Some new inequalities for the Cheeger constant

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Shape Optimization and Isoperimetric and Functional Inequalities

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Given $\Omega \subset \mathbb{R}^2$ with finite measure

$$h(\Omega) := \inf \left\{ \frac{\mathrm{Per}(A, \mathbb{R}^2)}{|A|} \ : \ A \text{ measurable}, \ A \subseteq \Omega \right\}.$$

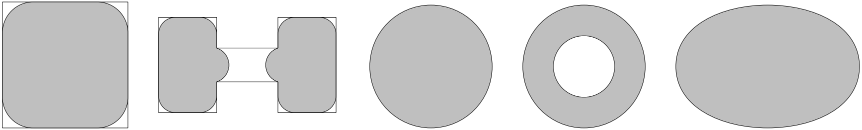
A solution is called a Cheeger set of Ω.

Relaxed formulation:

$$\inf \left\{ \frac{|Du|(\mathbb{R}^2)}{\int_\Omega |u|} \ : \ u \in BV(\mathbb{R}^2) \setminus \{0\}, \ u = 0 \text{ on } \mathbb{R}^2 \setminus \Omega \right\}.$$

[Alter, Buttazzo, Carlier, Caselles, Chambolle, Comte, Figalli, Fridman, Fusco, Kawohl, Krejčiřík, Lachand-Robert, Leonardi, Maggi, Novaga, Pratelli]

Some examples of Cheeger sets:



A quick overview

- *Existence/uniqueness:*

  a Cheeger set $C(\Omega)$ always exists, and it is unique if Ω is convex.

- *Regularity*:

  $\partial C(\Omega) \cap \Omega$ is made by arcs of circle, which meet $\partial\Omega$ tangentially.

- *Dependence on the domain*:

  $\Omega \mapsto h(\Omega)$ is monotone decreasing and homogeneous of degree -1.

- *Faber-Krahn inequality:*

  the ball minimizes the Cheeger constant under a volume constraint

$$h(\Omega) = \frac{\mathrm{Per}(C(\Omega), \mathbb{R}^2)}{|C(\Omega)|} \geq \frac{\mathrm{Per}(C^*(\Omega), \mathbb{R}^2)}{|C^*(\Omega)|} \geq h(\Omega^*) .$$

- *Relation with Dirichlet eigenvalues*:

  the first Dirichlet eigenvalue $\lambda_{1,p}$ of the p-Laplacian satisfies

  - $$\lambda_{1,p}(\Omega) \geq \Big(\frac{h(\Omega)}{p} \Big)^p$$

  - $$\lim_{p \to 1^+} \lambda_{1,p}(\Omega) = h(\Omega) .$$

A bit of mathematical faith [L.C.Evans]:

One important principle of mathematics is that *extreme cases reveal interesting structure*.

Outline

- I. Discrete Faber-Krahn inequality
- II. Reverse Faber-Krahn inequality
- III. Mahler inequality
- IV. Asymptotic behaviour of optimal partitions

joint works with D. BUCUR, B. VELICHKOV, G. VERZINI

The shape optimization problem:

$$\min \Big\{ h(\Omega) \ : \ \Omega \in \mathscr{P}_N \,, \quad |\Omega| = c \Big\} \,,$$

where $\mathscr{P}_N$ is the class of simple polygons with at most N sides.

A *simple polygon* is the open bounded planar region Ω delimited by a finite number of not self-intersecting line segments (called *sides*) which are pairwise joined (at their endpoints called *vertices*) to form a closed path.

Motivation: more than half a century ago, Pólya-Szego *conjectured* that the same problem for the first Dirichlet eigenvalue is solved by the *regular N-gon*:

$$\lambda_1(\Omega) \geq \lambda_1(\Omega_N^*) \qquad \forall \Omega \in \mathscr{P}_N .$$

- ▷ same conjecture also for other shape functionals, such as torsional rigidity;
- ▷ solved for $N = 3,4$; OPEN for $N \geq 5$;
- ▷ the problem for the *logarithmic capacity* has been solved for every N [Solynin-Zalgaller 04] .

Theorem [Bucur-F.]

Among all simple polygons with a given area and at most N sides, the regular N-gon minimizes the Cheeger constant.

$$h(\Omega) \geq h(\Omega_N^*) \qquad \forall \Omega \in \mathscr{P}_N \,.$$

Corollary: $\lambda_1(\Omega) \geq \gamma_N \, \lambda_1(\Omega_N^*) \quad \forall \Omega \in \mathscr{P}_N \quad (\gamma_N < 1)\,.$

Proof for convex polygons (easy!)

If Ω minimizes the Cheeger constant among *convex* polygons with the same area and at most N sides, it is *Cheeger regular*, and consequently

$$h(\Omega) = \frac{|\partial\Omega| + \sqrt{|\partial\Omega|^2 - 4|\Omega|(\Lambda(\Omega) - \pi)}}{2|\Omega|} \qquad \text{with } \Lambda(\Omega) = \sum_i \cot\left(\frac{\theta_i}{2}\right).$$

By the isoperimetric inequality for convex polygons $|\partial\Omega|^2 \geq 4\Lambda(\Omega)|\Omega|$, one gets

$$h(\Omega) \geq \frac{|\partial\Omega| + \sqrt{4\pi|\Omega|}}{2|\Omega|}.$$

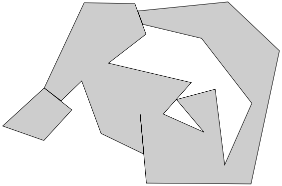
The conclusion follows since the regular N-gon is the unique minimizer of the perimeter among simple polygons with the same area and at most N-sides:

$$h(\Omega) \geq \frac{|\partial\Omega_N^*| + \sqrt{4\pi|\Omega_N^*|}}{2|\Omega_N^*|} = h(\Omega_N^*).$$

□

Hints of proof for general polygons

- Since $\mathscr{P}_N$ is not closed in the Hausdorff complementary topology, we enlarge the class of competitors to $\overline{\mathscr{P}_N}$ (thus allowing self-intersections).

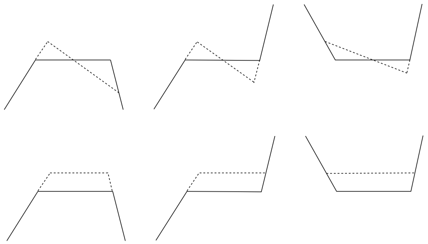


- For such "generalized polygons", we introduce a notion of *relaxed Cheeger constant*. In this framework, we obtain an *existence result*, and a *representation formula* for an optimal generalized polygon.

▷ We introduce a Lagrange multiplier μ and we use first order shape derivatives to get some *stationariety conditions*:

$$\frac{d}{d\varepsilon}\Big(h(\Omega_\varepsilon)+\mu|\Omega_\varepsilon|\Big)=0\ .$$

The deformations we use are *rotations* and *parallel movements* of one side.



▷ Via the stationariety conditions, we show that the boundary of an optimal generalized polygons contains *no self-intersections* and *no reflex angles*.

We are thus back to the case of simple convex polygons and we are done.

□

The case of the classical isoperimetric inequality

$\mathscr{K}^n :=$ n-dimensional convex bodies

- Balls solve

$$\inf_{K \in \mathscr{K}^n} \frac{|\partial K|}{|K|^{\frac{n-1}{n}}} .$$

The corresponding supremum equals $+\infty$.

- Regular simplexes and cubes solve respectively

$$\sup_{K \in \mathscr{K}^n} \inf_{T \in A_n} \frac{|\partial T(K)|}{|T(K)|^{\frac{n-1}{n}}} \qquad \sup_{K \in \mathscr{K}^n_*} \inf_{T \in GL_n} \frac{|\partial T(K)|}{|T(K)|^{\frac{n-1}{n}}} .$$

[K.Ball, Barthe, Gustin, Behrend]

An (almost) unexplored class of shape optimization problems

$$\sup_{K \in \mathscr{K}^n} \inf_{T \in A_n} f(T(K)),$$

where f is a continuous, translation invariant, 0-homogeneous functional, involving some *variational energy*.

Our goal: to reverse the FK inequality for the Cheeger constant, by taking

$$f(K) = h(K)|K|^{1/2}.$$

A central issue is to understand features of bodies in special *positions*.

A convex body is in *John position* if its John ellipsoid is a ball.

Theorem [Bucur-F.]

The regular triangle Δ and the square Q are optimal respectively for the reverse FK inequality for the Cheeger constant in $\mathscr{K}^2$ and in $\mathscr{K}^2_*$:

(i) for every $K \in \mathscr{K}^2$, if $\widetilde{K}$ is the image of K in John position, it holds

$$h(\widetilde{K})|\widetilde{K}|^{1/2} \leq h(\Delta)|\Delta|^{1/2};$$

(ii) for every $K \in \mathscr{K}^2_*$, if $\widetilde{K}$ is the image of K in John position, it holds

$$h(\widetilde{K})|\widetilde{K}|^{1/2} \leq h(Q)|Q|^{1/2}.$$

Remark: The same result for the first Dirichlet eigenvalue is OPEN.

Hints of proof for bodies in John position

- By approximation, we can work with polygons in John positions:

$$\sup_{K \in \mathscr{J} \cap \mathscr{P}_N} h(K)|K|^{1/2}.$$

- If K_0 is optimal, we prove:

  (1) John's ball is *contained* into $C(K_0)$ (by using Ball's volume ratio estimate);

  (2) K_0 is *Cheeger-regular* (by using (1));

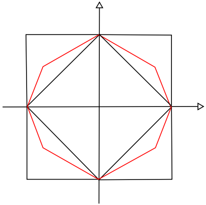
  (3) K_0 is *circumscribed* around B (by using (2) and BM inequality).

- We can use the representation formula:

$$h(K_0) = \frac{|\partial K_0| + \sqrt{4\pi|K_0|}}{2|K_0|} \quad \Rightarrow \quad h(K_0)|K_0|^{1/2} = \frac{|\partial K_0|}{2|K_0|^{1/2}} + \sqrt{\pi}$$

and we conclude via the reverse isoperimetric inequality by K. Ball. □

A convex axisymmetric body is in *$Q_{\pm}$-position* if it lies between Q_- and Q_+.



Theorem [Bucur-F.]

For every axisymmetric convex *octagon* K, if $\widetilde{K}$ is the image of K in $Q_{\pm}$-position, it holds

$$h(\widetilde{K})|\widetilde{K}|^{1/2} \leq h(Q)|Q|^{1/2}.$$

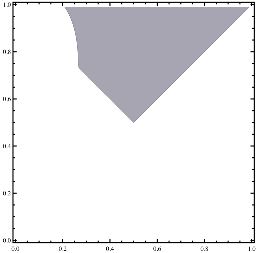
Remark: The same result holds the first Dirichlet eigenvalue [Bucur-F.] .

Hints of proof for axisymmetric octagons in $Q_\pm$ position

We have to show that

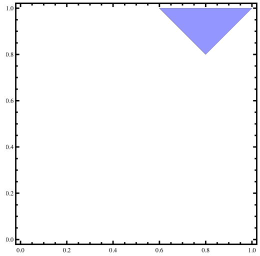
$$h(\Omega_{(a,b)})|\Omega_{(a,b)}|^{1/2} \leq h(Q)|Q|^{1/2} \qquad \forall (a,b) \in \mathscr{A}\,,$$

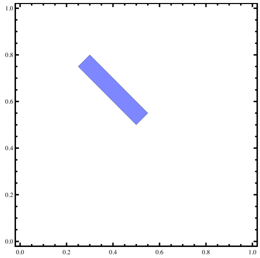
$\mathscr{A} = (a,b) \in (0,1)^2$ with $b > \max\{a, 1-a\}$ such that $\Omega_{(a,b)}$ is Cheeger-regular.



We use a mix of *theoretical* and *numerical* arguments.

- STEP 1 (*theoretical*): we justify the inequality analytically in a "confidence zone" near the maximum points.





- STEP 2 (*numerical*): we cover the complement of the confidence regions by a square grid, and we use a monotonicity argument together with explicit analytical bounds which are computable with machine precision.

$$\Omega_{\rm in} \subseteq \Omega_{(a,b)} \subseteq \Omega_{\rm out} \qquad \Rightarrow$$

$$h(\Omega_{(a,b)})|\Omega_{(a,b)}|^{1/2} \ \leq \ h(\Omega_{\rm in})|\Omega_{\rm out}|^{1/2} \leq h(Q)|Q|^{1/2}\,.$$

Another (almost) unexplored class of shape optimization problems

Minimize or maximize over $\mathscr{K}^n$ a product functional of the form

$$f(K)f(K^o),$$

where K^o is the *polar body*

$$K^o := \left\{ y \in \mathbb{R}^n \ : \ y \cdot x \leq 1 \quad \forall x \in K \right\}$$

and f is a some *variational energy*.

Our goal: to obtain a Mahler-type inequality for the Cheeger const., by taking

$$f(K) = h(K).$$

The case of the volume product $|K||K^o|$

- *Blaschke-Santaló inequality*:

$$|K||K^o| \text{ is maximal at balls}\,.$$

- *Mahler conjecture (1939)*:

$|K||K^o|$ is minimal at simplexes (over $\mathscr{K}^n$) or at cubes (over $\mathscr{K}^n_*$) .

STILL OPEN [Schneider, Tao]

The case of the Cheeger product $h(K)h(K^o)$

- Minimizers of $h(K)h(K^o)$ are balls (immediate using FK).
- Maximizers have to be studied modulo affinities

$$\sup_{K\in\mathscr{K}^n}\inf_{T\in A_n} h(T(K))\,h(T(K)^o)\,.$$

Theorem [Bucur-F.]

In dimension $n = 2$, and within *axisymmetric* convex bodies, the square Q is optimal in the Mahler inequality for the Cheeger constant:

for every $K \in \mathscr{K}^2_{axi}$, if $\widetilde{K}$ is the image of K in $Q_\pm$-position, it holds

$$h(\widetilde{K})h(\widetilde{K}^o) \leq h(Q)h(Q^o).$$

Remarks:

- ▷ The same result holds the first Dirichlet eigenvalue [Bucur-F.].
- ▷ The proof is a tricky consequence of the reverse FK inequality for *axisymmetric octagons* in $Q_\pm$ -position (inspired by [Meyer]).

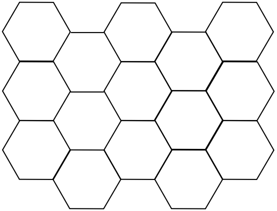
Motivation: almost ten years ago, Caffarelli-Lin *conjectured* that the optimal partition problem

$$\Lambda_k(\Omega) := \inf \left\{ \sum_{i=1}^{k} \lambda_1(E_i) \ : \ \{E_i\} \text{ k-partition of } \Omega \right\}$$

in the limit as $k \to +\infty$ is solved by a packing of regular hexagons, i.e.

$$\Lambda_k(\Omega) \sim \frac{k^2}{|\Omega|} \lambda_1(H) ,$$

where H is a unit area *regular hexagon*. STILL OPEN !



Advances in existence and regularity theory

- Optimal spectral partitions

$$\Lambda_k(\Omega) := \inf \left\{ \max_{i=1,\dots,k} \lambda_1(E_i) \ : \ \{E_i\} \text{ k-partition of } \Omega \right\}$$

[Bonaillie-Nöel, Bucur, Conti, Helffer, Hoffmann-Ostenhof, Ramos, Tavares, Terracini, Velichkov, Verzini, Vial]

- Optimal Cheeger partitions

$$H_k(\Omega) := \inf \left\{ \sum_{i=1}^{k} h(E_i) \ : \ \{E_i\} \text{ k-partition of } \Omega \right\}$$

[Caroccia]

Towards the honeycomb conjecture: a class of minimal convex partitions

$$m_k(\Omega) := \inf \left\{ \max_{i=1,\dots,k} F(E_i) \ : \ \{E_i\} \text{ convex k-partition of } \Omega \right\},$$

where:

- Ω is an open bounded smooth subset of $\mathbb{R}^2$;
- a convex k-partition of Ω is a family $\{E_i\}$ such that
  - $E_i \subset \Omega$ and $E_i \in \mathscr{K}^2$ for every i;
  - $0 < |E_i| < +\infty$ for every i;
  - $|E_i \cap E_j| = 0$ for every $i \neq j$;
- F is a given shape functional satisfying a few assumptions.

Theorem [Bucur-F.-Velichkov-Verzini]

There holds

$$\lim_{k\to+\infty} \frac{1}{k^{\alpha/2}} |\Omega|^{\alpha/2} m_k(\Omega) = F(H), \qquad \text{H} = \text{unit area regular hexagon}$$

provided:

(1) F is monotone decreasing under domain inclusion;

(2) F is homogeneous of degree $-\alpha$ under dilations;

(3) setting $\gamma_n := \min\Big\{F(P) \ : \ P \in \mathscr{P}_n, \ |P| = 1\}$, we have

(a) $\gamma_6 = F(H)$;

(b) $\frac{1}{k}\sum_{i=1}^{k} n_i \le 6 \ \Rightarrow \ \frac{1}{k}\sum_{i=1}^{k} {\gamma_{n_i}}^{2/\alpha} \ge {\gamma_6}^{2/\alpha}$.

Remark: If F satisfies a discrete FK inequality telling that $\gamma_n = F(P_n^*)$:
(3a) OK. (3b) OK if the map $n \mapsto F(P_n^*)^{2/\alpha}$ is convex and decreasing.

The case of the Cheeger constant

$$m_k(\Omega) := \inf \left\{ \max_{i=1,\dots,k} h(E_i) \ : \ \{E_i\} \text{ convex k-partition of } \Omega \right\}$$

The functional $F(\Omega) = h(\Omega)$ satisfies all the assumptions:

(1) monotone decreasing under inclusions;

(2) homogeneous of degree -1;

(3) satisfies the discrete FK inequality and the map $n \mapsto h(P_n^*)^2$ is decreasing and convex.

Hence,

$$\lim_{k\to+\infty} \frac{1}{k^{1/2}} |\Omega|^{1/2} m_k(\Omega) = h(H) \,.$$

The case of the first Dirichlet eigenvalue

$$m_k(\Omega) := \inf \left\{ \max_{i=1,\dots,k} \lambda_1(E_i) \ : \ \{E_i\} \text{ convex k-partition of } \Omega \right\}$$

The functional $F(\Omega) = \lambda_1(\Omega)$ satisfies assumptions (1) and (2), and we can give "simple" sufficient conditions for the validity of (3):

(1) monotone decreasing under inclusions;

(2) homogeneous of degree -2;

(3) TRUE IF

- $\gamma_6 = \lambda_1(H)$
- $\gamma_5 \geq a = 6.02\pi$
- $\gamma_7 \geq b := 5.82\pi$.

IF the conditions above are satisfied:

$$\lim_{k \to +\infty} \frac{1}{k} |\Omega| m_k(\Omega) = \lambda_1(H) .$$

Hints of proof.

- By an argument of [Bonaillie Nöel-Helffer-Vial], it is enough to show that

$$m_k(\Omega_k) = F(H)$$

whenever Ω_k is the tiling of k-copies of H.

- Given a convex k-partition $\{E_i\}$ of Ω_k, let us show that

$$\max_{i=1,\dots,k} F(E_i) \leq F(H) \quad \Rightarrow \quad \max_{i=1,\dots,k} F(E_i) = F(H).$$

- We consider a convex k-partitions made by *polygons* P_i such that

$$E_i \subseteq P_i \quad \forall i = 1, \dots, k \qquad \text{and} \qquad \sum_{i=1}^{k} |P_i| = k\,.$$

We have $F(P_i) \leq F(E_i) \leq F(H) \quad \forall i = 1, \dots, k$.

We are done if we prove that $F(P_i) \geq F(H)(= \gamma_6) \quad \forall i = 1, \dots, k$.

▷ The number of sides n_i of P_i satisfies the *mean value property*:

$$\frac{1}{k}\sum_{i=1}^{k} n_i \leq 6\,.$$

▷ We have:

$$\gamma_6 |P_i|^{\alpha/2} \geq F(P_i)|P_i|^{\alpha/2} \geq \gamma_{n_i} \qquad \forall i = 1, \dots, k\,.$$

Hence

$$F(P_i) \geq \frac{\gamma_{n_i}}{|P_i|^{\alpha/2}} \qquad \text{and} \qquad |P_i| \geq \Big(\frac{\gamma_{n_i}}{\gamma_6}\Big)^{2/\alpha}.$$

Summing over k, we get

$$k = \sum_{i=1}^{k} |P_i| \geq \sum_{i=1}^{k} \Big(\frac{\gamma_{n_i}}{\gamma_6}\Big)^{2/\alpha} \geq k\,.$$

We conclude that:

$$F(P_i) \geq \gamma_6 \qquad \forall i = 1, \dots, k\,.$$

□

Consequence (coming soon...)

The case of the first Robin eigenvalue of the Laplacian

$$\lambda_1(\Omega;\beta) := \inf_{u \in H^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 + \beta \int_{\partial\Omega} u^2}{\int_\Omega u^2}, \qquad \beta > 0.$$

Setting

$$m_k(\Omega;\beta) := \inf \Big\{ \max_{i=1,\dots,k} \lambda_1(E_i;\beta) \ : \ \{E_i\} \text{ convex k-partition of } \Omega \Big\},$$

there holds

$$\lim_{k \to +\infty} \frac{1}{k^{1/2}} |\Omega|^{1/2} m_k(\Omega;\beta) = \beta\, h(H).$$

Asymptotically, an optimal partition is obtained by the *Cheeger sets* of the cells of a regular hexagonal honeycomb.

References:

▷ Bucur-F., Journal of Geometric Analysis, 2016.

▷ Bucur-F., Proc. Royal Soc. Edinburgh, to appear.

▷ Bucur-F.-Velichkov-Verzini: preprint, 2016.

MANY THANKS FOR YOUR ATTENTION