = \min \left\{ \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx - \int_{\Omega} u \, dx : u \in W_0^{1,p}(\Omega) \right\}$$

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: Ω_t :

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Then we would like to prove that

$$e''(0) \ge \gamma \|\varphi\|_{H^{\frac{1}{2}}}^2 \qquad |e''(t) - e''(0)| \le \omega (\|\varphi\|_{H^{\frac{1}{2}}}) \|\varphi\|_{H^{\frac{1}{2}}}^2$$

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e'(t) is OK, but due to the degeneracy of the p-Laplacian,

$$e''(t)$$
 does not exist.....

Thus, for
$$\kappa \geq 0$$
 and $t \in (0,1)$ we set

Note that $e_0(t) = e(t)$.

 $\underline{e_{\kappa}}(t) = E_{\kappa}(\Omega_t) = \min_{u \in W_0^{1, p}(\Omega)} \left\{ \frac{1}{p} \int_{\Omega} (\kappa^2 + |\nabla u|^2)^{\frac{p}{2}} dx - \int_{\Omega} u dx \right\}$

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Thus, for $\kappa \geq 0$ and $t \in (0,1)$ we set

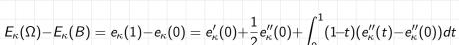
For $\kappa > 0$ we calculate

 $e_{\kappa}'(t) = \frac{1}{p} \int_{\Omega} \operatorname{div}((\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{p}{2}}X) - \int_{\Omega} \operatorname{div}((\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}} |\nabla u_{\kappa,t}|^2X)$

 $e_{\kappa}(t) = E_{\kappa}(\Omega_t) = \min_{u \in W_{\bullet}^{1, p}(\Omega)} \left\{ \frac{1}{p} \int_{\Omega} (\kappa^2 + |\nabla u|^2)^{\frac{p}{2}} dx - \int_{\Omega} u dx \right\}$









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$$E_{\kappa}(\Omega) - E_{\kappa}(B) = e_{\kappa}(1) - e_{\kappa}(0) = e_{\kappa}'(0) + \frac{1}{2}e_{\kappa}''(0) + \int_{0}^{1} (1-t)(e_{\kappa}''(t) - e_{\kappa}''(0))dt$$

$$E_{\kappa}(\Omega) - E_{\kappa}(B) = e_{\kappa}(1) - e_{\kappa}(0) = e'_{\kappa}(0) + \frac{1}{2}e''_{\kappa}(0) + \int_{0}^{\infty} (1-t)(e''_{\kappa}(t) - e''_{\kappa}(0))$$
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For
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$$\frac{1}{2} \int \operatorname{div}((\kappa^2 | \nabla u_{-}|^2)^{\frac{p}{2}} X) \int \operatorname{div}((\kappa^2 | |\nabla u_{-}|^2)^{\frac{p-2}{2}} |\nabla u_{-}|^2 X)$$

For
$$\kappa > 0$$
 we calculate
$$\frac{1}{2} \int_{0}^{\infty} \operatorname{div}((\kappa^{2} + |\nabla u_{k+1}|^{2})^{\frac{p}{2}} X) = \int_{0}^{\infty} \operatorname{div}((\kappa^{2} + |\nabla u_{k+1}|^{2})^{\frac{p-2}{2}} |\nabla u_{k+1}|^{2} X)$$

$$e_{\kappa}'(t) = \frac{1}{p} \int_{\Omega_t} \operatorname{div}((\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{p}{2}} X) - \int_{\Omega_t} \operatorname{div}((\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}} |\nabla u_{\kappa,t}|^2 X)$$

where

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where

 $e_{\kappa}'(t) = \frac{1}{\rho} \int_{\Omega_{\epsilon}} \operatorname{div}((\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{\rho}{2}} X) - \int_{\Omega_{\epsilon}} \operatorname{div}((\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{\rho-2}{2}} |\nabla u_{\kappa,t}|^2 X)$

 $\begin{cases} -\operatorname{div}((\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{\rho-2}{2}} \nabla u_{\kappa,t}) = 1 & \text{in } \Omega_t \\ u_{\kappa,t} = 0 & \text{on } \partial \Omega_t. \end{cases}$

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+

$$C^{1,lpha}$$
 estimates for the p -laplacian in $\overline{\Omega}_t$

 $\implies u_{\kappa,t}$ converge in $C^{1,\alpha}$ to u_t as $\kappa \to 0$, uniformly w.r.t. t

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 $C^{1,\alpha}$ estimates for the p-laplacian in $\overline{\Omega}_t$

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For $\kappa > 0$

$$\begin{aligned} & \boldsymbol{e}_{\kappa}'(t) = \frac{1}{p} \int_{\Omega_{t}} \operatorname{div} \left((\kappa^{2} + |\nabla u_{\kappa,t}|^{2})^{\frac{p}{2}} \boldsymbol{X} \right) - \int_{\Omega_{t}} \operatorname{div} \left((\kappa^{2} + |\nabla u_{\kappa,t}|^{2})^{\frac{p-2}{2}} |\nabla u_{\kappa,t}|^{2} \boldsymbol{X} \right) \\ & \text{where} \\ & \left\{ \begin{array}{l} -\operatorname{div} \left((\kappa^{2} + |\nabla u_{\kappa,t}|^{2})^{\frac{p-2}{2}} \nabla u_{\kappa,t} \right) = 1 & \text{in } \Omega_{t} \\ u_{\kappa,t} = 0 & \text{on } \partial \Omega_{t}. \end{array} \right. \end{aligned}$$

$$C^{1,lpha}$$
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 $\begin{cases} -\operatorname{div}(|\nabla u_t|^{p-2}\nabla u_t) = 1 & \text{in } \Omega_t \\ u_t = 0 & \text{on } \partial \Omega_t. \end{cases}$

$$\Longrightarrow \lim_{\kappa \to 0} e_{\kappa}'(t) = e_{0}'(t)$$
 uniformly w.r.t. t

For $\kappa > 0$, $t \in [0,1]$ we calculate

$$e_{\kappa}''(t) = \int_{\partial\Omega_t} (\kappa^2 + |
abla u_{\kappa,t}|^2)^{rac{p-2}{2}} (
abla W_{\kappa,t} \cdot
u_{\Omega_t}) W_{\kappa,t} d\mathcal{H}^{n-1}$$

$$+ (p-2) \int_{\partial\Omega_{t}} (\kappa^{2} + |\nabla u_{\kappa,t}|^{2})^{\frac{p-4}{2}} (\nabla W_{\kappa,t} \cdot \nabla u_{\kappa,t}) (\nabla u_{\kappa,t} \cdot \nu_{\Omega_{t}}) W_{\kappa,t} d\mathcal{H}^{n-1}$$

$$- \int_{\partial\Omega_{t}} (\kappa^{2} + |\nabla u_{\kappa,t}|^{2})^{\frac{p-2}{2}} (\nabla^{2} u_{\kappa,t} [\nabla u_{\kappa,t}] \cdot X_{\tau}) (X \cdot \nu_{\Omega_{t}}) d\mathcal{H}^{n-1}$$

$$-\int_{\partial\Omega_{t}} (\kappa^{2} + |\nabla u_{\kappa,t}|^{2})^{\frac{p-2}{2}} (\nabla^{2} u_{\kappa,t} [\nabla u_{\kappa,t}] \cdot X_{\tau}) (X \cdot \nu_{\Omega_{t}}) d\mathcal{H}^{n-1}$$

$$-(p-2) \int_{\partial\Omega_{t}} (\kappa^{2} + |\nabla u_{\kappa,t}|^{2})^{\frac{p-4}{2}} |\nabla u_{\kappa,t}|^{2} (\nabla^{2} u_{\kappa,t} [\nabla u_{\kappa,t}] \cdot X_{\tau}) (X \cdot \nu_{\Omega_{t}}) d\mathcal{H}^{n-1}$$

 $+ \int_{\partial\Omega_t} |\nabla u_{\kappa,t}| \big((\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}} |\nabla u_{\kappa,t}| \mathcal{H}_{\Omega_t} - 1\big) (X \cdot \nu_{\Omega_t})^2 d\mathcal{H}^{n-1},$

where
$$W_{\kappa,t}$$
 is the unique solution in $H^1(\Omega_t)$ of the equation

For $\kappa > 0$, $t \in [0,1]$ we calculate

$$\frac{e_{\kappa}''(t)}{e_{\kappa}''(t)} = \int_{\partial\Omega_{t}} (\kappa^{2} + |\nabla u_{\kappa,t}|^{2})^{\frac{p-2}{2}} (\nabla W_{\kappa,t} \cdot \nu_{\Omega_{t}}) W_{\kappa,t} d\mathcal{H}^{n-1}
+ (p-2) \int_{\partial\Omega_{t}} (\kappa^{2} + |\nabla u_{\kappa,t}|^{2})^{\frac{p-4}{2}} (\nabla W_{\kappa,t} \cdot \nabla u_{\kappa,t}) (\nabla u_{\kappa,t} \cdot \nu_{\Omega_{t}}) W_{\kappa,t} d\mathcal{H}^{n-1}$$

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$$-(p-2)\int_{\partial\Omega_{t}}(\kappa^{2}+|\nabla u_{\kappa,t}|^{2})^{\frac{p-4}{2}}|\nabla u_{\kappa,t}|^{2}(\nabla^{2}u_{\kappa,t}[\nabla u_{\kappa,t}]\cdot X_{\tau})(X\cdot\nu_{\Omega_{t}})d\mathcal{H}^{n-1}$$

$$+\int_{\partial\Omega_{t}}|\nabla u_{\kappa,t}|((\kappa^{2}+|\nabla u_{\kappa,t}|^{2})^{\frac{p-2}{2}}|\nabla u_{\kappa,t}|H_{\Omega_{t}}-1)(X\cdot\nu_{\Omega_{t}})^{2}d\mathcal{H}^{n-1},$$

 $\begin{cases} \operatorname{div} \left[(\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}} \left(\nabla W_{\kappa,t} + (p-2) \frac{\nabla u_{\kappa,t} \cdot \nabla W_{\kappa,t}}{\kappa^2 + |\nabla u_{\kappa,t}|^2} \nabla u_{\kappa,t} \right) \right] = 0 & \text{in } \Omega_t \\ \dot{u}_{\kappa,t} = -\nabla u_{\kappa,t} \cdot X & \text{on } \partial \Omega_t \end{cases}$

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and $\begin{cases} -\operatorname{div}((\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}} \nabla u_{\kappa,t}) = 1 & \text{in } \Omega_t \\ u_{\kappa,t} = 0 & \text{on } \partial \Omega_t. \end{cases}$

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$$E_{\kappa}(\Omega) - E_{\kappa}(B) = e_{\kappa}'(0) + \frac{1}{2}e_{\kappa}''(0) + \int_{0}^{1} (1-t)(e_{\kappa}''(t) - e_{\kappa}''(0))dt$$

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$$\text{and} \quad \begin{cases} -\operatorname{div} \left((\kappa^2 + |\nabla u_{\kappa,t}|^2)^{\frac{p-2}{2}} \nabla u_{\kappa,t} \right) = 1 & \text{in } \Omega_t \\ u_{\kappa,t} = 0 & \text{on } \partial \Omega_t. \end{cases}$$

$$\begin{array}{c} u_{\kappa,t}=0 & \text{on }\partial\Omega_t. \end{array}$$
 We know
$$u_{\kappa,0}\to c(1-|x|^{\frac{p}{p-1}}) \quad \text{in } C^{1,\alpha}, \qquad (\kappa^2+|\nabla u_{\kappa,0}|^2)^{\frac{p-2}{2}}\to c'|x|^{\frac{p-2}{p-1}} \end{array}$$

$$E_{\kappa}(\Omega) - E_{\kappa}(B) = e_{\kappa}'(0) + \frac{1}{2}e_{\kappa}''(0) + \int_{0}^{1}(1-t)(e_{\kappa}''(t) - e_{\kappa}''(0))dt$$

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where W is the unique weak solution in $H^1(B; \mu)$ of the equation

 $\begin{cases} \operatorname{div}\left(|x|^{\frac{p-2}{p-1}}\nabla W + (p-2)|x|^{\frac{p-2}{p-1}}\left(\frac{x}{|x|}\cdot\nabla W\right)\frac{x}{|x|}\right) = 0 & \text{in } B \\ W = n^{-\frac{1}{p-1}}X\cdot\nu_B & \text{on } B \end{cases}$

on ∂B

$$n^{\frac{p-2}{p-1}} \lim_{\kappa \to 0} e_{\kappa}''(0) = \int_{\mathbb{R}} |x|^{\frac{p-2}{p-1}} \left(|\nabla W|^2 + (p-2) \left| \frac{x}{|x|} \cdot \nabla W \right|^2 \right) dx - \int_{\partial \mathbb{R}} W^2 d\mathcal{H}^{n-1}$$









and the weight $\mu = |x|^{\frac{p-2}{p-1}}$

Lemma 1 (hard!)

$$E_{\kappa}(\Omega) - E_{\kappa}(B) = e_{\kappa}'(0) + rac{1}{2}e_{\kappa}''(0) + \int_{0}^{1}(1-t)(e_{\kappa}''(t) - e_{\kappa}''(0))dt$$

Lemma 1 (hard!)

$$n^{\frac{p-2}{p-1}} \lim_{\kappa \to 0} e_{\kappa}''(0) = \int_{B} |x|^{\frac{p-2}{p-1}} \left(|\nabla W|^{2} + (p-2) \left| \frac{x}{|x|} \cdot \nabla W \right|^{2} \right) dx - \int_{\partial B} W^{2} d\mathcal{H}^{n-1}$$

where W is the unique weak solution in $H^1(B;\mu)$ of the equation

$$\begin{cases} \operatorname{div}\left(|x|^{\frac{p-2}{p-1}}\nabla W + (p-2)|x|^{\frac{p-2}{p-1}}\left(\frac{x}{|x|}\cdot\nabla W\right)\frac{x}{|x|}\right) = 0 & \text{in } B\\ W = n^{-\frac{1}{p-1}}X\cdot\nu_B & \text{on } \partial B \end{cases}$$

and the weight $\mu=|x|^{\frac{p-2}{p-1}}$

Lemma 2 (harder!) For
$$\kappa, t \in (0, 1]$$
 we have

$$|e_{\kappa}''(t) - e_{\kappa}''(0)| \le \omega(\|\varphi\|_{C^{2,\alpha}} + \kappa) \|X \cdot \nu_B\|_{H^{\frac{1}{2}}(\partial B)}^2$$

$$E_{\kappa}(\Omega) - E_{\kappa}(B) = e_{\kappa}'(0) + \frac{1}{2}e_{\kappa}''(0) + \int_{0}^{1}(1-t)(e_{\kappa}''(t) - e_{\kappa}''(0))dt$$

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and then we get
$$E(\Omega) - E(B) > \gamma_{n,n} \int |x|^{\frac{p-2}{p-1}} (|\nabla W|^2 + (p-2)|^{\frac{x}{p-2}} \cdot |\nabla W|^2) dx$$

and then we get
$$E(\Omega) - E(B) \ge \gamma_{n,p} \int_{B} |x|^{\frac{p-2}{p-1}} \left(|\nabla W|^2 + (p-2) \left| \frac{x}{|x|} \cdot \nabla W \right|^2 \right) dx$$

$$= \gamma \int_{B} W^2 d\mathcal{H}^{n-1} = \langle y(|y_0||_{C^{2-p}(\Omega)}) ||X + y_0||^2$$

$$-\gamma_{n,p} \int_{\partial B} W^2 d\mathcal{H}^{n-1} - \omega(\|\varphi\|_{C^{2,\alpha}(\partial B)}) \|X \cdot \nu_B\|_{H^{1/2}(\partial B)}^2$$

$$\geq \qquad \qquad (\text{writing } W \text{ in terms of spherical harmonics})$$

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$$\geq \frac{ \text{(writing } W \text{ in terms of spherical harmonics)} }{ \geq \gamma' \|X \cdot \nu_B\|_{H^{1/2}(\partial B)}^2 - \omega(\|\varphi\|_{C^{2,\alpha}(\partial B)}) \|X \cdot \nu_B\|_{H^{1/2}(\partial B)}^2 }$$

$$E_{\kappa}(\Omega) - E_{\kappa}(B) = e_{\kappa}'(0) + \frac{1}{2}e_{\kappa}''(0) + \int_{0}^{1}(1-t)(e_{\kappa}''(t) - e_{\kappa}''(0))dt$$

Using the two previous lemmas and $e_0'(0)=0$, let us take $\kappa \to 0^+$ and then we get

$$E(\Omega) - E(B) \ge \gamma_{n,p} \int_{B} |x|^{\frac{p-2}{p-1}} \left(|\nabla W|^{2} + (p-2) \left| \frac{x}{|x|} \cdot \nabla W \right|^{2} \right) dx$$

$$- \gamma_{n,p} \int_{\partial B} W^{2} d\mathcal{H}^{n-1} - \omega(\|\varphi\|_{C^{2,\alpha}(\partial B)}) \|X \cdot \nu_{B}\|_{H^{1/2}(\partial B)}^{2}$$

$$\ge \qquad \text{(writing W in terms of spherical harmonics)}$$

 $\geq \gamma' \|X \cdot \nu_B\|_{H^{1/2}(\partial B)}^2 - \omega(\|\varphi\|_{C^{2,\alpha}(\partial B)}) \|X \cdot \nu_B\|_{H^{1/2}(\partial B)}^2$

Thus if $\|\varphi\|_{C^{2,\alpha}(\partial B)}$ is small we conclude

$$E_{\kappa}(\Omega) - E_{\kappa}(B) = e_{\kappa}'(0) + \frac{1}{2}e_{\kappa}''(0) + \int_{0}^{1} (1-t)(e_{\kappa}''(t) - e_{\kappa}''(0))dt$$

Using the two previous lemmas and $e_0'(0) = 0$, let us take $\kappa \to 0^+$ and then we get

$$E(\Omega) - E(B) \ge \gamma_{n,p} \int_{B} |x|^{\frac{p-2}{p-1}} \left(|\nabla W|^{2} + (p-2) \left| \frac{x}{|x|} \cdot \nabla W \right|^{2} \right) dx$$

$$- \gamma_{n,p} \int_{\partial B} W^{2} d\mathcal{H}^{n-1} - \omega(\|\varphi\|_{C^{2,\alpha}(\partial B)}) \|X \cdot \nu_{B}\|_{H^{1/2}(\partial B)}^{2}$$

$$\ge \qquad \text{(writing } W \text{ in terms of spherical harmonics)}$$

$$\ge \gamma' \|X \cdot \nu_{B}\|_{H^{1/2}(\partial B)}^{2} - \omega(\|\varphi\|_{C^{2,\alpha}(\partial B)}) \|X \cdot \nu_{B}\|_{H^{1/2}(\partial B)}^{2}$$

Thus if $\|\varphi\|_{C^{2,\alpha}(\partial B)}$ is small we conclude

$$E(\Omega) - E(B) \ge c \|X \cdot \nu_B\|_{H^{1/2}(\partial B)}^2 \ge c \|\varphi\|_{H^{1/2}(\partial B)}^2$$

Back to Step 4 (Reduction to nearly spherical sets via regularity)

Thus we need to show that if $|\Omega| = |B|$, $\Omega \subset B_R$, $D(\Omega) \leq \delta$, then

$$D(\Omega) := E(\Omega) - E(B) \ge \gamma_0 \alpha(\Omega)$$

$$\alpha(\Omega) = \int_{\Omega \Delta B_1(x_{\Omega})} |1 - |x - x_{\Omega}| | dx$$

where

Back to Step 4 (Reduction to nearly spherical sets via regularity)

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Assume that there exists a sequence
$$\Omega_h \subset B_R, |\Omega_h| = |B|$$
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Solution that there exists a sequence
$$2L_{\eta} \subset D_{R}$$
, $|2L_{\eta}| = |D|$ with

$$\delta_h = D(\Omega_h) o 0$$
 but $E(\Omega_h) - E(B) \le \sigma^4 \alpha(\Omega)$ for some $0 < \sigma < 1$

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 $\delta_h = D(\Omega_h) \to 0$ but $E(\Omega_h) - E(B) \le \sigma^4 \alpha(\Omega)$ for some $0 < \sigma < 1$

$$D(\Omega_h) o 0$$
 but $E(\Omega_h) - E(B) \le \sigma^4 \alpha(\Omega)$ for some $0 < \sigma$

(*) $\inf \left\{ E(U) + \sqrt{\delta_h^2 + \sigma^2(\alpha_h(U) - \delta_h)^2} + P(|U| - |B|) : U \subset B_R \right\}$

Back to Step 4 (Reduction to nearly spherical sets via regularity)

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 with

Assume that there exists a sequence $\Omega_h \subset B_R$, $|\Omega_h| = |B|$ with

$$\delta_h = D(\Omega_h) o 0$$
 but $E(\Omega_h) - E(B) \le \sigma^4 \alpha(\Omega)$ for some $0 < \sigma < 1$

$$O_h = D(\Omega_h) \rightarrow 0$$
 But $L(\Omega_h) = L(D) \le O(\Omega_h)$ for some $0 \le O(\Omega_h)$

(*) Inf
$$\left\{ E(U) + \sqrt{\delta_h^2 + \sigma^2(\alpha_h(U) - \delta_h)^2} + P(|U| - |B|) : U \subset B_R \right\}$$

We show that
$$(*)$$
 has a minimizer U_h , that $D(U_h) o 0$ and

(**)

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Back to Step 4 (Reduction to nearly spherical sets via regularity)

Thus we need to show that if $|\Omega|=|B|$, $\Omega\subset B_R$, $D(\Omega)\leq \delta$, then $D(\Omega):=E(\Omega)-E(B)\geq \gamma_0\,\alpha(\Omega)$

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$$\ (*)$$
 has a minimizer $\ U_h$, that $\ D(U_h) o 0$ and

$$(**) E(U_h) - E(B) \le c(n, p)\sigma \alpha(U_h)$$

Moreover, U_h is smooth and $U_h o B$ in C^k for all k

Back to Step 4 (Reduction to nearly spherical sets via regularity) Thus we need to show that if $|\Omega| = |B|$, $\Omega \subset B_R$, $D(\Omega) \leq \delta$, then

$$D(\Omega) := E(\Omega) - E(B) \geq \gamma_0 \, lpha(\Omega)$$

Assume that there exists a sequence $\Omega_h \subset B_R$, $|\Omega_h| = |B|$ with

$$\delta_h = D(\Omega_h) o 0$$
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(*) $\inf \left\{ E(U) + \sqrt{\delta_h^2 + \sigma^2(\alpha_h(U) - \delta_h)^2} + P(|U| - |B|) : U \subset B_R \right\}$

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We show that (*) has a minimizer U_h , that $D(U_h) \rightarrow 0$ and $E(U_h) - E(B) \le c(n, p)\sigma \alpha(U_h)$ (**)

Moreover, U_h is smooth and $U_h \to B$ in C^k for all k Then (**) gives a contradiction if σ is sufficiently small!

Is there an open set
$$U \subset B_R$$
 minimizing
$$E(U) + \sqrt{\delta_h^2 + \sigma^2(\alpha_h(U) - \delta_h)^2} + P(|U| - |B|)$$
?

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$$U$$
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U p-quasiopen iff
$$U=\{u^*>0\}$$
 for some $u\in W^{1,p}_0({\mathbb R}^n)$
Then the problem

(*)
$$\min \left\{ E(U) + \sqrt{\delta_h^2 + \sigma^2(\alpha_h(U) - \delta_h)^2 + P(|U| - |B|)} : \right\}$$

$$U \subset B_R, \ U$$
 p-quasiopen $\Big\}$

has a solution
$$U_h$$
!

Is there an open set $U \subset B_R$ minimizing

$$E(U) + \sqrt{\delta_h^2 + \sigma^2(\alpha_h(U) - \delta_h)^2 + P(|U| - |B|)}$$
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We consider a larger class of sets the *p*-quasiopen sets

$$U$$
 p-quasiopen iff $U=\{u^*>0\}$ for some $u\in W^{1,p}_0(\mathbb{R}^n)$

Then the problem

(*)
$$\min \left\{ E(U) + \sqrt{\delta_h^2 + \sigma^2(\alpha_h(U) - \delta_h)^2 + P(|U| - |B|)} : U \subset B_R, U \text{ p-quasiopen} \right\}$$

has a solution $U_h!$ Observe that if $v\in W^{1,p}_0(B_R)$ then

$$E(U_h) + \sqrt{\delta_h^2 + \sigma^2(\alpha_h(U_h) - \delta_h)^2} + P(|U_h| - |B|)$$

$$\leq E(\{v>0\}) + \sqrt{\delta_h^2 + \sigma^2(\alpha_h(\{v>0\}) - \delta_h)^2} + P(|\{v>0\}| - |B|)$$

If
$$v \in W^{1,p}_0(B_R)$$
 then

$$E(U_h) + \sqrt{\delta_h^2 + \sigma^2(\alpha_h(U_h) - \delta_h)^2} + P(|U_h| - |B|)$$





 $\leq E(\{v>0\}) + \sqrt{\delta_h^2 + \sigma^2(\alpha_h(\{v>0\}) - \delta_h)^2 + P(|\{v>0\}| - |B|)}$

If $v \in W_0^{1,p}(B_R)$ then

$$=(11.)+\sqrt{\delta^2+\sigma^2(\alpha_1(11.))}$$

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 $\pm \sqrt{\delta^2 \pm \sigma^2(\alpha_1(11))}$

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 $\leq E(\{v>0\}) + \sqrt{\delta_h^2 + \sigma^2(\alpha_h(\{v>0\}) - \delta_h)^2 + P(|\{v>0\}| - |B|)}$

 $\leq \frac{1}{p} \int |\nabla v|^p dx - \int v dx + \sqrt{\delta_h^2 + \sigma^2(\alpha_h(\{v > 0\}) - \delta_h)^2} + P(|\{v > 0\}| - |B|)$

If $v \in W_0^{1,p}(B_R)$ then

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$$E(U_h) + \sqrt{\delta_h^2 + \sigma^2(\alpha_h(U_h) - \delta_h)^2} + P(|U_h| - |B|)$$

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Therefore if u_h is the function minimizing $E(U_h)$ we have

 $\leq \frac{1}{n} \int |\nabla v|^p dx - \int v dx + \sqrt{\delta_h^2 + \sigma^2(\alpha_h(\{v > 0\}) - \delta_h)^2} + P(|\{v > 0\}| - |B|)$





If $v \in W_0^{1,p}(B_R)$ then

$$E(U_h) + \sqrt{\delta_h^2 + \sigma^2(\alpha_h(U_h) - \delta_h)^2} + P(|U_h| - |B|)$$

for all $v \in W_0^{1,p}(B_R)$

$$\frac{1}{52}$$

$$(B_R)$$
 then

 $\leq E(\{v>0\}) + \sqrt{\delta_h^2 + \sigma^2(\alpha_h(\{v>0\}) - \delta_h)^2 + P(|\{v>0\}| - |B|)}$

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 $\frac{1}{n} \int |\nabla u_h|^p dx - \int u_h dx + \sqrt{\delta_h^2 + \sigma^2(\alpha_h(\{u_h > 0\}) - \delta_h)^2} + P(|\{u_h > 0\}| - |B|)$

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Since u_h satisfies

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 $\leq \frac{1}{p} \int |\nabla v|^p dx - \int v dx + \sqrt{\delta_h^2 + \sigma^2(\alpha_h(\{v > 0\}) - \delta_h)^2} + P(|\{v > 0\}| - |B|)$

$$\frac{1}{2}\int |\nabla u_h|^p dx - \int u_h |\nabla u_h|^p dx$$

for all $v \in W_0^{1,p}(B_R)$

Since u_h satisfies

for all $v \in W_0^{1,p}(B_R)$

$$\frac{1}{p} \int |\nabla u_h|^p dx - \int u_h dx + \sqrt{\delta_h^2 + \sigma^2(\alpha_h(\{u_h > 0\}) - \delta_h)^2} + P(|\{u_h > 0\}| - |B|)$$

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.... There exists a nonnegative Borel function q_{u_h} such that

where $U_h = \{u_h > 0\}$ and for all $x \in \partial U_h$

 $-\mathrm{div}(|\nabla u_h(x)|^{p-2}\nabla u_h(x)) = \chi_{U_h}(x) - q_{u_h}(x)^{p-1}\mathcal{H}^{n-1} \sqcup \partial U_h$

 $\frac{1}{C} \leq q_{u_j}(x) \leq C$

in B_R